Orbital dyanmics: Calculus

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(Also `~/Problems/Analysis/Calculus/tunnel.latex`)

Abbreviations. For common numbers, use

$$\boldsymbol{\pi} := 2\pi \qquad \text{and} \qquad \boxed{2} := \sqrt{2}\,.$$

(Mnemonically: *Doubled Pi* and □*root* 2.)

Use ⓜ, ⓓ, ⓣ for abstract units of *mass*, *distance*, *time*.

Use i.p.t for "is proportional to", with $\asymp$ as symbol. Use SoG for "Source of Gravity"; the center of an inverse-square rotationally symmetric gravity field; a planet or a sun.

Language: A planet ***rotates*** about its axis, and ***revolves*** about its sun.

Tools from physics. The kinetic energy of an object of mass m traveling at speed s is $\frac{1}{2}ms^2$; the units are ⓜⓓ^{2}/ⓣ^{2}. We will do all our computations in terms of ***energy-density***, that is, *energy per unit mass*. Restating,

1: *The energy-density of a speed-s object is $\frac{1}{2}s^2$.*

Newton tells us, at distance ρ from a SoG, that

2: $$\text{Accel. from gravity} \;=\; \mathsf{K}/\rho^2\,,$$

where K is a "constant of proportionality" that depends on the planet; it has units ⓓ^{3}/ⓣ^{2}.[♡1]

[♡1] This K is called the ***Standard gravitational parameter*** of the SoG. (When the SoG is a planet, this is also called the ***Geocentric gravitational constant***.) It is the product of the mass of the Planet/Sun times the Universal Gravitational Constant. One symbol for the *Standard gravitational parameter* is μ, but I will use K in these notes.

Aside: I've arranged the computation so we'll never need to know the mass of a planet, nor the Univ-Grav-Const.

Calculus. Let $\mathbf{u}{:}\mathbb{R}\to\mathbb{R}^2$ denote the "unit speed" parametrization of a circle. Twice differentiating w.r.t time yields that

3: $$\mathbf{u}''() \;=\; -\mathbf{u}()\,,$$

either by directly applying the defn of derivative, or else differentiating the coordinate formula

3*b*: $$\mathbf{u}(t) \;:=\; \cos(t)\cdot\hat{\boldsymbol{\imath}} \;+\; \sin(t)\cdot\hat{\boldsymbol{\jmath}}\,.$$

For a constant-speed object traveling in a circle [the ***period*** is the time to go once around], we use

4: $\ell :=$ period, $\qquad \rho :=$ radius, $\qquad s :=$ speed.
Hence $\; s\cdot\ell \;=\; \rho\cdot\boldsymbol{\pi}\,.$

To see that a parametrization of a speed-s object traveling in a radius-ρ circle is

3*c*: $$\mathbf{F}(t) \;:=\; \rho\cdot\mathbf{u}\Big(t\cdot\frac{s}{\rho}\Big)\,,$$

note that $\mathbf{F}'(t)$ equals $\rho\cdot\frac{s}{\rho}\cdot\mathbf{u}(t\cdot\frac{s}{\rho})$ by the Chain rule. I.e $\mathbf{F}'(t) = s\cdot\mathbf{u}'(t\cdot\frac{s}{\rho})$. So $\|\mathbf{F}'\| = s\cdot\|\mathbf{u}'\| = s$.

Twice time-differentiating gives

3*d*: $$\mathbf{F}''(t) \;:=\; \rho\cdot\tfrac{s}{\rho}\cdot\tfrac{s}{\rho}\cdot\mathbf{u}''\Big(t\cdot\tfrac{s}{\rho}\Big) \;=\; \frac{s^2}{\rho}\cdot\mathbf{u}''\Big(t\cdot\tfrac{s}{\rho}\Big)\,.$$

Taking norms shows that the magnitude of the acceleration of the object is

2′: $$\text{Accel. of motion} \;=\; s^2/\rho\;.$$

Equations (**??**′) & (2), together, show that an s,ρ,ℓ-orbit satisfies

5: $$\mathsf{K} \;=\; \rho\cdot s^2 \;\underset{}{\overset{\text{by (4)}}{=\!=}}\; \frac{\rho^3}{\ell^2}\cdot\boldsymbol{\pi}^2\,, \quad \text{i.e,}$$
$$\rho^3 \;=\; \mathsf{K}\ell^2/\boldsymbol{\pi}^2\,.$$

Planet/Sun notation. Planet ***Pal*** has

$\mathsf{D} :=$ Day, $\quad \mathsf{R} :=$ Radius, $\qquad \mathsf{A} :=$ Sur-Acc.

Pal is in orbit about sun ***Sol***, with

$\mathsf{Y} :=$ Year, $\quad \mathsf{U} :=$ OrbitalRad<u>i</u>us.

The constants of proportionality, for Pal and for Sol, are thus

$$\text{6:}\quad \begin{aligned} \mathsf{K}_{\text{Pal}} &= \mathsf{A}\cdot\mathsf{R}^2\,; \\ \mathsf{K}_{\text{Sol}} &= \frac{\mathsf{U}^3}{\mathsf{Y}^2}\cdot\boldsymbol{\pi}^2\,. \end{aligned}$$

These follow from (5a), that is, the top line of (5).

Special orbits

A geosynchronous orbit, courtesy (5b), has radius and speed

$$\text{7a:}\quad \begin{aligned} \rho_{\text{Syn}} &= \sqrt[3]{\mathsf{AR}^2\cdot\mathsf{D}^2/\boldsymbol{\pi}^2} \overset{\text{Earth}}{\approx} 26{,}369\,\mathsf{mi}\,, \\ s_{\text{Syn}} &= \sqrt[3]{\mathsf{AR}^2\cdot\mathsf{D}^{-1}\cdot\boldsymbol{\pi}} \overset{\text{Earth}}{\approx} 6903\,\mathsf{mph}\,, \end{aligned}$$

respectively. This (7ab) follows from (7aa) and (4).

The ***golden-snitch orbit***[♡2] is an orbit around Pal at Pal's surface. But $s^2 = \mathsf{K}/\rho$, so (6) gives

$$\text{7b:}\quad \begin{aligned} s_{\text{Gold}} &= \sqrt{\mathsf{AR}} \overset{\text{Earth}}{\approx} 17{,}725\,\tfrac{\mathsf{mi}}{\mathsf{hr}}\,. \quad \text{From (4),} \\ \ell_{\text{Gold}} &= \frac{\mathsf{R}\cdot\boldsymbol{\pi}}{s_{\text{Gold}}} = \sqrt{\frac{\mathsf{R}}{\mathsf{A}}}\cdot\boldsymbol{\pi} \overset{\text{Earth}}{\approx} 1.414\,\mathsf{hr} \end{aligned}$$

is the period of the golden-snitch. Playing "chicken" with the snitch, I now stand somewhere on Pal's equator, with the snitch in orbit around the equator. How many times a day must I duck?

Let N be the **"duck number"** **if** Pal didn't rotate. On a rotating Pal, then, I duck $N\mp 1$ times, depending on whether the snitch revolves in the *same/opposite* direction that Pal rotates. Computing,

$$\text{7c:}\quad N = \frac{\mathsf{D}}{\ell_{\text{Gold}}} = \mathsf{D}\cdot\sqrt{\frac{\mathsf{A}}{\mathsf{R}}}\Big/\boldsymbol{\pi} \overset{\text{Earth}}{\approx} 16.92\,\text{times.}$$

Escape speed

Use $\mathcal{E}$ for ***energy density*** (energy per unit mass); here, I'll just call this "energy".

[♡2] In homage to the Harry Potter books.

At distance z from SoG, a test mass feels acceleration K/z^2. Dropping a test mass from radius ρ_2, it reaches radius ρ_1 with kinetic energy (energy-density, actually)

$$\text{8.1:}\quad \mathcal{E} = \int_{\rho_1}^{\rho_2}\frac{\mathsf{K}}{z^2}\,\mathrm{d}z = \Big[\frac{1}{\rho_1}-\frac{1}{\rho_2}\Big]\cdot\mathsf{K}\,.$$

Dropping from infinity therefore gives the escape energy. Since $\mathcal{E}_{\text{Esc}} = \frac{1}{2}\cdot[s_{\text{Esc}}]^2$, we conclude that to escape to ∞ from radius $\rho := \rho_1$ requires energy

$$\mathcal{E}_{\text{Esc}}(\rho) = \frac{\mathsf{K}}{\rho} \quad\text{and}\quad s_{\text{Esc}}(\rho)^2 = 2\cdot\frac{\mathsf{K}}{\rho}\,.$$

Courtesy (6), the escape-speeds are:

$$\text{8.2:}\quad \begin{aligned} \mathsf{S}_{\text{Pal}} &= \sqrt{2\mathsf{AR}} \overset{\text{Earth}}{\approx} 7\,\mathsf{mi/sec}\,; && \text{(from Pal at surface.)} \\ \mathsf{S}_{\text{Sol}} &= \tfrac{\mathsf{U}}{\mathsf{Y}}\cdot\boldsymbol{\pi}\cdot\boxed{2} \overset{\text{Sun}}{\approx} 26\,\tfrac{\mathsf{mi}}{\mathsf{sec}}\,; && \text{(from Sol at Pal's orbit.)} \end{aligned}$$

From Pal's surface, suppose we fire a cannonball at a speed $s \geqslant s_{\text{Esc}}$. The cannonball goes to infinity, slowing down asymptotically to speed

$$s_{\text{Asymp}} = \sqrt{s^2 - \mathsf{S}_{\text{Pal}}{}^2}\,;$$

this, since kinetic energy varies as the *square* of speed. Similarly, if our cannonball must escape <u>both</u> Pal and Sol, then

$$s_{\text{EscBoth}} = \sqrt{\mathsf{S}_{\text{Sol}}{}^2 + \mathsf{S}_{\text{Pal}}{}^2}$$

is the necessary escape-speed.

Energy comparison. Comparing S_{Pal} with (7b), we see that the escape-energy, from Pal's surface, is exactly *twice* its golden-snitch energy. So

8.3: *For a satellite in orbit about a SoG, its escape-energy is precisely twice the orbital kinetic-energy of the satellite.*

In particular, to escape Sol when in Pal's orbit, one needs twice the orbital kinetic-energy.

With Pal's surface the zero of potential energy, what is the total energy needed to put an object in orbit at radius $\rho \geqslant \mathsf{R}$? Well $\mathcal{E}_{\text{Kin}} = \frac{1}{2}s^2$, which equals $\frac{1}{2} \cdot \frac{\mathsf{K}}{\rho}$, from (5a). And $\mathcal{E}_{\text{Pot}} = [\frac{1}{\mathsf{R}} - \frac{1}{\rho}]\mathsf{K}$, by (8.1). Adding together gives

$$\text{8.4:} \qquad \mathcal{E}_{\text{Total}}(\rho) := \left[\frac{2}{\mathsf{R}} - \frac{1}{\rho}\right]\frac{\mathsf{K}}{2}.$$

When $\rho = \infty$, there is no kinetic energy and so $\mathcal{E}_{\text{Total}}(\infty)$ should simply be esc-energy, $\mathcal{E}_{\text{Esc}}(\mathsf{R})$. And indeed $\mathcal{E}_{\text{Total}}(\infty)$ is precisely twice $\mathcal{E}_{\text{Total}}(\mathsf{R})$.

Kinetic energy to help escape. In escaping from Pal's surface, or from Sol, we ignored both Pal's rotational energy, and its orbital energy. Are they significant in helping escape?

From (8.3), we see that the orbital kinetic energy is significant in escaping Sol. How about escaping Pal's surface —does the *rotational* energy of Pal help?

At Pal's equator the rotational speed is $\boldsymbol{\pi}\mathsf{R}/\mathsf{D}$, so the rotational energy $\asymp \frac{\mathsf{R}^2}{\mathsf{D}^2}\boldsymbol{\pi}^2$.

Escape path. Fire a cannonball directly away from the earth's surface. The distance $y = y(t)$ satisfies DE

$$\text{8.5:} \qquad y'' = -\mathsf{K}/y^2.$$

(Natch', this is an autonomous DE.[♡3]) Seeking a specific solution, we try the form $y(t) := m \cdot t^\alpha$. Solving gives $\alpha = \frac{2}{3}$. We then solve for the multiplier m to get

$$\text{8.6:} \qquad \begin{aligned} m &= \sqrt[3]{\tfrac{9}{2}\mathsf{K}} = \sqrt[3]{\tfrac{9}{2}\mathsf{A} \cdot \mathsf{R}^2}. \quad \text{Thus} \\ y(t) &= t^{2/3} \cdot m = \sqrt[3]{\tfrac{9}{2}\mathsf{A} \cdot t^2 \cdot \mathsf{R}^2}. \end{aligned}$$

The speed, $\frac{dy}{dt}$, equals $\frac{2}{3}m\big/t^{1/3}$. Hence it goes to zero as $t\nearrow\infty$. Thus

> *This solution is the "escape speed" soln. At each point on the escape path, the object is travelling at escape speed.*

[♡3] An alternative DE has form $\frac{1}{2}[y']^2 + \frac{-\mathsf{K}}{y} = \text{Const}$, since its LhS is $\mathcal{E}_{\text{Kin}} + \mathcal{E}_{\text{Pot}}$.

Computing a time τ so that $y(\tau) = \mathsf{R}$ gives $\tau = \sqrt{\frac{2}{9} \cdot \frac{\mathsf{R}}{\mathsf{A}}}$. So the trajectory which departs Pal's surface at time-zero is

$$\text{8.7:} \qquad y_{\text{Surf}}(t) = \left[t\mathsf{R} \cdot \mathsf{A}^{1/2} \cdot \tfrac{3}{\sqrt{2}} + \mathsf{R}^{3/2}\right]^{2/3}.$$

Transit of Venus

When Venus passing directly between the Sun and the Earth, then we see Venus "in transit" across the Sun's surface. Our goal is to use this transit to estimate the distance from Earth to Sun. (This is called the ***A.U***, for "astronomical unit").

Use subscripts 1 and 0 for *Earth* and *Venus*, we have from (5) that

$$\left[\frac{\rho_1}{\rho_0}\right]^3 = \left[\frac{\ell_1}{\ell_0}\right]^2,$$

which is one of Kepler's laws.

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