Hall's Marriage Lemma

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Entrance. We start with the classic thm of Philip Hall. We have sets $\mathcal{B}$ and $\mathcal{G}$ ["boys" and "girls", possibly infinite] and a bipartite graph $\Gamma = ((\mathcal{B}, \mathcal{G}), \mathsf{E})$. We write $b\mathsf{E}g$ if boy b and girl g know each other. Let ♀(b) be the *set* of girls known by b. And for a set $B \subset \mathcal{B}$, use

$$♀(B) \quad := \quad \bigcup_{b \in B} ♀(b)$$

for the girls known by at least one B-boy. Analogously, use ♂(g) and ♂(G) for boys known by girls.

A "***marriage*** for the boys" is an *injection* $f\colon \mathcal{B} \hookrightarrow \mathcal{G}$ st. for each b, we have $b\mathsf{E}f(b)$.

The ***Hall condition*** on Γ is:

HC: *Each* $B \subset \mathcal{B}$ *has* $|♀(B)| \geq |B|$.

Evidently HC is a necessary condition for a marriage.

1: Marriage lemma (Philip Hall, 1935). *Suppose bipartite graph* $\Gamma = ((\mathcal{B}, \mathcal{G}), \mathsf{E})$ *has* $\mathcal{B}$ *finite. Then there is a marriage for the boys IFF* Γ *satisfies HC.* ◊

1a: *Remark.* When $\mathcal{B}$ is infinite, HC does not imply a marriage. Consider boys $\mathbb{N}$ and girls $\mathbb{Z}_+$. *Boy* b_0 *knows all the girls, and each other* b_n *knows only* g_n. Each *proper* subset $B \subset \mathbb{N}$ can be married-off: Pick $K \in \mathbb{N} \setminus B$ and marry b_0 to g_K. For the remaining $n \in B$, marry b_n to g_n.

OTOHand, we can't marry-off *all* the boys; the wife g_K of b_0 leaves poor b_K with no-one to marry.

The below proof uses induction on $|\mathcal{B}|$, doing divorces to marry-in the new boy. The above CEX shows that there can*not* be an induction Proof-by-Extension; the divorces are necessary, even with look-ahead. □

Pf of (1). Suppose we have married-off finite set $\mathcal{B}$ into [possibly infinite] $\mathcal{G}$. We have a new boy $b_0 \notin \mathcal{B}$ whom we wish to marry-off. Our goal is to find a ***chain***

$$*: \quad b_0 \rightarrow g_1 \boldsymbol{\rightarrow} b_1 \rightarrow g_2 \boldsymbol{\rightarrow} b_2 \rightarrow \ldots \rightarrow g_{K-1} \boldsymbol{\rightarrow} b_{K-1} \rightarrow g_K\,,$$

where: *Girl* g_K *is unmarried, each other* g_n *is married to* b_n*, and each* b_{j-1} *knows* g_j. Divorce these married-girls, then marry each b_{j-1} to g_j. Now all boys in $\mathcal{B} \sqcup \{b_0\}$ are married.

Producing a chain. "Mark" b_0. Iteratively mark additional girls and boys as follows:

- Mark each girl known by a marked boy.
- Mark each boy married to a marked girl.

This process must eventually stabilize, as $\mathcal{B}$ is finite. At this point, let B and G be the sets of marked boys and girls. By defn $G = ♀(B)$, so the Hall condition says $|G| \geq |B|$.

Each B-boy *except* b_0 is married; so G has precisely $|B| - 1$ wives. Thus *there is some unmarried* G*-girl.* Pick one. In the marking-process, she was introduced at some stage, K. Hence she is g_K of some $(*)$-chain. ♦

Distinct-cards Problem

For the cards in a playing-deck, denote the ranks A,2,…,J,Q,K by $r_1, r_2, \ldots, r_{13}$.

2: Distinct-cards thm. *Deal a randomized deck into* 13 *piles of four cards apiece. Now* <u>*remove*</u> *some three cards. Then it is always possible to choose one card-per-pile so that all* 13 *ranks were chosen.* ◊

Proof. Imagine that each pile of cards is on its own little tray. The trays are the "boys", the ranks are the "girls" and the cards are the $52-3 = 49$ edges of the bipartite graph. Does this graph satisfy Philip Hall's condition?

In a set, $\mathcal{C}$, of n many cards, no rank occurs on more than 4 cards, so the number of ranks occurring in $\mathcal{C}$ is at least $\lceil n/4 \rceil$. A collection of K many trays has at least $n := 4K - 3$ many cards, so this collection has at least $\left\lceil \frac{[4K]-3}{4} \right\rceil \overset{\text{note}}{=\!=} K$ many ranks. I.e, each set of K boys "knows" at least K many girls. ♦

Land matching

3: The Hunter/Farmer problem. *There is an island which, from time immemorial, has been divided into N equal-area farming regions, taking up the whole island. It is also divided into N equal-area hunting tracts, taking up the whole island.*

There are N married couples on the island; the wives hunt and the husbands farm. We would like to be able to assign tracts to wives and farms to husbands so that each couple could build a house on territory common to both. Indeed, territory with at least area $\boxed{\delta_N \cdot \text{Area}(\textit{Island})}$. Determine the largest $\delta = \delta_N$ which that works for every *division of the island into tracts/regions.* ♢

Proof. ???