

## Projections on a line : LinearAlg

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### Problem

Let  $P = P_{30^\circ}$  be the  $2 \times 2$  matrix whose lefthand action is ortho-projection on the angle= $30^\circ$  line through the origin. Compute  $P$ .

Support your answer in three conceptually-different ways.

### Strategy

**Step S0.** A simplifying notation:  
Use  $c := \cos(30^\circ)$  and  $s := \sin(30^\circ)$ .

**Step S1.** Use similar triangles to compute  $P \cdot \mathbf{e}_1$ ; this product gives the first column of  $P$ . Use similar triangles to compute  $P \cdot \mathbf{e}_2$ ; this equals the second column of  $P$ .

**Step S2.** Checking our result: Each vector  $\mathbf{v}$  on the  $30^\circ$  line should be *fixed* (not moved) by the projection. So  $P \cdot \mathbf{v}$  better equal  $\mathbf{v}$ .

**Step S3.** “Variation of parameters”: Generalize the result of Step 1, replacing “ $30^\circ$ ” by a general angle  $\omega$ . Easily

$$1: \quad P_{0^\circ} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{90^\circ} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .$$

Now we will check that our  $P_\omega$  formula gives these two matrices when  $\omega$  is set equal to  $0^\circ$  and  $90^\circ$ .

**Step S4.** “The Dictionary Idea”: Instead, let us obtain  $P_{30^\circ}$  by *conjugation* by  $Y_{30^\circ}$ . So we check that

$$P_\omega \stackrel{?}{=} Y_\omega \cdot P_{0^\circ} \cdot Y_{-\omega} .$$

(Aside: Here, given two square matrices  $B$  and  $E$ , let “ $B$  conjugated by  $D$ ” mean  $D \cdot B \cdot D^{-1}$ .)

### Applying the Strategy

**Doing S1.** Similar triangles yields

$$2: \quad P = \begin{bmatrix} c \cdot c & c \cdot s \\ c \cdot s & s \cdot s \end{bmatrix} .$$

**Doing S2.** The unit-vectors on the  $30^\circ$  line are  $\pm \mathbf{v}$ , where  $\mathbf{v} := \begin{bmatrix} c \\ s \end{bmatrix}$ . Define  $\begin{bmatrix} x \\ y \end{bmatrix} := P \cdot \begin{bmatrix} c \\ s \end{bmatrix}$ . The Pythagorean thm yields that

$$x = [c \cdot c] \cdot c + [c \cdot s] \cdot s = c \cdot [c^2 + s^2] \stackrel{\text{Pythag}}{=} c .$$

Similarly  $y = s$ . Thus we see that  $P \cdot \mathbf{v}$  indeed equals  $\mathbf{v}$ .

**Doing S3.** The derivation of (2) applies when  $30^\circ$  is replaced by a general angle  $\omega$ . Re-defining  $c := \cos$  and  $s := \sin$  thus gives

$$3: \quad P_\omega = \begin{bmatrix} c(\omega) \cdot c(\omega) & c(\omega) \cdot s(\omega) \\ c(\omega) \cdot s(\omega) & s(\omega) \cdot s(\omega) \end{bmatrix} .$$

So  $P_{0^\circ}$  is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , which indeed equals  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , the  $P_{0^\circ}$  from (1). The check for  $P_{90^\circ}$  similarly works. Note that we could also have used  $P_{45^\circ}$  as an easy check.

**Doing S4.** Letting  $c$  and  $s$  denote  $\cos(\omega)$  and  $\sin(\omega)$ , recall our rotation matrix  $Y_\omega = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ . Since  $\cos()$  is an even fnc, and  $\sin()$  is odd, automatically  $Y_{-\omega} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ . Multiplying the matrices  $Y_\omega \cdot P_{0^\circ} \cdot Y_{-\omega}$  indeed yields RhS(2), as hoped.

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