

Base- \mathcal{V} numerals

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Note: I intend to eventually precede this with a discussion of base- \mathcal{V} representations of just *integers*.

Numerals. Fix a *base* $\mathcal{V} \in [2.. \infty)$. A *vigit* [base- \mathcal{V} digit] is an element of $J_{\mathcal{V}} := [0.. \mathcal{V})$. A *base- \mathcal{V} numeral* is an infinite string, σ , of form

$$\dagger: \quad \sigma := d_K \cdots d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots, \text{ with each } d_j \in J_{\mathcal{V}}$$

and $K \in [-1.. \infty)$. We call the “.” the *vig-point*. When $\mathcal{V}=10$ we may also call “.” the *decimal point*. [Note: $K = -1$ means there are no vigits before the point.] We regard two numerals as the *same numeral*, if we can alter one into the other by adjoining/removing leading zero-vigits. I.e.,

$$7.2945 \cdots \quad \text{and} \quad 007.2945 \cdots$$

are regarded as the same numeral.

An overbar indicates to ∞ ly repeat a finite string of vigits. E.g.,

$$34.2\overline{581}$$

abbreviates the $34.2581581581 \cdots$ numeral. The *eventual-period* of this string is 3, since string “581” has 3 vigits. When $\mathcal{V}=\text{ten}$, period=1 and the repetition string is “0”, e.g, $2.74\overline{0}$, then some folks write this as 2.74 and call it a “terminating decimal”, though other conventions for the missing digits are possible.

Numbers. The number represented by σ in (\dagger) is

$$\dagger: \quad \text{Num}_{\mathcal{V}}(\sigma) := \sum_{j \in (-\infty.. K]} d_j \cdot \mathcal{V}^j.$$

For example, in base $\mathcal{V}=\text{ten}$, the numeral $0.7\overline{9}$ represents $\frac{7}{10} + \sum_{k=2}^{\infty} [9/10^k]$, i.e $\frac{7}{10} + \frac{1}{10}$, that is, $\frac{4}{5}$. Another base-ten representation for $\frac{4}{5}$ is the $0.8\overline{0}$ numeral. Numerals

$$0.7\overline{9} \quad \text{and} \quad 0.8\overline{0} \quad (\text{in base-ten})$$

are decidedly **not** the same *numeral*, but they do name the same *number*.

***1: Base-twelve example.** Using $0, 1, \dots, 9, A, B$ for base-twelve vigits, these two base-twelve numerals

$$.04\overline{B} \quad \text{and} \quad .05\overline{0}$$

are names for the number $5/144$. □

The *\mathcal{V} -adic rationals* are those that can be written in form N/\mathcal{V}^k , with $N \in \mathbb{Z}$ and $k \in \mathbb{N}$.

***2: Multiple-rep Thm.** Fix \mathcal{V} . A *posreal* has multiple base- \mathcal{V} numerals IFF it is a \mathcal{V} -adic rational.

Moreover, a (positive) \mathcal{V} -adic rational x has precisely two base- \mathcal{V} reps. Write $y := \mathcal{V}^S x$ for the unique integer S which places y in interval $[\frac{1}{\mathcal{V}}, 1)$. Use $H := \mathcal{V} - 1$ for the largest vigit. Then one rep for y looks like

$$R_H: \quad .d_1 d_2 \cdots d_{L-1} f \overline{H}, \quad \text{with } L \in \mathbb{N},$$

where each $d_j \in [0.. H]$ and $f \in [0.. H)$. Numeral

$$R_0: \quad .d_1 d_2 \cdots d_{L-1} [f+1] \overline{0}$$

is the other base- \mathcal{V} representation. **Pf. Exercise E1.** ◇

Fraction \rightarrow eventually-periodic. Consider $x := \frac{N}{D}$, where $N, D \in \mathbb{Z}_+$. Write each posint in base- \mathcal{V} ,

$$N = f_L \cdots f_1 f_0 \quad \text{and} \quad D = d_K \cdots d_1 d_0,$$

then divide numeral $d_K \cdots d_0$ into $f_L \cdots f_0$, using the long-division algorithm. You will obtain ^{♥1} an eventually-periodic base- \mathcal{V} rep of x . Moreover (**exercise E2**), its eventual-period is less than D .

***3: Example: Eventually-periodic \rightarrow fraction.** We'll write $x := 3.511\overline{27}$ (base-ten) as a fraction.

Let $y := 10^3 x$, to move the periodic part to the vig-point. The periodic part has period $P=2$. So

$$10^2 \cdot y = 351127.\overline{27} \quad \text{and} \\ y = 3511.\overline{27}.$$

Subtracting yields that

$$[10^2 - 1]y = 351127 - 3511 \stackrel{\text{note}}{=} 347616.$$

^{♥1}When x is a \mathcal{V} -adic rational, you will obtain the (R_0) rep. To obtain the (R_H) rep: At stage L in the long-division, subtract one less copy of D than you would have. Stage $L+1$ will have you dividing D into $\mathcal{V} \cdot D$. Subtract only H copies of D and proceed to stage $L+2$. Continue.

Solving for y gives $y = \frac{347616}{99} \stackrel{\text{note}}{=} \frac{38624}{11}$. Hence

$$x = \frac{y}{10^3} = \frac{38624}{11 \cdot 1000} = \frac{4828}{1375},$$

as was sought. \square

Thus $x = \frac{5A9}{BB0}$. Dividing out 3 from numer & denom,

$$x = \frac{1B7}{3B8},$$

with numerator coprime to denominator. \square

General: Eventually-periodic \rightarrow fraction. For your eventually-periodic base- \mathcal{V} numeral for x , take $S \in \mathbb{Z}$ so that $y := \mathcal{V}^S \cdot x$ has numeral

$$y = d_K \cdots d_1 d_0 \cdot \overline{f_1 \cdots f_P}.$$

Subtracting the bottom from the top, in

$$\begin{aligned} \mathcal{V}^P \cdot y &= d_K \cdots d_1 d_0 f_1 \cdots f_P \cdot \overline{f_1 \cdots f_P} \quad \text{and} \\ y &= d_K \cdots d_1 d_0 \cdot \overline{f_1 \cdots f_P}, \end{aligned}$$

gives a posint N which is this difference

$$[d_K \cdots d_0 f_1 \cdots f_P] - [d_K \cdots d_0]$$

of base- \mathcal{V} numerals. Thus $[\mathcal{V}^P - 1]y = N$, so

$$*4: \quad x = \frac{N}{[\mathcal{V}^P - 1] \cdot \mathcal{V}^S}.$$

Finally, put RhS(*4) in standard form, so that the numerator and denominator are integers with no common factor.

***5: Base-twelve computation.** With $\mathcal{V} := 12$ and using $0, 1, \dots, 9, A, B$ for twigits, our goal is to write

$$x := .5\overline{B2}$$

as a ratio of posints (each written in, say, base-ten).

From above, $S = 1$ and $P = 2$. Moreover, N equals

$$5B2 - 5 \stackrel{\text{note}}{=} [5 \cdot 12^2 + 11 \cdot 12 + 2] - 5 = 849.$$

Thus x equals

$$\frac{849}{[12^2 - 1] \cdot 12^1} = \frac{849}{143 \cdot 12} = \frac{283}{143 \cdot 4} = \frac{283}{572},$$

as desired.

If we wish x written as a ratio of base-twelve numerals, we could convert $\frac{283}{572}$. Alternatively, simply do the above computations in base-twelve. E.g, $5B2 - 5 = 5A9$. And

$$[\mathcal{V}^P - 1] \cdot \mathcal{V}^S = [100 - 1] \cdot 10 = BB \cdot 10 = BB0.$$