Linear Recurrence using matrices

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See also Problems/NumberTheory/fibonacci.latex

Notation. Use \( \sim \) for row-equivalence: I.e., \( K \times N \) matrices \( M \sim M' \) iff we can get from \( M \) to \( M' \) using row operations. Use \( \cong \) for column-equivalence.

Two \( N \times N \) matrices \( B \sim C \) are similar, or conjugate to each other, if \([\text{there exists}]\) an invertible matrix \( U \) such that

\[
U^{-1} \cdot B \cdot U = C.
\]

Use \( B \sim C \) for the similarity equivalence relation.

Call a matrix OTForm \( sI \) note \([0 \ 2] \), a dilation; its action on the plane is simply to scale-uniformly by factor \( s \). Every matrix commutes with \( I \), and so:

1: A dilation is only conjugate to itself.

I.e., for invertible \( U \), necessarily \( U^{-1} \cdot sI \cdot U = sI \).

For \( T \) a square matrix (or a tran from a vectorspace to itself) and a complex number \( \alpha \), define the subspace

\[
E_{T,\alpha} := \{ \text{vectors } v \mid Tv = \alpha v \}.
\]

Saying that “\( \alpha \) is a \( T \)-eigenvalue” is the same as saying that \( \text{Dim}(E_{T,\alpha}) \geq 1 \).

Linear recurrence

A fibonacci-like sequence \( z := (z_n)_{n=\infty} \) is specified using (complex) numbers \( S \) and \( P \), with \( P \) non-zero\(^\ddagger\), by

\[
z_{n+2} := Sz_{n+1} - Pz_n,
\]

and some initial condition \( (z_1, z_0) \). With

\[
G := \begin{bmatrix} S & -P \\ 1 & 0 \end{bmatrix}, \text{ then } [z_{n+1}^n] = G^n \cdot [z_1^0],
\]

for each integer \( n \). We want to diagonalize \( G \).

\(^\ddagger\) Actually, recurrence (3) can be run backwards, giving \( z_n \) values when \( n \) is negative, as soon as \( (S, P) \neq (0, 0) \). However, the \( P\neq 0 \) case is the interesting case.

Its char-poly is \( f(x) = f_G(x) := x^2 - Sx + P \). Factor this as \( f(x) = [x - \alpha][x - \beta] \), with \( \alpha, \beta \in \mathbb{C} \).

Equating coeffs in the polynomial gives

\[
\begin{align*}
\alpha + \beta &= S; \quad \text{(Sum)} \\
\alpha \cdot \beta &= P. \quad \text{(Product)}
\end{align*}
\]

Since \( P \neq 0 \), necessarily \([\alpha \neq 0] \) and \([\beta \neq 0] \). Each of \( \alpha, \beta \) is a root of \( f \), hence

\[
\alpha^2 = S\alpha - P \quad \text{and} \quad \beta^2 = S\beta - P.
\]

Finding an \( \alpha \)-eigenvector. An \( \alpha \)-evec (for \( G \)) is an element of \( \text{Nul}(G - \alpha I) \). Applying row operations,

\[
G - \alpha I = \begin{bmatrix} S - \alpha & -P \\ 1 & -\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & -\alpha \\ 0 & 0 \end{bmatrix}.
\]

This last \( \sim \) requires no computation, since the rows must be linearly-dependent (since \( \alpha \) is an eigenvalue of \( G \)).

This last matrix has one free column, and evidently multiplies \([\alpha] \) to the zero-vector. So

\[
\begin{align*}
7: & \quad \text{The singleton } \{ [\alpha] \} \text{ is a basis for the } \alpha \text{-eigenspace of } G, \text{ which is one-dimensional.} \\
\end{align*}
\]

Instead of row-ops to show that \( E_{G,\alpha} \) (the \( \alpha \)-eigenspace of \( G \)) is one-dim’al, we could have argued as follows: Were \( E_{G,\alpha} \) two-dim’al, then \( G \sim \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \).

By (1), then, \( G \) would equal \( \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \) —which it doesn’t!

When is \( \alpha = \beta \)? This happens when the discriminant of \( f \) is zero. Its discrm is \([S]^2 - 4 \cdot 1 \cdot P \), i.e \([\alpha^2 + \beta^2 + 2\alpha\beta] - 4\alpha\beta \). Hence

\[
8: \quad \text{Discr}(f) = S^2 - 4P \not\equiv [\alpha - \beta]^2.
\]

Since (7) also applies to eigenvalue \( \beta \), we conclude:

\[
9: \quad \text{Matrix } G = \begin{bmatrix} S & -P \\ 1 & 0 \end{bmatrix} \text{ is diagonalizable } \iff \text{G has distinct eigenvalues; i.e } S^2 \neq 4P.
\]
**Distinct eigenvalues**

When $\alpha \neq \beta$, our (7) implies that $G$ is conjugate to diagonal-matrix

10: $D := \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ via matrix

11: $U := \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}$. Note $U^{-1} = \frac{1}{\alpha - \beta} \begin{bmatrix} 1 & -\beta \\ 1 & \alpha \end{bmatrix}$.

We populated $U$ with evecs, using the same order of evals in $D$. One can check that $(U^{-1}GU = D)$. Hence $G = UDU^{-1}$. So for each integer $n$,

12: $G^n = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \cdot \begin{bmatrix} 1 & -\beta \\ 1 & \alpha \end{bmatrix}$.

Multiplying from the right by initial-condition $\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$ will produce $\begin{bmatrix} z_{n+1} \\ z_n \end{bmatrix}$. Extracting the $z_n$, we get the nice formula

13: $z_n = \frac{1}{\alpha - \beta} \cdot [z_1 - z_0 \beta] \alpha^n - [z_1 - z_0 \alpha] \beta^n$.

This formula is symmetric in $\alpha$ & $\beta$, as it must be.

**Equal eigenvalues**

Let’s take a look at the $\alpha = \beta$ case. This means that

14: $G = \begin{bmatrix} 2\alpha & -\alpha^2 \\ 1 & 0 \end{bmatrix}$.

While we can’t conjugate $G$ to a diagonal matrix, we can conjugate to its Jordan canonical form. The JCF of (14) is

$J = J_G := \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$.

Mysteriously pulling the below $C$ from a hat, multiplication verifies that

15: $J = C^{-1}GC$, where $C := \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}$

and $C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}$. Leaving a derivation of $C$ to later, let’s use this to get a formula for $z_n$.

Induction on $n$ shows that

$J^n := \alpha^{n-1} \begin{bmatrix} \alpha & n \\ 0 & \alpha \end{bmatrix}$, for each $n \in \mathbb{Z}$.

As before, $G = CJC^{-1}$ so $G^n = CJ^nC^{-1}$. That is,

$G^n = \alpha^{n-1} \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & n \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}$.

Multiplying by $\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$; the bottom entry in the resulting column-vector is

17: $z_n = \alpha^{n-1} \begin{bmatrix} n z_1 + [1 - n] \alpha z_0 \end{bmatrix}$.