

Nullspaces of commuting transformations : LinearAlg

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ABSTRACT: Conditions under which the nullspace of a composition $B \circ A$ is the span of the two nullspaces. This is then applied to constant-coefficient linear differential equations.

§A Entrance

The setting is a vectorspace \mathbf{V} and two linear transformations $A, B: \mathbf{V} \rightarrow \mathbf{V}$. Use both $\text{Nul}(A)$ and A° for the nullspace of trn A .
Use C for the composition $C := BA$.

1: **Fact.** Suppose $B, A: \mathbf{V} \rightarrow \mathbf{V}$. Then

$$2: \quad \text{Dim}(B^\circ) + \text{Dim}(A^\circ) \stackrel{1'}{\geq} \text{Dim}(\text{Nul}(BA)) \\ \geq \text{Dim}(A^\circ). \quad \diamond$$

Now suppose that $A \rightleftharpoons B$ (the trns commute) so that $C = BA = AB$. Then, automatically,

$$3: \quad C^\circ \supset \text{Spn}(B^\circ, A^\circ).$$

We explore when we have equality in (1'), and when in (3). For two subspaces $\mathbf{W}, \mathbf{W}' \subset \mathbf{V}$, let $\mathbf{W} \perp \mathbf{W}'$ mean that these $\{\mathbf{W}, \mathbf{W}'\}$ is a (linearly) independent set, as in (*), below. [In particular, \mathbf{W} and \mathbf{W}' only intersect in the singleton $\{\mathbf{0}\}$.] More generally, $\perp_{j=1}^K \mathbf{W}_j$ indicates mutual independence of the subspaces in that the *only* solution to

$$*: \quad \mathbf{w}_1 + \cdots + \mathbf{w}_K = \mathbf{0} \text{ with each } \mathbf{w}_j \in \mathbf{W}_j,$$

is to have every \mathbf{w}_j be $\mathbf{0}$.

4: **Commuting Thm.** Suppose that $A \rightleftharpoons B$. If $B^\circ \perp A^\circ$ and at least one of the nullspaces is finite-dim'al, then

$$3': \quad C^\circ = \text{Spn}(B^\circ, A^\circ). \quad \diamond$$

5: **Corollary.** Consider $C := B_1^{R_1} B_2^{R_2} \cdots B_K^{R_K}$ where $R_j \in \mathbb{N}$, and the B_1, \dots, B_K are commuting trns with at most one B_j having an ∞ -dim'al nullspace. If $\perp_{j=1}^K \text{Nul}(B_j^{R_j})$ then

$$4': \quad C^\circ = \text{Spn}(\text{Nul}(B_1^{R_1}), \dots, \text{Nul}(B_K^{R_K})).$$

Further suppose that each B_j has a 1-dim'al nullspace. If each $\text{Nul}(B_j^{R_j})$ has dimension at least R_j (and thus has dimension exactly R_j , by (1')) then $\text{Dim}(C^\circ)$ is precisely $R_1 + \cdots + R_K$. \diamond

CEX: C1. We do not get equality in (1') even if the trns commute and \mathbf{V} is finite-dim'al: Let B be the idempotent matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A := B$.

Moreover, equality (1') and commutativity and $\text{Dim}(\mathbf{V}) < \infty$ are not enough for (3'): Let B be the nilpotent matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A := B$ and B commute, yet $\text{Nul}(B^2)$ is 2-dim'al.

In both examples, the nullspaces of B and A are *not* linearly independent. \square

CEX: C2. Here is an example of $A \rightleftharpoons B$ with linearly-indep nullspaces, yet inclusion (3) is strict. (Necessarily, this \mathbf{V} is ∞ -dim'al):

Let $\mathcal{E} := \{\mathbf{w}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \dots\}$ be a basis for \mathbf{V} . Define trn B to map

$$\begin{aligned} \dagger: & \quad \mathbf{w} \mapsto \mathbf{a}_0 \text{ and each } \mathbf{a}_j \mapsto \mathbf{a}_{j+1}, \\ \ddagger: & \quad \text{and each } \mathbf{b}_j \mapsto \mathbf{0}. \end{aligned}$$

Restricted to $\mathbf{W} := \text{Spn}(\mathbf{w}, \mathbf{a}_0, \dots)$, trn B has no nullspace, courtesy (\dagger). Since B maps \mathbf{W}° , we have equality

$$B^\circ = \text{Spn}(\mathbf{b}_0, \mathbf{b}_1, \dots).$$

Define trn A on basis \mathcal{E} using (\dagger, \ddagger) analogously, reversing the roles of vectors $\mathbf{a}_?$ and $\mathbf{b}_?$; thus A° is $\text{Spn}(\mathbf{a}_0, \mathbf{a}_1, \dots)$. So $A^\circ \perp B^\circ$. And easily BA and AB are each the zero operator. Yet \mathbf{w} is not in $\text{Spn}(B^\circ, A^\circ)$. \square

Proof of Commuting Thm, (4). Since $A \rightleftharpoons B$, we can WLOG shrink V to be C° ; so our goal is to show that every vector is in $\text{Spn}(B^\circ, A^\circ)$. WLOG A° is finite dim'al. So fixing a vector $w \in V$, ISTFind an $a \in A^\circ$ so that

6: $\text{Trn } B \text{ sends } w - a \text{ to } 0.$

$\text{Trn } B$ maps all of V into A° (by commutativity). Thus $B|_{A^\circ}$ maps $A^\circ \rightarrow A^\circ$. But $\text{Dim}(A^\circ)$ is finite, so $B|_{A^\circ}$ maps A° onto A° . (Otherwise $B|_{A^\circ}$ would have to fail to be 1-to-1, ie, it would have nullspace. But the nullspaces of B and A are linearly indep, by hypothesis.)

The upshot is that $B(w)$ is in A° , and there is a vector $a \in A^\circ$ which is also mapped to $B(w)$ by B . Hence (6). ♦

§B Application to Differential Eqns

Let D and I be the differentiation and identity operators. That the nullspace of $D - I$ is $\text{Spn}(e^x)$ follows easily from the Mean Value Thm; in particular, this nullspace is one-dimensional. (Here, we are using expression e^x to mean the fnc $[x \mapsto e^x]$.) More generally,

7:
$$x^j e^x \xrightarrow{D-I} j \cdot x^{j-1} e^x, \text{ for } j \in \mathbb{N}.$$

Courtesy (1), the dimension of $\text{Nul}([D-I]^7)$ is at most 7. A small effort shows that the set of fncs $\{e^x, xe^x, x^2e^x, \dots, x^6e^x\}$ is lin-indep, and so its span is precisely $\text{Nul}([D-I]^7)$.

A linear substitution now shows that

8:
$$\text{Nul}([D-9I]^7) = \text{Spn}(e^{9x}, xe^{9x}, x^2e^{9x}, \dots, x^6e^{9x}).$$

Here “9” represents an arbitrary complex number, and “7” represents an arbitrary posint.

Consider an arbitrary complex monic polynomial $p(D) = D^N + b_{N-1}D^{N-1} + \dots + b_0I$. Factor it as

$$p(D) = [D - z_1I]^{R_1} \cdot \dots \cdot [D - z_KI]^{R_K}$$

with distinct zeros $z_1, \dots, z_K \in \mathbb{C}$. These K operators $[D - z_jI]^{R_j}$ commute and satisfy the remaining hypotheses of (5). Hence Corollary 5 applies to show that the set of fncs f satisfying this DE,

9:
$$f^{(N)} + b_{N-1}f^{(N-1)} + \dots + b_1f' + b_0f = 0,$$

is the span of

9':
$$\begin{aligned} & e^{z_1x}, xe^{z_1x}, x^2e^{z_1x}, \dots, x^{R_1-1}e^{z_1x}, \\ & e^{z_2x}, xe^{z_2x}, x^2e^{z_2x}, \dots, x^{R_2-1}e^{z_2x}, \\ & \vdots \\ & e^{z_Kx}, xe^{z_Kx}, x^2e^{z_Kx}, \dots, x^{R_K-1}e^{z_Kx}. \end{aligned}$$

And this is the standard result from a diffyq course. In particular, the soln-set to (9) has dimension N .

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