There does not exist $f{:}\mathbb{R}\circlearrowleft$ continuous exactly on $\mathbb{Q}$: Topology,BCT

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
23 September, 2017 (at *16:17*)

ABSTRACT: Gives various applications of BCT, the Baire Category theorem.

Note. The ***ruler function*** $\mathcal{R}{:}\mathbb{R}\to[0,1]$,

$$\mathcal{R}(x) := \begin{Bmatrix} 0 & \text{if } x \text{ irrational;} \\ \frac{1}{q} & \text{if } x \text{ has form } \frac{p}{q} \text{ in lowest terms} \end{Bmatrix}$$

is continuous precisely on the irrationals. The next thm shows that the opposite of this behavior is not possible. □

1: Theorem. *There does not exist a function $f{:}\mathbb{R}\to\mathbb{R}$ with $\mathrm{Cty}(f) = \mathbb{Q}$.* ♢

Pf. The set $\mathrm{Cty}(f)$ is always a $\mathcal{G}_\delta$ (exer., or see *notes-AdvCalc.pdf*). Were $\mathrm{Cty}(f) = \mathbb{Q}$, it would be a *dense* $\mathcal{G}_\delta$, hence residual. But $\mathbb{Q}$, being countable, is meager. ♦

2: Theorem. *Suppose we have sets $A,B \subset \mathbb{R}$, each $\mathbb{R}$-dense, and continuous functions $f_n{:}\mathbb{R}\to\mathbb{R}$ such that*

$$\dagger: \quad f_n\lfloor_A \xrightarrow{n\to\infty} \mathbf{0}\lfloor_A \quad \text{and} \quad f_n\lfloor_B \xrightarrow{n\to\infty} \mathbf{1}\lfloor_B\,,$$

where each convergence is pointwise. Then this set

$$\ddagger: \quad D := \left\{ x\in\mathbb{R} \;\middle|\; \begin{array}{l} \limsup_n f_n(x) \geqslant 1 \quad \text{and} \\ \liminf_n f_n(x) \leqslant 0 \end{array} \right\}$$

is residual in $\mathbb{R}$. ♢

Proof. For a value $\mathsf{v}\in\mathbb{R}$ and posint K, the set

$$U_{\mathsf{v},K} := \left\{ x\in\mathbb{R} \;\middle|\; \exists n \geqslant K \text{ s.t } |f_n(x) - \mathsf{v}| < \tfrac{1}{K} \right\}$$

is open. Thus $G_\mathsf{v} := [\bigcap_{K=1}^\infty U_{\mathsf{v},K}]$ is a $\mathcal{G}_\delta$ set. And

2a: *G_v comprises those x whose sequence $(f_n(x))_{n=0}^\infty$ has v as a limit-point.*

Since $G_1 \supset B$ and $G_0 \supset A$, each G_i is dense, hence residual. Thus $G_1 \cap G_0$ is residual. And $G_1 \cap G_0 \subset D$. ♦

2b: *The proof shows more.* Consider denumerably many values $(\mathsf{v}_k)_{k=1}^\infty$ and sets $(A_k)_1^\infty$, each dense, s.t for every point $y\in A_k$: $\lim(\vec{\mathbf{f}}(y)) = \mathsf{v}_k$.

The proof shows that the following set, $\widetilde{D}$, is residual, where $x \in \widetilde{D}$ IFF each value in $\{\mathsf{v}_k\}_1^\infty$ is a limit-point of the $(f_n(x))_{n=1}^\infty$ sequence. □

2c: *Question.* In ($\ddagger$), can the "$\geqslant$" and "$\leqslant$" each be replaced by "$=$"?

No! We'll make $A,B,\vec{\mathbf{f}}$, as in (2), such that

$$*: \quad x\in\widetilde{D} \quad\Longrightarrow\quad \limsup_{n\to\infty} f_n(x) = +\infty\,.$$

for a particular residual set $\widetilde{D}$.

Let $A{=:}A_0, B{=:}A_1, A_2, \ldots, A_k, \ldots$ be pairwise-disjoint sets, each countable and $\mathbb{R}$-dense. For $k = 0,1,2,\ldots$, fix an enumeration of A_k.

For each n, we can construct a piecewise-linear f_n which, for each of $k = 0,1,\ldots,n$, takes the value $\mathsf{v}_k := k$ on the first n members of A_k.

Apply (2b). This produces a residual set $\widetilde{D}$ s.t for each $x \in \widetilde{D}$: Value $[\limsup_n f_n(x)]$ dominates each posint k. Thus $(*)$. □

Filename: `Problems/Topology/no_fnc_cts_on_Q.latex`
As of: *Friday 24Apr2009.* Typeset: *23Sep2017 at 16:17.*