

Mean Value Theorem and L'Hôpital's Rule

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Abbrevs. IVT; *Intermediate Value Thm.* MVT; *Mean Value Thm.* FTC; *Fund. Thm of Calculus.* Use cts for “continuous” and cty for “continuity”.

The exponential fnc exp can also be written $\exp(x) = e^x$. (So $\log \circ \exp = Id_{\mathbb{R}}$ and $\exp \circ \log = Id_{\mathbb{R}_+}$.) Use *NevZ* to mean “never-zero”; e.g “exp() is *NevZ* on \mathbb{R} ”.

Prolegomena. Use $\overline{\mathbb{R}}$ for the *extended reals* $[-\infty, +\infty]$. With DNE denoting “Does Not Exist”, adjoin a point to $\overline{\mathbb{R}}$ to create

$$\mathbb{R}^* := [-\infty, +\infty] \sqcup \{\text{DNE}\}.$$

Use *diff'able* for “differentiable”. A fnc $f: \mathbb{R} \rightarrow \mathbb{R}$ is *ext-diff'able* (for *extended diff'able*) at point 6 if the $\lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6}$ exists in $[-\infty, +\infty]$. Use f'_{LH} for the *lefthand ext-derivative*, from $\lim_{x \nearrow 6}$. Use f'_{RH} for the *right-hand ext-derivative*, $\lim_{x \searrow 6}$.

The result below apply to fncs on a closed bounded interval J . For specificity, I will use $J := [4, 6]$ and will use $J^\circ := (4, 6)$ for its interior.

A fnc $h: J \rightarrow \mathbb{R}$ has a (global) *max-point* $P \in J$ if

$$\forall x \in J: \quad h(P) \geq h(x);$$

and the number $h(P)$ is the “*max-value* of h on J ”. Weaker, P is a *local max-point* of h (on J) if there exists a J -open set $U \ni P$, so that P is a global max-point of $h|_U$. Imagine analogous defns for *min-point*, *min-value* and *local min-point* of h .

1: Tool. A *continuous* $h: J \rightarrow \mathbb{R}$ has a *max-point* and a *min-point*. *Proof.* Interval J is compact, etc. \diamond

2: Lemma. Suppose *continuous* $h: J \rightarrow \mathbb{R}$ has a *local-extremum* at a point $\tau \in J^\circ$. If h is *extended-differentiable* at τ , then $h'(\tau) = 0$. \diamond

Proof. WLOG τ is a local-min of h . So for all $x > \tau$ with x suff. close to τ , necessarily $h(x) - h(\tau) \geq 0$; thus $h'_{RH}(\tau) \in [0, +\infty]$. Similarly, $h'_{LH}(\tau) \in [-\infty, 0]$. By hypothesis, $h'_{LH}(\tau) = h'_{RH}(\tau)$. So $h'(\tau)$ is zero. \diamond

3: Rolle's Thm. Suppose a *continuous* $h: [4, 6] \rightarrow \mathbb{R}$ is *ext-diff'able* on $(4, 6)$. If $h(4) = h(6)$, then there exists a point $\tau \in (4, 6)$ such that $h'(\tau) = 0$. \diamond

Pf. Courtesy (2), WLOG $h()$ has no global-max in J° , so its global-max on J must be the common value of $h(4) = h(6)$. Thus h has a global-min in J° ; pick one, and call it τ . Now (2) gives that $h'(\tau) = 0$. \diamond

4: MVThm. Suppose *cts* $f: [4, 6] \rightarrow \mathbb{R}$ is *ext-diff'able* on $(4, 6)$. Then there exists a $\tau \in (4, 6)$ such that

$$f'(\tau) = \frac{f(6) - f(4)}{6 - 4}. \quad \diamond$$

Pf. With $S := \frac{f(6) - f(4)}{6 - 4}$, the slope of the chord, let

$$h(x) := f(x) - [x - 4] \cdot S.$$

Since $h(4) = f(4) = h(6)$, Rolle's thm applies to assert a point $\tau \in (4, 6)$ with $0 = h'(\tau) \stackrel{\text{note}}{=} f'(\tau) - 1 \cdot S$. \diamond

E.g. Fnc $f(x) := \sqrt[3]{x}$ has $f'(0) = +\infty$. MVT applies to assert a point $\tau \in (-8, 27)$ where $f'(\tau)$ equals $\frac{3 - (-2)}{27 - (-8)}$.

For a beautiful MVT-application due to Liouville, see *Liouville's Theorem* on my TeachingPage. \square

5: Prop'n. On an interval $H \subset \mathbb{R}$, suppose fnc g is *extended-differentiable*. If g' is *NevZ* on H , then g is *strictly-monotone* on H . (We do not assume that g' is *continuous*.) \diamond

Proof. Were g not strictly-monotone then, WLOG, there are points $a < b < c$, in H , with $g(a) \leq g(b)$ yet $g(b) \geq g(c)$. WLOG $g(a) \leq g(c)$. Applying IVT to g on $[a, b]$ yields point $z \in [a, b]$ with $g(z) = g(c)$. Now Rolle's thm produces a $\tau \in (z, c)$ with $g'(\tau) = 0$. \diamond

6: Cauchy MVT. *Continuous* fncs $f, g: [4, 6] \rightarrow \mathbb{R}$ are *diff'able* on $(4, 6)$. Then there exists a $\tau \in (4, 6)$ st.

$$6a: \quad f'(\tau) \cdot [g(6) - g(4)] = g'(\tau) \cdot [f(6) - f(4)].$$

And if g' is never-zero on $(4, 6)$, then

$$6b: \quad \frac{f'(\tau)}{g'(\tau)} = \frac{f(6) - f(4)}{g(6) - g(4)}. \quad \diamond$$

Proof of (6a). Apply Rolle's theorem to

$$7: \quad h(x) := f(x) \cdot [g(6) - g(4)] - g(x) \cdot [f(6) - f(4)],$$

noting that $h(4) = f(4)g(6) - f(6)g(4) = h(6)$. ♦

Proof of (6b). Since g' is NevZ on $(4, 6)$, the MVT forces $g(6) \neq g(4)$. So we can cross-divide in (6a). ♦

L'Hôpital's rule

I'll usually state the thms for a one-sided limits, $\lim_{x \searrow T}$ or $\lim_{x \nearrow T}$, where $T \in \overline{\mathbb{R}}$. I'll use "Lim(f)" to mean $\lim_{x \searrow T} f(x)$ or $\lim_{x \nearrow T} f(x)$, as appropriate.

First some **cautionary tales**:

8: No end. What is $\lim_{x \searrow -\infty} \frac{\sqrt{x^2+1}}{x}$? Well, $x < 0$ implies that $-\sqrt{x^2} = x$, so

$$\frac{\sqrt{x^2+1}}{x} = -\sqrt{1 + \frac{1}{x^2}}.$$

Hence the limit is -1. But applying L'H to $\frac{\sqrt{x^2+1}}{x}$ will (after algebra) give $\frac{x}{\sqrt{x^2+1}}$. Which L'H sends to the original $\frac{\sqrt{x^2+1}}{x}$. □

9: BttDB: Back to the Drawing Board. Certainly

$$\lim_{x \searrow -\infty} \frac{x^2 + \sin(x)}{x^2} = 1.$$

L'H examines $\frac{2x + \cos(x)}{2x}$; still $\lim=1$. But applying L'H again gives $\frac{2 - \sin(x)}{2}$, which has *no limit*. There is no error here; if the f'/g' limit doesn't exist in $\overline{\mathbb{R}}$, then we can draw **no conclusion** about the f/g limit. □

10: L'Hôpital's Thm. Fix a "target" $T \in [-\infty, +\infty)$ and consider real-valued fncs f, g defined on part of \mathbb{R} .

Suppose there exists a real number $B > T$ for which

10†: On $(T, B]$: $f \& g$ are diff'able, and g' is NevZ. ♥¹

Recalling that $\mathbb{R}^* = [-\infty, +\infty] \sqcup \{\text{DNE}\}$, define

$$L := \lim_{x \searrow T} \frac{f(x)}{g(x)} \in \mathbb{R}^* \quad \text{and}$$

$$\Lambda := \lim_{x \searrow T} \frac{f'(x)}{g'(x)} \in \mathbb{R}^*.$$

^{♥1}Courtesy (5), our g is zero at most once on $(T, B]$. Consequently, we can move B closer to T so that g is NevZ on $(T, B]$.

Suppose, as $x \searrow T$, that **either**

$$10_0: \quad g(x) \rightarrow 0 \text{ and } f(x) \rightarrow 0, \quad \text{or}$$

$$10_\infty: \quad g(x) \rightarrow \pm\infty.$$

If $\Lambda \neq \text{DNE}$, then $L = \Lambda$. ♦

Reduction to T finite. If $T = -\infty$, then define

$$\varphi(x) := f\left(\frac{-1}{x}\right) \quad \text{and}$$

$$\gamma(x) := g\left(\frac{-1}{x}\right).$$

So $\lim_{x \searrow 0} \frac{\varphi}{\gamma}(x) = L$. But the Chain Rule gives

$$\varphi'(x) = \frac{1}{x^2} \cdot f'\left(\frac{-1}{x}\right) \quad \text{and}$$

$$\gamma'(x) = \frac{1}{x^2} \cdot g'\left(\frac{-1}{x}\right).$$

Thus $\lim_{x \searrow 0} \frac{\varphi'}{\gamma'}(x) = \Lambda$. Proving L'H for $(\varphi, \gamma \text{ at } 0)$ will thus establish L'H for $(f, g \text{ at } -\infty)$.

In the proofs below, we will take $(T, B] = (4, 6]$. □

Proof using (10₀). We may extend f by continuity so that f is cts on $[4, 6]$; hence $f(4) = 0$. Ditto for g .

Each $x \in (4, 6]$, gives a point $\dot{x} \in (4, x)$ with

$$\frac{f'(\dot{x})}{g'(\dot{x})} \stackrel{\text{Cauchy-MVT}}{=} \frac{f(x) - f(4)}{g(x) - g(4)} \stackrel{\text{note}}{=} \frac{f(x)}{g(x)}.$$

Sending $x \searrow 4$ forces $\dot{x} \rightarrow 4$. ♦

Proof using (10_∞). Assume, say, $\Lambda=7$. (The $\Lambda=\pm\infty$ case just changes notation.) Fixing an $\varepsilon > 0$, ISTEablish that

$$\text{£1:} \quad \limsup_{x \searrow 4} \frac{f}{g}(x) \leq 7 + \varepsilon.$$

Inequality $\liminf \frac{f}{g} \geq 7 - \varepsilon$ will follow analogously.

Take $B > 4$ so close to 4 that

$$\text{£2:} \quad \forall y \in (4, B] : \quad \left| \frac{f'}{g'}(y) - 7 \right| \leq \varepsilon.$$

For each $x \in (4, B]$, the Cauchy-MVT gives a point $\dot{x} \in (x, B)$ with

$$\text{£3:} \quad \frac{f'(\dot{x})}{g'(\dot{x})} = \frac{f(B) - f(x)}{g(B) - g(x)} \stackrel{\text{note}}{=} \frac{\frac{f(x)}{g(x)} - \frac{f(B)}{g(B)}}{1 - \frac{g(B)}{g(x)}}.$$

In (*), note that g' is zero **infinitely** often, due to the initial $\cos()$ factor. This contravenes the (10†) condition, despite the snow-job that I perpetrated in (**).

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