Mean Value Theorem and L'Hôpital's Rule

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Abbrevs. IVT: Intermediate Value Thm. MVT: Mean Value Thm. FTC; Fund. Thm of Calculus. Use cts for “continuous” and cty for “continuity.”

The exponential fnc exp can also be written exp(x) = e^x. (So log ◦ exp = I_d) Use NevZ to mean “never-zero”, e.g “exp() is NevZ on ℝ”.

Prolegomena. Use ℝ denote the extended reals [−∞, +∞]. With DNE denoting “Does Not Exist”, adjoin a point to ℝ to create

\[ ℝ^\ast := [−∞, +∞] ∪ \{DNE\} \]

Use diff’able for “differentiable”. A fnc f: ℝ→ ℝ is ext-diff’able (for extended differentiable) if

\[ \lim_{x→−∞} f(x) \] exists in [−∞, +∞].

Use \( f_{\text{LH}} \) for the left-hand ext-derivative, from \( \lim_{x→−6} \). Use \( f_{\text{RH}} \) for the right-hand ext-derivative, \( \lim_{x→6} \).

The result below apply to fncs on a closed bounded interval J. For specificity, I will use J := [4,6] and will use J^o := (4,6) for its interior.

A fnc h: J→ ℝ has a (global) max-point P ∈ J if

\[ \forall x ∈ J: \quad h(P) ≥ h(x) \]

and the number h(P) is the “max-value of h on J”. Weaker, P is a local max-point of h (on J) if there exists a J-open set U ⊃ P, so that P is a global max-point of h|_U. Imagine analogous defns for min-point, min-value and local min-point of h.

1: Tool. A continuous h: J→ ℝ has a max-point and a min-point. Proof. Interval J is compact, etc.

2: Lemma. Suppose continuous h: J→ ℝ has a local-extremum at a point τ ∈ J^o. If h is extended-differentiable at τ, then \( h'(τ) = 0 \).

Proof. WLOG τ is a local-min of h. So for all x > τ with x suff. close to τ, necessarily h(x) − h(τ) ≥ 0; thus h'_{RH}(τ) = 0. Similarly, h'_{LH}(τ) = 0. By hypothesis, h'_{LH}(τ) = h'_{RH}(τ). So h'(τ) is zero.

3: Rolle’s Thm. Suppose a continuous h: [4,6]→ ℝ is ext-diff’able on (4,6). If h(4) = h(6), then there exists a point τ ∈ (4,6) such that \( h'(τ) = 0 \).

Proof. Courtesy (2), WLOG h() has no global-max in J^o, so its global-max on J must be the common value of h(4) = h(6). Thus h has a global-min in J^o; pick one, and call it τ. Now (2) gives that h'(τ) = 0.

4: MVT. Suppose cts f: [4,6]→ ℝ is ext-diff’able on (4,6). Then there exists a τ ∈ (4,6) such that

\[ f'(τ) = \frac{f(6)−f(4)}{6−4} \]

Proof. With S := \( \frac{f(6)−f(4)}{6−4} \), the slope of the chord, let

\[ h(x) := f(x) − |x−4| \cdot S \].

Since h(4) = f(4) = h(6), Rolle’s thm applies to assert a point τ ∈ (4,6) with 0 = h'(τ) = f'(τ) − 1⋅S.

E.g: Fnc \( f(x) := \sqrt{x} \) has \( f'(0) = +∞ \). MVT applies to assert a point τ ∈ (−8,27) where \( f'(τ) = \frac{3}{2\cdot\sqrt{3}} \).

For a beautiful MVT-application due to Liouville, see Liouville’s Theorem on my TeachingPage.

5: Prop’n. On an interval H⊂ ℝ, suppose fnc g is extended-differentiable. If g’ is NevZ on H, then g is strictly-monotone on H. (We do not assume that g’ is continuous.)

Proof. Were g not strictly-monotone then, WLOG, there are points a<b<c, in H, with g(a) ≤ g(b) yet g(b) ≥ g(c). WLOG g(a) ≤ g(c). Applying IVT to g on [a,b] yields point z ∈ [a,b] with g(z) = g(c). Now Rolle’s thm produces a τ ∈ (z,c) with g'(τ) = 0.

6: Cauchy MVT. Continuous fncs f,g: [4,6]→ ℝ are diff’able on (4,6). Then there exists a τ ∈ (4,6) st.

6a: \( f'(τ) \cdot [g(6)−g(4)] = g'(τ) \cdot [f(6)−f(4)] \).

And if g’ is never-zero on (4,6), then

6b: \[ \frac{f'(τ)}{g'(τ)} = \frac{f(6)−f(4)}{g(6)−g(4)} \].
Proof of (6a). Apply Rolle's theorem to
\[ 7: \quad h(x) := f(x) \cdot |g(6) - g(4)| - g(x) \cdot |f(6) - f(4)|, \]
noting that \( h(4) = f(4)g(6) - f(6)g(4) = h(6) \). ♦

Proof of (6b). Since \( g' \) is NewZ on \((4, 6)\), the MVT forces \( g(6) \neq g(4) \). So we can cross-divide in (6a). ♦

L'Hôpital's Rule

I'll usually state the thms for a one-sided limits,
\[ \lim_{x \to T} \text{ or } \lim_{x \not\to T}, \] where \( T \in \mathbb{R} \). I'll use “Lim(\( f \))”
to mean \( \lim f(x) \) or \( \lim f(x) \), as appropriate.

First some cautionary tales:

8: No end. What is \( \lim_{x \to \infty} \sqrt{\frac{x^2+1}{x}} \)?
Well, \( x < 0 \) implies that \( -\sqrt{x^2} = x \), so
\[
\sqrt{\frac{x^2+1}{x}} = \frac{\sqrt{x^2+1}}{-\sqrt{x^2}} = -\sqrt{1 + \frac{1}{x^2}}.
\]
Hence the limit \( x \to \infty \) is -1. But applying L'Hôpital to \( \sqrt{\frac{x^2+1}{x}} \)
will (after algebra) give \( \frac{x}{\sqrt{x^2+1}} \). Which L'Hôpital sends to the
original \( \sqrt{\frac{x^2+1}{x}} \).

9: BtDB: Back to the Drawing Board. Certainly
\[
\lim_{x \to \infty} \frac{x^2+\sin(x)}{x^2} = 1.
\]
L'Hôpital examines \( \frac{2x+\cos(x)}{2x} \); still limit=1. But applying L'Hôpital
again gives \( \frac{2-\sin(x)}{2} \), which has no limit. There is no error here; if the \( f'/g' \) limit doesn't exist in \( \mathbb{R} \),
then we can draw no conclusion about the \( f/g \) limit. [4] [5]

10: L'Hôpital's Thm. Fix a "target" \( T \in [\infty, +\infty) \) and
consider real-valued fns \( f, g \) defined on part of \( \mathbb{R} \).

Suppose there exists a real number \( B > T \) for which
\[ 10\): On \((T, B)\): \( f \& g \) are diff'ble, and \( g' \) is NewZ.\(^{\diamond1} \)

Recalling that \( \mathbb{R}^\circ = [\infty, +\infty) \cup \{DNE\} \), define
\[
L := \lim_{x \to T} \frac{f(x)}{g(x)} \in \mathbb{R}^\circ \quad \text{and} \quad \Lambda := \lim_{x \to T} \frac{f'(x)}{g'(x)} \in \mathbb{R}^\circ.
\]

\( ^{\diamond1} \) Courtesy (5), our \( g \) is zero at most once on \((T, B)\). Consequently, we can move \( B \) closer to \( T \) so that \( g \) is NewZ on \((T, B)\).

Suppose, as \( x \searrow T \), that either
\[ 10_0: \quad g(x) \to 0 \text{ and } f(x) \to 0, \quad \text{or} \]
\[ 10_\infty: \quad g(x) \to \pm \infty. \]

If \( \Lambda \neq \text{DNE} \), then \( L = \Lambda \). ♦

Reduction to \( T \) finite. If \( T = -\infty \), then define
\[
\varphi(x) := \frac{1}{g(x)}, \quad \gamma(x) := \frac{1}{f(x)}.
\]
So \( \lim_{x \searrow 0} \varphi'(x) = \Lambda \). But the Chain Rule gives
\[
\varphi'(x) = \frac{1}{g(x)} \cdot f'(x), \quad \gamma'(x) = -\frac{1}{f(x)} \cdot g'(x).
\]
Thus \( \lim_{x \searrow 0} \frac{\varphi'(x)}{\gamma'(x)} = \Lambda \). Proving L'Hôpital for \( (\varphi, \gamma \atop 0) \) will
thus establish L'Hôpital for \( (f, g \atop -\infty) \).

In the proofs below, we will take \( (T, B) = (4, 6) \). [4] [5]

Proof using \( 10_0 \). We may extend \( f \) by continuity so that \( f \) is cts on \([4, 6]\); hence \( f(4) = 0 \). Ditto for \( g \).

Each \( x \in (4, 6) \), gives a point \( \bullet \in (4, x) \) with
\[
\frac{f'(\bullet)}{g'(\bullet)} \overset{\text{Cauchy-MVT}}{\Longrightarrow} \frac{f(x) - f(4)}{g(x) - g(4)} \overset{\text{note}}{\Longrightarrow} \frac{f(x)}{g(x)}.
\]
Sending \( x \searrow 4 \) forces \( \bullet \to 4 \).

Proof using \( 10_\infty \). Assume, say, \( \Lambda = 7 \). (The \( \Lambda = \pm \infty \) case
just changes notation.) Fixing an \( \varepsilon > 0 \), ISTEstablish that
\[ 11: \limsup_{x \searrow 4} \frac{f(x)}{g(x)} \leq 7 + \varepsilon. \]

Inequality \( \liminf_{x \searrow 4} \frac{f(x)}{g(x)} \geq 7 - \varepsilon \) will follow analogously.

Take \( B > 4 \) so close to 4 that
\[ 12: \quad \forall y \in (4, B): \quad \left| \frac{f'(y)}{g'(y)} - 7 \right| \leq \varepsilon. \]

For each \( x \in (4, B) \), the Cauchy-MVT gives a point \( \bullet \in (x, B) \) with
\[ 13: \quad \frac{f'\left(\bullet_{x}\right)}{g'\left(\bullet_{x}\right)} = \frac{f(B) - f(x)}{g(B) - g(x)} \overset{\text{note}}{=\Longrightarrow} \frac{f(x)}{g(x)} - 1 = \frac{g(B)}{g(x)}. \]
And \( \frac{f(B)}{g(x)} \to 0 \), by \( 10_\infty \), as \( x \searrow 4 \).

Taking limsup in (\( \ell 3 \)) gives

\[ \limsup_{x \searrow 4} \frac{f'(x)}{g'} = \limsup_{x \searrow 4} \frac{f(x)}{g(x)}. \]

Now we don’t know that \( \bullet \to 4 \). But we do know that each \( \bullet \in (4, B] \). Thus LhS(\( \bullet \)) \( \leq 7 + \varepsilon \), by (\( \ell 2 \)). Hence (\( \ell 1 \)).

\[ \star: \quad \limsup_{x \searrow 4} \frac{f'(x)}{g'}(x) = \limsup_{x \searrow 4} \frac{f(x)}{g(x)}. \]

\[ \star \star: \quad \limsup_{x \searrow 4} \frac{f'(x)}{g'}(x) = \limsup_{x \searrow 4} \frac{f(x)}{g(x)}. \]

A remark on the above proof.

11: Can \( L = 1 \) yet \( \Lambda = \text{DNE} \)? Yes. For each sequence \( x_n \searrow 4 \), nec. \( \frac{f'(x_n)}{g'(x_n)} \to 1 \) and therefore \( \frac{f'}{g'}(\bullet_n) \to 1 \). Yet there could be a seq. \( z_n \searrow 4 \) where, say,

\[ \frac{f'(z_n)}{g'(z_n)} \to \infty. \]

Yet it might that every \( \bullet_n \) sequence misses the dissenting \( z_n \) sequence. Indeed, this is what happens in example (BttDB), above.

12: Nifty application: Does \( h' \to 0? \) Given a differentiable function with finite limit, say \( \lim_{x \to 1} h(x) = 7 \), suppose \( L := \lim_{x \to 1} h'(x) \) exists in \( \mathbb{R} \). A picture suggests that \( L \), “must” be 0. Can we prove it by L'Hôpital’s theorem?

Let \( f := h \cdot \exp \) and \( g := \exp \). So \( \text{Lim} (\frac{f}{g}) = 7 \). But, \( f' = [h' + h] \cdot \exp \). So \( \text{Lim} (\frac{f'}{g'}) = L + 7 \). Thus L'Hôpital applies to tell us that \( 7 = L + 7 \). Hence \( L = 0 \).□

13: FT Calculus. The conditions in (\( 10_0 \)) apply to

\[ \gamma: \quad \lim_{x \searrow 0} \frac{\int_0^x \sin(\sin(t)) \, dt}{x^2}. \]

Since the FT Calculus applies to the numerator, L'Hôpital yields \( \frac{f'}{g'}(x) = \frac{\sin(h(x))}{x} \). Applying L'H again gives

\[ \frac{f''}{g''}(x) = \frac{\cos(h(x)) \cdot \cos(x)}{2}. \]

As \( \frac{f''}{g''}(0) = \frac{\cos(0) \cdot \cos(0)}{2} = \frac{1}{2} \), the limit in (\( 13 \gamma \)) is \( \frac{1}{2} \).□

Here is the famous example by O. Stolz, published in 1871, generalized by R. P. Boas, 1986.

Stolz example: (1871). Let

\[ f(x) := \int_0^x \cos(t)^2 \, dt \overset{\text{note}}{=} \frac{1}{2} \cdot [x + \cos(x) \cdot \sin(x)]. \]

Easily \( f(x) \searrow \infty \) as \( x \searrow \infty \). Evidently

\[ g(x) := f(x) \cdot e^{\sin(x)} \]

\( \Rightarrow \infty \), since \( e^{\sin(x)} \) is banded below by \( 1/e \), which is positive. Now \( f' = \cos^2 \).

Thus

\[ \star: \quad g'(x) = \cos(x)^2 \cdot e^{\sin(x)} + f(x) \cdot e^{\sin(x) \cdot \cos(x)} \]

\[ = \cos(x) \cdot e^{\sin(x) \cdot [\cos(x) + f(x)]}. \]

Dividing produces the to-zero-going quantity

\[ \star \star: \quad \frac{f'}{g'}(x) = \frac{\cos(x)}{e^{\sin(x)} \cdot [\cos(x) + f(x)]}, \]

since \( \text{denom} \Rightarrow \infty \) and \( \text{numerator} \) is banded. Thus \( \Lambda = 0 \).

But \( L \) is the limit of \( \frac{f(x)}{g(x)} \overset{\text{def}}{=} \frac{1}{e^{\sin(x)}} \), which is oscillatory, i.e. DNE. What went wrong?

Give up? Don’t give up — examine the hypotheses of L’Hôpital with a finely-toothed comb. If desperate, read further . . .

\[ \text{(**) in } \text{Job that } I \text{ performed in } \text{outside of } \text{course, despite the show's name. The answer is}\]

\[ \text{In other words, note that } f' = \text{zero once.} \]

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In (*), note that $g'$ is zero **infinitely** often, due to the initial $\cos()$ factor. This contravenes the (10†) condition, despite the snow-job that I perpetrated in (**).