# Mean Value Theorem and L'Hôpital's Rule

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**Abbrevs.** IVT; *Intermediate Value Thm.* MVT; *Mean Value Thm.* FTC; *Fund. Thm of Calculus.* Use cts for "continuous" and cty for "continuity".

The exponential fnc exp can also be written $\exp(x) = e^x$. (So $\log \circ \exp = Id_{\mathbb{R}}$ and $\exp \circ \log = Id_{\mathbb{R}_+}$.) Use *NevZ* to mean "never-zero"; e.g "exp() is *NevZ* on $\mathbb{R}$".

**Prolegomena.** Use $\overline{\mathbb{R}}$ for the ***extended reals*** $[-\infty, +\infty]$. With DNE denoting "Does Not Exist", adjoin a point to $\overline{\mathbb{R}}$ to create

$$\mathbb{R}^{\circledast} \;:=\; [-\infty, +\infty] \sqcup \{\text{DNE}\}\,.$$

Use ***diff'able*** for "differentiable". A fnc $f{:}\mathbb{R}\to\mathbb{R}$ is ***ext-diff'able*** (for ***extended diff'able***) at point 6 if the $\lim_{x\to 6} \frac{f(x)-f(6)}{x-6}$ exists in $[-\infty, +\infty]$. Use $f'_{LH}$ for the ***lefthand ext-derivative***, from $\lim_{x\nearrow 6}$. Use $f'_{RH}$ for the ***righthand ext-derivative***, $\lim_{x\searrow 6}$.

The result below apply to fncs on a closed bounded interval $J$. For specificity, I will use $J := [4,6]$ and will use $J^\circ := (4,6)$ for its interior.

A fnc $h{:}\,J\to\mathbb{R}$ has a (global) ***max-point*** $P \in J$ if

$$\forall x \in J\colon \qquad h(P) \;\geqslant\; h(x)\,;$$

and the number $h(P)$ is the "***max-value*** of $h$ on $J$". Weaker, $P$ is a ***local max-point*** of $h$ (on $J$) if there exists a $J$-open set $U \ni P$, so that $P$ is a global max-point of $h{\restriction}_U$. Imagine analogous defns for ***min-point***, ***min-value*** and ***local min-point*** of $h$.

**1: Tool.** *A continuous $h{:}\,J\to\mathbb{R}$ has a max-point and a min-point.* *Proof.* Interval $J$ is compact, etc. ♢

**2: Lemma.** *Suppose continuous $h{:}\,J\to\mathbb{R}$ has a local-extremum at a point $\tau \in J^\circ$. If $h$ is extended-differentiable at $\tau$, then $\boxed{h'(\tau) = 0}$.* ♢

*Proof.* WLOG $\tau$ is a local-min of $h$. So for all $x > \tau$ with $x$ suff. close to $\tau$, necessarily $h(x) - h(\tau) \geqslant 0$; thus $h'_{RH}(\tau) \in [0, +\infty]$. Similarly, $h'_{LH}(\tau) \in [-\infty, 0]$. By hypothesis, $h'_{LH}(\tau) = h'_{RH}(\tau)$. So $h'(\tau)$ is zero. ♦

**3: Rolle's Thm.** *Suppose a continuous $h{:}\,[4,6]\to\mathbb{R}$ is ext-diff'able on $(4,6)$. If $h(4) = h(6)$, then there exists a point $\tau \in (4,6)$ such that $\boxed{h'(\tau) = 0}$.* ♢

*Pf.* Courtesy (2), WLOG $h()$ has no global-max in $J^\circ$, so its global-max on $J$ must be the common value of $h(4) = h(6)$. Thus $h$ has a global-*min* in $J^\circ$; pick one, and call it $\tau$. Now (2) gives that $h'(\tau) = 0$. ♦

**4: MVThm.** *Suppose cts $f{:}\,[4,6]\to\mathbb{R}$ is ext-diff'able on $(4,6)$. Then there exists a $\tau \in (4,6)$ such that*

$$f'(\tau) \;=\; \frac{f(6)-f(4)}{6-4}\,. \qquad ♢$$

*Pf.* With $S := \frac{f(6)-f(4)}{6-4}$, the slope of the chord, let

$$h(x) \quad:=\quad f(x) \;-\; [x-4]\cdot S\,.$$

Since $h(4) = f(4) = h(6)$, Rolle's thm applies to assert a point $\tau \in (4,6)$ with $0 = h'(\tau) \overset{\text{note}}{=\!=} f'(\tau) - 1{\cdot}S$. ♦

*E.g:* Fnc $f(x) := \sqrt[3]{x}$ has $f'(0) = +\infty$. MVT applies to assert a point $\tau \in (-8, 27)$ where $f'(\tau)$ equals $\frac{3--2}{27--8}$.

For a beautiful MVT-application due to Liouville, see *Liouville's Theorem* on my TeachingPage. □

**5: Prop'n.** *On an interval $H \subset \mathbb{R}$, suppose fnc $g$ is extended-differentiable. If $g'$ is NevZ on $H$, then $g$ is strictly-monotone on $H$.* (We do **not** assume that $g'$ is continuous.) ♢

*Proof.* Were $g$ not strictly-monotone then, WLOG, there are points $a<b<c$, in $H$, with $g(a) \leqslant g(b)$ yet $g(b) \geqslant g(c)$. WLOG $g(a) \leqslant g(c)$. Applying IVT to $g$ on $[a,b]$ yields point $z \in [a,b]$ with $g(z) = g(c)$. Now Rolle's thm produces a $\tau \in (z,c)$ with $g'(\tau) = 0$. ♦

**6: Cauchy MVT.** *Continuous fncs $f, g{:}\,[4,6]\to\mathbb{R}$ are diff'able on $(4,6)$. Then there exists a $\tau \in (4,6)$ st.*

6a: $$f'(\tau)\cdot[g(6) - g(4)] \;=\; g'(\tau)\cdot[f(6) - f(4)]\,.$$

*And if $g'$ is never-zero on $(4,6)$, then*

6b: $$\frac{f'(\tau)}{g'(\tau)} \;=\; \frac{f(6) - f(4)}{g(6) - g(4)}\,. \qquad ♢$$

*Proof of* (6a). Apply Rolle's theorem to

7: $$h(x) := f(x)\cdot[g(6)-g(4)] - g(x)\cdot[f(6)-f(4)],$$

noting that $h(4) = f(4)g(6) - f(6)g(4) = h(6)$. ♦

*Proof of* (6b). Since $g'$ is *NevZ* on $(4,6)$, the MVT forces $g(6) \neq g(4)$. So we can cross-divide in (6a). ♦

## L'Hôpital's rule

I'll usually state the thms for a one-sided limits, $\lim_{x\searrow T}$ or $\lim_{x\nearrow T}$, where $T \in \overline{\mathbb{R}}$. I'll use **"**$\text{Lim}(f)$**"** to mean $\lim\limits_{x\searrow T} f(x)$ or $\lim\limits_{x\nearrow T} f(x)$, as appropriate.

First some *cautionary tales*:

8: *No end.* What is $\lim_{x\searrow -\infty} \frac{\sqrt{x^2+1}}{x}$? Well, $x<0$ implies that $-\sqrt{x^2} = x$, so

$$\frac{\sqrt{x^2+1}}{x} = -\sqrt{1+\frac{1}{x^2}}\,.$$

Hence the limit is -1. But applying L'H to $\frac{\sqrt{x^2+1}}{x}$ will (after algebra) give $\frac{x}{\sqrt{x^2+1}}$. Which L'H sends to the original $\frac{\sqrt{x^2+1}}{x}$. □

9: *BttDB: Back to the Drawing Board.* Certainly

$$\lim_{x\searrow -\infty} \frac{x^2+\sin(x)}{x^2} = 1\,.$$

L'H examines $\frac{2x+\cos(x)}{2x}$; still lim=1. But applying L'H again gives $\frac{2-\sin(x)}{2}$, which has *no limit*. There is no error here; if the $f'/g'$ limit doesn't exist in $\overline{\mathbb{R}}$, then we can draw **no conclusion** about the $f/g$ limit. □

10: L'Hôpital's Thm. *Fix a "target" $T \in [-\infty, +\infty)$ and consider real-valued fncs $f, g$ defined on part of $\mathbb{R}$.*

*Suppose there exists a real number $B>T$ for which*

10†: *On $(T,B]$: $f$ & $g$ are diff'able, and $g'$ is NevZ.*[♡1]

*Recalling that $\mathbb{R}^{\circledast} = [-\infty, +\infty] \sqcup \{DNE\}$, define*

$$L := \lim_{x\searrow T} \frac{f(x)}{g(x)} \in \mathbb{R}^{\circledast} \quad \text{and}$$

$$\Lambda := \lim_{x\searrow T} \frac{f'(x)}{g'(x)} \in \mathbb{R}^{\circledast}\,.$$

[♡1] Courtesy (5), our $g$ is zero at most once on $(T,B]$. Consequently, we can move $B$ closer to $T$ so that $g$ is *NevZ* on $(T,B]$.

*Suppose, as $x \searrow T$, that* ***either***

$10_0$: $g(x) \to 0$ *and* $f(x) \to 0$, ***or***

$10_\infty$: $g(x) \to \pm\infty$.

*If $\Lambda \neq$ DNE, then $L = \Lambda$.* ◊

*Reduction to $T$ finite.* If $T = -\infty$, then define

$$\varphi(x) := f(\tfrac{-1}{x}) \quad \text{and}$$
$$\gamma(x) := g(\tfrac{-1}{x})\,.$$

So $\lim\limits_{x\searrow 0} \frac{\varphi}{\gamma}(x) = L$. But the Chain Rule gives

$$\varphi'(x) = \tfrac{1}{x^2}\cdot f'(\tfrac{-1}{x}) \quad \text{and}$$
$$\gamma'(x) = \tfrac{1}{x^2}\cdot g'(\tfrac{-1}{x})\,.$$

Thus $\lim\limits_{x\searrow 0} \frac{\varphi'}{\gamma'}(x) = \Lambda$. Proving L'H for $\boxed{\varphi, \gamma \text{ at } 0}$ will thus establish L'H for $\boxed{f, g \text{ at } -\infty}$.

*In the proofs below, we will take* $(T,B] = (4,6]$. □

*Proof using* $(10_0)$. We may extend $f$ by continuity so that $f$ is cts on $[4,6]$; hence $f(4) = 0$. Ditto for $g$.

Each $x \in (4,6]$, gives a point $\overset{\bullet}{x} \in (4,x)$ with

$$\frac{f'(\overset{\bullet}{x})}{g'(\overset{\bullet}{x})} \overset{\text{Cauchy-MVT}}{=\!=} \frac{f(x)-f(4)}{g(x)-g(4)} \overset{\text{note}}{=\!=} \frac{f(x)}{g(x)}\,.$$

Sending $x \searrow 4$ forces $\overset{\bullet}{x} \to 4$. ♦

*Proof using* $(10_\infty)$. Assume, say, $\Lambda=7$. (The $\Lambda=\pm\infty$ case just changes notation.) Fixing an $\varepsilon > 0$, ISTEstablish that

£1: $$\limsup_{x\searrow 4} \frac{f}{g}(x) \leqslant 7+\varepsilon\,.$$

Inequality $\liminf \frac{f}{g} \geqslant 7-\varepsilon$ will follow analogously.

Take $B>4$ so close to 4 that

£2: $$\forall y \in (4,B]: \quad \left|\frac{f'}{g'}(y) - 7\right| \leqslant \varepsilon\,.$$

For each $x \in (4,B]$, the Cauchy-MVT gives a point $\overset{\bullet}{x} \in (x,B)$ with

£3: $$\frac{f'(\overset{\bullet}{x})}{g'(\overset{\bullet}{x})} = \frac{f(B)-f(x)}{g(B)-g(x)} \overset{\text{note}}{=\!=} \frac{\frac{f(x)}{g(x)} - \frac{f(B)}{g(x)}}{1 - \frac{g(B)}{g(x)}}\,.$$

And $\frac{f(B)}{g(x)}, \frac{g(B)}{g(x)} \to 0$, by $(10_\infty)$, as $x \searrow 4$.

Taking limsups in (£3) gives

$$*: \qquad \limsup_{x\searrow 4} \frac{f'}{g'}(\dot{x}) \quad = \quad \limsup_{x\searrow 4} \frac{f}{g}(x)\,.$$

Now we ***don't know*** that $\dot{x} \to 4$. But we do know that each $\dot{x} \in (4, B]$. Thus LhS$(*) \leqslant 7 + \varepsilon$, by (£2). Hence (£1). ♦

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## *A remark on the above proof.*

**11:** *Can L=1 yet Λ=DNE?* Yes. For each sequence $x_n \searrow T$, nec. $\frac{f}{g}(x_n) \to 1$ and therefore $\frac{f'}{g'}(\dot{x}_n) \to 1$. Yet there *could* be a seq. $z_n \searrow T$ where, say,

$$\frac{f'}{g'}(z_n) \xrightarrow{n\to\infty} 927\,. \quad \text{(I.e, some number } \neq 1.)$$

Yet it might that *every* $(\dot{x}_n)_{n=1}^\infty$ sequence misses the dissenting $(z_n)_{n=1}^\infty$ sequence. Indeed, this is what happens in example (BttDB), above. □

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**12:** *Nifty Application: Does $h' \to 0$?* Given a differentiable function with *finite* limit, say $\lim_{x\to\infty} h(x) = 7$, suppose $L := \lim_{x\to\infty} h'(x)$ **exists** in $\overline{\mathbb{R}}$. A picture suggests that $L$ "must" be 0. Can we *prove it* by L'Hôpital's theorem?

Let $f := h\cdot\exp$ and $g := \exp$. So $\text{Lim}(\frac{f}{g}) = 7$. But, $f' = [h' + h]\cdot\exp$. So $\text{Lim}(\frac{f'}{g'}) = L + 7$. Thus L'H applies to tell us that $7 = L + 7$. Hence $L = 0$. □

**13:** *FTCalculus.* The conditions in $(10_0)$ apply to

$$¥: \qquad \lim_{x\searrow 0} \frac{\int_0^x \sin(\sin(t))\,dt}{x^2}\,.$$

Since the FTCalculus applies to the numerator, L'H yields $\frac{f'}{g'}(x) = \frac{\sin(\sin(x))}{2x}$. Applying L'H again gives

$$\frac{f''}{g''}(x) \quad = \quad \frac{\cos(\sin(x))\cdot\cos(x)}{2}\,.$$

As $\frac{f''}{g''}(0) = \frac{\cos(0)\cdot\cos(0)}{2} = \frac{1}{2}$, the limit in (13¥) is $\frac{1}{2}$. □

Here is the famous example by *O. Stolz*, published in 1871, generalized by *R.P. Boas*, 1986.

*Stolz example: (1871).* Let

$$f(x) := \int_0^x \cos(t)^2\,dt \overset{\text{note}}{=\!=} \tfrac{1}{2}\cdot[x + \cos(x)\sin(x)]\,.$$

Easily $f(x) \nearrow \infty$ as $x \nearrow \infty$. Evidently

$$g(x) \;:=\; f(x)\cdot e^{\sin(x)}$$

goes $\to\infty$, since $e^{\sin(x)}$ is bnded-below by $1/e$, which is positive. Now $f' = \cos^2$. Thus

$$*: \quad \begin{aligned} g'(x) &= \cos(x)^2\cdot e^{\sin(x)} + f(x)\cdot e^{\sin(x)}\cdot\cos(x) \\ &= \cos(x)\cdot e^{\sin(x)}\cdot\big[\cos(x) + f(x)\big]\,. \end{aligned}$$

Dividing produces the *to-zero-going* quantity

$$**: \qquad \frac{f'}{g'}\Big(x\Big) \;=\; \frac{\cos(x)}{e^{\sin(x)}\cdot\big[\cos(x) + f(x)\big]}\,, \qquad □$$

since denom$\to\infty$ and numer is bnded. Thus $\Lambda = 0$. But $L$ is the limit of $\frac{f(x)}{g(x)} \overset{\text{def}}{=\!=} 1/e^{\sin(x)}$, which is oscillatory, i.e DNE. *What went wrong?*

*Give up? Don't give up —examine the hypotheses of* L'H *with a finely-toothed comb. If desperate, read further. . .*

In $(*)$, note that $g'$ is zero **infinitely** often, due to the initial cos() factor. This con-traverses the (10†) condition, despite the snow-job that I perpetrated in $(**)$.

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