Projections on a line : LinearAlg

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Problem

Let $\mathsf{P} = \mathsf{P}_{30^\circ}$ be the 2×2 matrix whose lefthand action is ortho-projection on the angle=30° line through the origin. Compute P.

Support your answer in three conceptually-different ways.

Strategy

***Step** S0.* A simplifying notation:
Use $c := \cos(30^\circ)$ and $s := \sin(30^\circ)$.

***Step** S1.* Use similar triangles to compute $\mathsf{P}\cdot\mathbf{e}_1$; this product gives the first column of P. Use similar triangles to compute $\mathsf{P}\cdot\mathbf{e}_2$; this equals the second column of P.

***Step** S2.* Checking our result: Each vector $\mathbf{v}$ on the 30° line should be *fixed* (not moved) by the projection. So $\mathsf{P}\cdot\mathbf{v}$ better equal $\mathbf{v}$.

***Step** S3.* "Variation of parameters": Generalize the result of Step 1, replacing "30°" by a general angle ω. Easily

$$\text{1:}\quad \mathsf{P}_{0^\circ} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad\text{and}\quad \mathsf{P}_{90^\circ} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Now we will check that our P_ω formula gives these two matrices when ω is set equal to 0° and 90°.

***Step** S4.* "The Dictionary Idea": Instead, let us obtain P_{30° by *conjugation* by Y_{30°. So we check that

$$\mathsf{P}_\omega \stackrel{?}{=\!=} \mathsf{Y}_\omega\cdot\mathsf{P}_{0^\circ}\cdot\mathsf{Y}_{-\omega}\;.$$

(Aside: Here, given two square matrices B and E, let "B conjugated by D" mean $\mathsf{D}\cdot\mathsf{B}\cdot\mathsf{D}^{-1}$.)

Applying the Strategy

***Doing** S1.* Similar triangles yields

$$\text{2:}\qquad \mathsf{P} = \begin{bmatrix}c\cdot c & c\cdot s\\ c\cdot s & s\cdot s\end{bmatrix}.$$

***Doing** S2.* The unit-vectors on the 30° line are $\pm\mathbf{v}$, where $\mathbf{v} := \left[\begin{smallmatrix}c\\ s\end{smallmatrix}\right]$. Define $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] := \mathsf{P}\cdot\left[\begin{smallmatrix}c\\ s\end{smallmatrix}\right]$. The Pythagorean thm yields that

$$x = [c\cdot c]\cdot c + [c\cdot s]\cdot s = c\cdot[c^2+s^2] \stackrel{\text{Pythag}}{=\!=} c\,.$$

Similarly $y = s$. Thus we see that $\mathsf{P}\cdot\mathbf{v}$ indeed equals $\mathbf{v}$.

***Doing** S3.* The derivation of (2) applies when 30° is replaced by a general angle ω. Redefining $c := \cos$ and $s := \sin$ thus gives

$$\text{3:}\qquad \mathsf{P}_\omega = \begin{bmatrix}c(\omega)\cdot c(\omega) & c(\omega)\cdot s(\omega)\\ c(\omega)\cdot s(\omega) & s(\omega)\cdot s(\omega)\end{bmatrix}.$$

So P_{0° is $\left[\begin{smallmatrix}1\cdot 1 & 1\cdot 0\\ 1\cdot 0 & 0\cdot 0\end{smallmatrix}\right]$, which indeed equals $\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$, the P_{0° from (1). The check for P_{90° similarly works. Note that we could also have used P_{45° as an easy check.

***Doing** S4.* Letting c and s denote $\cos(\omega)$ and $\sin(\omega)$, recall our rotation matrix $\mathsf{Y}_\omega = \left[\begin{smallmatrix}c & -s\\ s & c\end{smallmatrix}\right]$. Since cos() is an even fnc, and sin() is odd, automatically $\mathsf{Y}_{-\omega} = \left[\begin{smallmatrix}c & s\\ -s & c\end{smallmatrix}\right]$. Multiplying the matrices $\mathsf{Y}_\omega\cdot\mathsf{P}_{0^\circ}\cdot\mathsf{Y}_{-\omega}$ indeed yields RhS(2), as hoped.

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