Algorithms in Number Theory

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Iterated Lightning-bolt (Euclidean algorithm)

Fix integers J_0 and J_1, and set $D := \text{Gcd}(J_0, J_1)$. A pair (S,T) of integers is "a ***Bézout pair*** for J_0, J_1" if

1a: $$SJ_0 + TJ_1 \quad = \quad D\,.$$

Bézout's lemma says: *There always exists a Bézout pair.* (Alternative term: S and T are ***Bézout multipliers***.)

A Bézout pair (S,T) is not unique; it is (except in the boring J_0=0=J_1 case) part of a one-parameter family

1b: $$\begin{aligned} S_{\langle k\rangle} &:= S + \left[k \cdot \frac{J_1}{D}\right] \quad \text{and} \\ T_{\langle k\rangle} &:= T - \left[k \cdot \frac{J_0}{D}\right], \end{aligned}$$

of Bézout pairs $(S_{\langle k\rangle}, T_{\langle k\rangle})$, for each $k \in \mathbb{Z}$.

1c: *Exercise.* Prove that (1b) describes *all* the Bézout pairs for J_0,J_1. □

Gcd of several integers. Given a list of integers, $\vec{\mathbf{J}} = (J_0, J_1, \ldots, J_L)$, use

2a: $$\text{Gcd}(J_0, J_1, \ldots, J_L) \text{ or } \text{Gcd}(\vec{\mathbf{J}})$$

to denote the greatest common divisor, D, of the list. Our goal is to simultaneously compute D and a Bézout-tuple $\vec{\mathbf{s}} := (S_0, \ldots, S_L)$ such that

2b: $$\sum_{\ell=0}^{L} [S_\ell \cdot J_\ell] \quad = \quad D\,.$$

We'll accomplish this with L applications of LBolt:

$$D \overset{\text{note}}{=\!=} \text{Gcd}\Big(\ldots \text{Gcd}(\text{Gcd}(J_0, J_1), J_2) \ldots, J_L\Big)\,.$$

Algorithm: From integers $\vec{\mathbf{J}} = (J_0, J_1, \ldots, J_{L-1}, J_L)$, set

$$C := \text{Gcd}(J_0, J_1, \ldots, J_{L-1}) \quad \textit{and}$$
$$D := \text{Gcd}(J_0, J_1, \ldots, J_{L-1}, J_L) \overset{\text{note}}{=\!=} \text{Gcd}(C, J_L)\,.$$

Apply LBolt $L-1$ times to produce integers $\mathsf{v}_0, \ldots, \mathsf{v}_{L-1}$ with $\sum_{\ell=0}^{L-1} [\mathsf{v}_\ell \cdot J_\ell] = C$, and an L^{th} time to produce $\alpha,\beta \in \mathbb{Z}$ with $\alpha C + \beta J_L = D$. Then

2c: $$S_0 := \alpha\mathsf{v}_0,\ S_1 := \alpha\mathsf{v}_1,\ \ldots,\ S_{L-1} := \alpha\mathsf{v}_{L-1},\ S_L := \beta,$$

gives a tuple $\vec{\mathbf{s}}$ satisfying (2b).

Proof. From the above defns of $\vec{\mathbf{v}}$, and of α and β,

$$\begin{aligned} D \;=\; \alpha C + \beta J_\ell \;&=\; \Big[\alpha \cdot \textstyle\sum_{\ell=0}^{L-1} [\mathsf{v}_\ell \cdot J_\ell]\Big] + \beta J_L \\ &=\; \Big[\textstyle\sum_{\ell=0}^{L-1} \alpha\mathsf{v}_\ell \cdot J_\ell\Big] + \beta J_L\,. \end{aligned}$$ ♦

Shorthand. Given two equal-length tuples of numbers, $\vec{\mathbf{a}} = (a_0, a_1, \ldots, a_N)$ and $\vec{\mathbf{c}} = (c_0, c_1, \ldots, c_N)$, define their ***dot product*** to be

$$\vec{\mathbf{a}} \bullet \vec{\mathbf{c}} \;:=\; \sum_{n=0}^{N} a_n \cdot c_n\,.$$

Worked LBolt. Consider $\vec{\mathbf{J}} := (525, 150, 350, 210)$. Using 3 applications of LBolt, we will compute a Bézout tuple $\vec{\mathbf{s}}^3$ such that $\vec{\mathbf{s}}^3 \bullet \vec{\mathbf{J}} = \text{Gcd}(\vec{\mathbf{J}})$.

We LBolt to compute gcd and multipliers:

$$\begin{aligned} D_1 &:= \text{Gcd}(J_0, J_1) = \text{Gcd}(525, 150) = 75\,; \\ (\alpha, \beta) &:= (1, -3)\,. \end{aligned}$$

Setting $\vec{\mathbf{s}}^1 := (1, -3)$, then, $\vec{\mathbf{s}}^1 \bullet (525, 150) = 75$.

Apply the algorithm again to produce

$$\begin{aligned} D_2 &:= \text{Gcd}(D_1, J_2) = \text{Gcd}(75, 350) = 25\,; \\ (\alpha, \beta) &:= (5, -1)\,. \end{aligned}$$

Multiply $\alpha \cdot \vec{\mathbf{s}}^1 \overset{\text{note}}{=\!=} (5, -15)$, then adjoin β, to produce $\vec{\mathbf{s}}^2 := (5, -15, -1)$. So now $\vec{\mathbf{s}}^2 \bullet (525, 150, 350) = 25$.

A third application of LBolt gives

$$\begin{aligned} D_3 &:= \text{Gcd}(D_2, J_3) = \text{Gcd}(25, 210) = 5\,; \\ (\alpha, \beta) &:= (17, -2)\,. \end{aligned}$$

Multiply $\alpha \cdot \vec{\mathbf{s}}^2 \overset{\text{note}}{=\!=} (85, -255, -17)$, then adjoin β, to produce $\vec{\mathbf{s}}^3 := (85, -255, -17, -2)$.

The upshot is that

$$\begin{aligned} \vec{\mathbf{s}}^3 \bullet \vec{\mathbf{J}} \quad &\overset{\text{def}}{=\!=} \quad \vec{\mathbf{s}}^3 \bullet (525, 150, 350, 210) \\ &= \quad 5 \quad = \quad \text{Gcd}(\vec{\mathbf{J}})\,, \end{aligned}$$

as desired.

Remark. Each of the three Bézout (α, β) pairs is actually part of a 1-parameter family specified by (1b). It follows that the above $\vec{\mathbf{s}}^3$ is just one member of a

3-parameter family of (integer) 4-tuples $\vec{\mathbf{s}}$ that satisfy $\vec{\mathbf{s}}^3 \bullet \vec{\mathbf{J}} = \text{Gcd}(\vec{\mathbf{J}})$.

In other words, there is an injective (i.e, 1-to-1) function $f\!:\!\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ with the property that

$$f(a,b,c) \bullet \vec{\mathbf{J}} \quad = \quad \text{Gcd}(\vec{\mathbf{J}})\,,$$

for each triple a,b,c of integers. □

Solving a linear congruence

Having fixed a *modulus* $M \in \mathbb{Z}_+$, as well as a *coefficient* and *target* B,$T \in \mathbb{Z}$, our goal is to find all solutions x to

3: $$B{\cdot}x \;\equiv_M\; T, \quad \text{where } x \in [0\,..\,M).$$

Our algorithm has three Steps.

This congruence has a solution IFF there exists a *pair* (x,k) solving eqn

3∗: $$B{\cdot}x \;+\; M{\cdot}k \;=\; T, \quad \text{where } x,k \in \mathbb{Z}.$$

Evidently $D := \text{Gcd}(B,M)$ divides LhS(3∗). Hence if $T \nmid D$, then (3∗) has no soln-pair. Whence

Step A: If $D := \text{Gcd}(B,M)$ fails to divide T, then (3) has no soln. Else, define

$$\beta := \tfrac{B}{D},\ \mu := \tfrac{M}{D} \text{ and } \tau := \tfrac{T}{D}$$

and study this "reduced congruence":

4: $$\beta{\cdot}y \;\equiv_\mu\; \tau, \quad \text{where } y \in [0\,..\,\mu).$$

We have gained that $\beta \perp \mu$.

Step B: Use LBolt to compute a mod-μ multiplicative-inverse, I, of β; so $I{\cdot}\beta \;\equiv_\mu\; 1$. Thus

$$y \;\equiv_\mu\; I{\cdot}\beta{\cdot}y \;\equiv_\mu\; I{\cdot}\tau\,.$$

Let y_0 be the unique value in $[0\,..\,\mu)$ st. $y_0 \equiv_\mu I{\cdot}\tau$.

This y_0 is in the *unique* mod-μ residue class solving (4). But mod-M, this residue class splits into D many residue classes. So here is the last step:

Step C: The D many solutions to (3) are

$$x \;=\; y_0,\, y_1,\, y_2,\, y_3,\, \ldots,\, y_{D-2},\, y_{D-1}\,,$$

where $y_k := y_0 + [k\mu]$.

A worked example. I use an arrow over a letter to abbreviate a sequence, e.g

$$\vec{\mathbf{b}} \;:=\; (b_0, b_1, b_2, \ldots)\,.$$

We consider

$$35x \;\equiv_{21}\; 55\,.$$

Let's apply Step A. Since $\text{Gcd}(35,21) \overset{\text{note}}{=\!=} 7$ does not divide 55, the above congruence has no soln. (The same computation shows that congr. $21x \equiv_{35} 55$ has no solution.)

A congruence with soln. Consider congruence

3′: $$33{\cdot}x \;\equiv_{114}\; 18\,, \quad \text{where } x \in [0\,..\,114).$$

For Step A, we compute just the $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ columns:

n	r_n	q_n
0	114	—
1	33	3
2	15	2
3	3	5
4	0	∞

Since $D := \text{Gcd}(33,114) \overset{\text{note}}{=\!=} 3$ divides the target, 18, we divide each of the numbers in (3′) by D=3 to obtain the reduced congruence

4′: $$11{\cdot}y \;\equiv_{38}\; 6, \quad \text{where } y \in [0\,..\,38).$$

For Step B, we compute (using $\vec{\mathbf{q}}$) just[♡1] the $\vec{\mathbf{t}}$ column. (Note: We have $\vec{\mathbf{q}}$ from the previous table.)

n	r_n	q_n	s_n	t_n
0	38	—	1	0
1	11	3	0	1
2	5	2	1	-3
3	1	5	-2	7
4	0	∞	11	-38

So the mod-38 reciprocal of 11 is 7. From Step B, then,

$$y \;\equiv_{38}\; 7 \cdot 6 \;=\; 42 \;\equiv_{38}\; 4\,.$$

So we set $\boxed{y_0 := 4}$.

[♡1] We do not need to compute $\vec{\mathbf{s}}$ nor $\vec{\mathbf{r}}$. Of course, the new $\vec{\mathbf{r}}$ is just the old $\vec{\mathbf{r}}$ divided by D. I have grayed-out the superfluous columns.

Finally, STEP C tells us that these three,

$$\mathbf{4}, \quad 4+38 \overset{\text{note}}{=\!=} \mathbf{42}, \quad 42+38 \overset{\text{note}}{=\!=} \mathbf{80}$$

are the D=3 many solutions to (3′).

Checking. We calculate:

$$\begin{aligned} 33\cdot\mathbf{4} &= 132 = 1{\cdot}114+18\,; \\ 33\cdot\mathbf{42} &= 1386 = 12{\cdot}114+18\,; \\ 33\cdot\mathbf{80} &= 2640 = 23{\cdot}114+18\,. \end{aligned}$$

Copasetic!

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