

**There does not exist  $f:\mathbb{R}\circlearrowleft$  continuous exactly on  $\mathbb{Q}$ : Topology,BCT**

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ABSTRACT: Gives various applications of BCT, the Baire Category theorem.

*Note.* The **ruler function**  $\mathcal{R}:\mathbb{R}\rightarrow[0,1]$ ,

$$\mathcal{R}(x) := \left\{ \begin{array}{l} 0 \quad \text{if } x \text{ irrational;} \\ \frac{1}{q} \quad \text{if } x \text{ has form } \frac{p}{q} \text{ in lowest terms} \end{array} \right\}$$

is continuous precisely on the irrationals. The next thm shows that the opposite of this behavior is not possible.  $\square$

**1: Theorem.** *There does not exist a function  $f:\mathbb{R}\rightarrow\mathbb{R}$  with  $\text{Cty}(f) = \mathbb{Q}$ .*  $\diamond$

*Pf.* The set  $\text{Cty}(f)$  is always a  $\mathcal{G}_\delta$  (exer., or see [notes-AdvCalc.pdf](#)). Were  $\text{Cty}(f) = \mathbb{Q}$ , it would be a dense  $\mathcal{G}_\delta$ , hence residual. But  $\mathbb{Q}$ , being countable, is meager.  $\blacklozenge$

**2: Theorem.** *Suppose we have sets  $A,B \subset \mathbb{R}$ , each  $\mathbb{R}$ -dense, and continuous functions  $f_n:\mathbb{R}\rightarrow\mathbb{R}$  such that*

$$\dagger: \quad f_n|_A \xrightarrow{n\rightarrow\infty} \mathbf{0}|_A \quad \text{and} \quad f_n|_B \xrightarrow{n\rightarrow\infty} \mathbf{1}|_B,$$

where each convergence is pointwise. Then this set

$$\dagger: \quad D := \left\{ x \in \mathbb{R} \mid \begin{array}{l} \limsup_n f_n(x) \geq 1 \quad \text{and} \\ \liminf_n f_n(x) \leq 0 \end{array} \right\}$$

is residual in  $\mathbb{R}$ .  $\diamond$

*Proof.* For a value  $v \in \mathbb{R}$  and posint  $K$ , the set

$$U_{v,K} := \left\{ x \in \mathbb{R} \mid \exists n \geq K \text{ s.t. } |f_n(x) - v| < \frac{1}{K} \right\}$$

is open. Thus  $G_v := [\bigcap_{K=1}^\infty U_{v,K}]$  is a  $\mathcal{G}_\delta$  set. And

2a:  $G_v$  comprises those  $x$  whose sequence  $(f_n(x))_{n=0}^\infty$  has  $v$  as a limit-point.

Since  $G_1 \supset B$  and  $G_0 \supset A$ , each  $G_i$  is dense, hence residual. Thus  $G_1 \cap G_0$  is residual. And  $G_1 \cap G_0 \subset D$ .  $\blacklozenge$

**2b: The proof shows more.** Consider denumerably many values  $(v_k)_{k=1}^\infty$  and sets  $(A_k)_1^\infty$ , each dense, s.t for every point  $y \in A_k$ :  $\lim(\vec{f}(y)) = v_k$ .

The proof shows that the following set,  $\widetilde{D}$ , is residual, where  $x \in \widetilde{D}$  IFF each value in  $\{v_k\}_1^\infty$  is a limit-point of the  $(f_n(x))_{n=1}^\infty$  sequence.  $\square$

**2c: Question.** In  $(\dagger)$ , can the “ $\geq$ ” and “ $\leq$ ” each be replaced by “ $=$ ”?

**No!** We’ll make  $A,B,\vec{f}$ , as in  $(??)$ , such that

$$*: \quad x \in \widetilde{D} \implies \limsup_{n \rightarrow \infty} f_n(x) = +\infty.$$

for a particular residual set  $\widetilde{D}$ .

Let  $A=:A_0, B=:A_1, A_2, \dots, A_k, \dots$  be pairwise-disjoint sets, each countable and  $\mathbb{R}$ -dense. For  $k = 0, 1, 2, \dots$ , fix an enumeration of  $A_k$ .

For each  $n$ , we can construct a piecewise-linear  $f_n$  which, for each of  $k = 0, 1, \dots, n$ , takes the value  $v_k := k$  on the first  $n$  members of  $A_k$ .

Apply  $(??)$ . This produces a residual set  $\widetilde{D}$  s.t for each  $x \in \widetilde{D}$ : Value  $[\limsup_n f_n(x)]$  dominates each posint  $k$ . Thus  $(*)$ .  $\square$

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