Linear Recurrence using matrices

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See also *Problems/NumberTheory/fibonacci.latex*

Notation. Use "$\overset{r}{\sim}$" for row-equivalence: I.e, $K \times N$ matrices $\mathsf{M} \overset{r}{\sim} \mathsf{M}'$ IFF we can get from M to M' using row operations. Use "$\overset{c}{\sim}$" for column-equivalence.

Two $N \times N$ matrices B & C are ***similar***, or ***conjugate*** to each other, if $\boxed{\textit{there exists}}$ an invertible matrix U such that

$$\mathsf{U}^{-1} \cdot \mathsf{B} \cdot \mathsf{U} \quad = \quad \mathsf{C}\,.$$

Use $\mathsf{B} \overset{\text{sim}}{\sim} \mathsf{C}$ for the similarity equiv-relation.

Call a matrix OTForm $s\mathbf{I} \overset{\text{note}}{=\!=} \left[\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right]$, a ***dilation***; its action on the plane is simply to scale-uniformly by factor s. *Every* matrix commutes with $\mathbf{I}$, and so:

1: *A dilation is only conjugate to itself.*

I.e, for invertible U, necessarily $\mathsf{U}^{-1} \cdot s\mathbf{I} \cdot \mathsf{U} = s\mathbf{I}$.

For T a square matrix (or a trn from a vectorspace to itself) and a complex number α, define the subspace

2: $$\mathbb{E}_{\mathsf{T},\alpha} \;\coloneqq\; \{\text{vectors } \mathbf{v} \mid \mathsf{T}\mathbf{v} = \alpha\mathbf{v}\}\,.$$

Saying that "*α is a T-eigenvalue*" is the same as saying that $\operatorname{Dim}(\mathbb{E}_{\mathsf{T},\alpha}) \geqslant 1$.

Linear recurrence

A fibonacci-like sequence $\vec{\mathbf{z}} \coloneqq (z_n)_{n=-\infty}^{\infty}$ is specified using (complex) numbers $\mathcal{S}$ and $\mathcal{P}$, with $\mathcal{P}$ non-zero[♡1], by

3: $$z_{n+2} \;\coloneqq\; \mathcal{S} z_{n+1} - \mathcal{P} z_n\,,$$

and some initial condition (z_1, z_0). With

$$\mathsf{G} \;\coloneqq\; \left[\begin{smallmatrix} \mathcal{S} & -\mathcal{P} \\ 1 & 0 \end{smallmatrix}\right], \quad \text{then } \left[\begin{smallmatrix} z_{n+1} \\ z_n \end{smallmatrix}\right] = \mathsf{G}^n \cdot \left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right],$$

for each integer n. We want to diagonalize G.

[♡1] Actually, recurrence (3) can be run backwards, giving z_n values when n is negative, as soon as $(\mathcal{S}, \mathcal{P}) \neq (0,0)$. However, the $\mathcal{P} \neq 0$ case is the interesting case.

Its char-poly is $f(x) = f_{\mathsf{G}}(x) \coloneqq x^2 - \mathcal{S}x + \mathcal{P}$. Factor this f as $f(x) = [x - \boldsymbol{\alpha}][x - \boldsymbol{\beta}]$, with $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}$. Equating coeffs in the polynomial gives

4: $$\begin{aligned} \boldsymbol{\alpha} + \boldsymbol{\beta} &= \mathcal{S}\,; && \text{(Sum)} \\ \boldsymbol{\alpha} \cdot \boldsymbol{\beta} &= \mathcal{P}\,. && \text{(Product)} \end{aligned}$$

Since $\mathcal{P} \neq 0$, necessarily $\boxed{\boldsymbol{\alpha} \neq 0}$ and $\boxed{\boldsymbol{\beta} \neq 0}$. Each of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is a root of f, hence

5: $$\boldsymbol{\alpha}^2 = \mathcal{S}\boldsymbol{\alpha} - \mathcal{P} \quad \textit{and} \quad \boldsymbol{\beta}^2 = \mathcal{S}\boldsymbol{\beta} - \mathcal{P}\,.$$

Finding an $\boldsymbol{\alpha}$-eigenvector. An $\boldsymbol{\alpha}$-evec (for G) is an element of $\operatorname{Nul}(\mathsf{G} - \boldsymbol{\alpha}\mathbf{I})$. Applying row operations,

6: $$\mathsf{G} - \boldsymbol{\alpha}\mathbf{I} \;=\; \begin{bmatrix} \mathcal{S} - \boldsymbol{\alpha} & -\mathcal{P} \\ 1 & -\boldsymbol{\alpha} \end{bmatrix} \;\overset{r}{\sim}\; \begin{bmatrix} 1 & -\boldsymbol{\alpha} \\ \mathcal{S} - \boldsymbol{\alpha} & -\mathcal{P} \end{bmatrix} \;\overset{r}{\sim}\; \begin{bmatrix} 1 & -\boldsymbol{\alpha} \\ 0 & 0 \end{bmatrix}.$$

This last $\overset{r}{\sim}$ requires **no** computation, since the rows *must* be linearly-dependent (since $\boldsymbol{\alpha}$ is an eigenvalue of G).

This last matrix has one free column, and evidently multiplies $\left[\begin{smallmatrix} \boldsymbol{\alpha} \\ 1 \end{smallmatrix}\right]$ to the zero-vector. So

7: *The singleton $\left\{\begin{bmatrix} \boldsymbol{\alpha} \\ 1 \end{bmatrix}\right\}$ is a basis for the $\boldsymbol{\alpha}$-eigenspace of G, which is one-dimensional.*

Instead of row-ops to show that $\mathbb{E}_{\mathsf{G},\boldsymbol{\alpha}}$ (the $\boldsymbol{\alpha}$-eigenspace of G) is one-dim'al, we could have argued as follows: *Were $\mathbb{E}_{\mathsf{G},\boldsymbol{\alpha}}$ two-dim'al, then $\mathsf{G} \overset{\text{sim}}{\sim} \left[\begin{smallmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{smallmatrix}\right]$. By (1), then, G would equal $\left[\begin{smallmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{smallmatrix}\right]$ —which it doesn't!*

When is $\boldsymbol{\alpha} = \boldsymbol{\beta}$? This happens when the discriminant of f is zero. Its discrim is $[-\mathcal{S}]^2 - 4 \cdot 1 \cdot \mathcal{P}$, i.e $[\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 + 2\boldsymbol{\alpha}\boldsymbol{\beta}] - 4\boldsymbol{\alpha}\boldsymbol{\beta}$. Hence

8: $$\operatorname{Discr}(f) \;=\; \mathcal{S}^2 - 4\mathcal{P} \;\overset{\text{note}}{=\!=}\; [\boldsymbol{\alpha} - \boldsymbol{\beta}]^2\,.$$

Since (7) also applies to eigenvalue $\boldsymbol{\beta}$, we conclude:

9: *Matrix $\mathsf{G} = \begin{bmatrix} \mathcal{S} & -\mathcal{P} \\ 1 & 0 \end{bmatrix}$ is diagonalizable IFF G has distinct eigenvalues; i.e $\mathcal{S}^2 \neq 4\mathcal{P}$.*

Distinct eigenvalues

When $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, our (7) implies that G is conjugate to diagonal-matrix

10: $\mathsf{D} := \begin{bmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\beta} \end{bmatrix}$ via matrix

11: $\mathsf{U} := \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ 1 & 1 \end{bmatrix}$. Note $\mathsf{U}^{-1} = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \cdot \begin{bmatrix} 1 & -\boldsymbol{\beta} \\ -1 & \boldsymbol{\alpha} \end{bmatrix}$.

We populated U with evecs, using the same order of evals in D. One can check that $\boxed{\mathsf{U}^{-1}\mathsf{G}\mathsf{U} = \mathsf{D}}$. Hence $\mathsf{G} = \mathsf{U}\mathsf{D}\mathsf{U}^{-1}$. So for each integer n,

12: $$\mathsf{G}^n = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\alpha}^n & 0 \\ 0 & \boldsymbol{\beta}^n \end{bmatrix} \cdot \begin{bmatrix} 1 & -\boldsymbol{\beta} \\ -1 & \boldsymbol{\alpha} \end{bmatrix}.$$

Multiplying from the right by initial-condition $\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]$ will produce $\left[\begin{smallmatrix} z_{n+1} \\ z_n \end{smallmatrix}\right]$. Extracting the z_n, we get the nice formula

13: $$z_n = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \cdot \Big[[z_1 - z_0\boldsymbol{\beta}]\boldsymbol{\alpha}^n - [z_1 - z_0\boldsymbol{\alpha}]\boldsymbol{\beta}^n \Big].$$

This formula is symmetric in $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$, as it must be.

Equal eigenvalues

Let's take a look at the $\boldsymbol{\alpha}=\boldsymbol{\beta}$ case. This means that

14: $$\mathsf{G} = \begin{bmatrix} 2\boldsymbol{\alpha} & -\boldsymbol{\alpha}^2 \\ 1 & 0 \end{bmatrix}.$$

While we can't conjugate G to a *diagonal* matrix, we can conjugate to its ***Jordan canonical form***. The JCF of (14) is

$$\mathsf{J} = \mathsf{J}_{\mathsf{G}} := \begin{bmatrix} \boldsymbol{\alpha} & 1 \\ 0 & \boldsymbol{\alpha} \end{bmatrix}.$$

Mysteriously pulling the below C from a hat, multiplication verifies that

15: $$\mathsf{J} = \mathsf{C}^{-1}\mathsf{G}\mathsf{C}, \quad \text{where} \quad \mathsf{C} := \begin{bmatrix} \boldsymbol{\alpha} & 1 \\ 1 & 0 \end{bmatrix}$$

and $\mathsf{C}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\boldsymbol{\alpha} \end{bmatrix}$. Leaving a derivation of C to later, let's use this to get a formula for z_n.

Induction on n shows that

$$\mathsf{J}^n := \boldsymbol{\alpha}^{n-1} \cdot \begin{bmatrix} \boldsymbol{\alpha} & n \\ 0 & \boldsymbol{\alpha} \end{bmatrix}, \quad \text{for each } n \in \mathbb{Z}.$$

As before, $\mathsf{G} = \mathsf{C}\mathsf{J}\mathsf{C}^{-1}$ so $\mathsf{G}^n = \mathsf{C}\mathsf{J}^n\mathsf{C}^{-1}$. That is,

16: $$\begin{aligned} \mathsf{G}^n &= \boldsymbol{\alpha}^{n-1} \cdot \begin{bmatrix} \boldsymbol{\alpha} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} & n \\ 0 & \boldsymbol{\alpha} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\boldsymbol{\alpha} \end{bmatrix} \\ &= \boldsymbol{\alpha}^{n-1} \cdot \begin{bmatrix} [1+n]\boldsymbol{\alpha} & -n\boldsymbol{\alpha}^2 \\ n & [1-n]\boldsymbol{\alpha} \end{bmatrix}. \end{aligned}$$

Multiplying by $\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]$; the bottom entry in the resulting column-vector is

17: $$z_n = \boldsymbol{\alpha}^{n-1} \cdot \Big[nz_1 + [1-n]\boldsymbol{\alpha} z_0 \Big].$$

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