The Laplace Transform: Calculus

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ABSTRACT: This first gives a defn of *exponential order* which is better adapted to convolution. Following, is a discussion of the "tapping on a bell" problem; one text called this "soldiers marching on a bridge". Both interpretations need a grain-of-salt...

Prelims. On a (possibly infinite) interval $J \subset \mathbb{R}$, a function $f: J \to \mathbb{C}$ is ***locally-integrable*** if, for each bounded subinterval $[a, b] \subset J$, integral $\int_a^b f$ exists and is finite. A *sufficient* (but not necessary) condition is that on each $[a, b]$, our f is bounded with only finitely-many discontinuities.

For a fnc $f: [0, \infty) \to \mathbb{C}$ and complex number s we define the "Laplace transform of f", evaluated at s,

1: $$\widehat{f}(s) = [\mathcal{L}(f)](s) := \int_0^\infty \mathrm{e}^{-st} \cdot f(t) \cdot \mathrm{d}t\,,$$

for those values of s where this integral exists.

For complex numbers α and β, let $\alpha \succ \beta$ mean $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$. For a real $\boldsymbol{\mu}$, say that "f has ***exponential order*** $\boldsymbol{\mu}$", written $\boxed{f \in \mathrm{Ord}(\boldsymbol{\mu})}$, if $f: [0, \infty) \to \mathbb{C}$ and f is locally-integrable *and*

†: $$\forall Q > \boldsymbol{\mu}: \quad \lim_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} = 0\,.$$

One can replace (†) by the seemingly weaker

‡: $$\forall Q > \boldsymbol{\mu}: \quad \limsup_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} < \infty\,,$$

but they are [exercise] equivalent.

2: Lemma. *Consider an $f \in \mathrm{Ord}(\boldsymbol{\mu})$. Then $\widehat{f}(s)$ exists for each s with $\mathrm{Re}(s) > \boldsymbol{\mu}$. Indeed the integrand in (1) is absolutely integrable.* ♢

Proof. Fix s with $x := \mathrm{Re}(s) > \boldsymbol{\mu}$, then pick Q with $x > Q > \boldsymbol{\mu}$. Our t is real, so $|\mathrm{e}^{-st}| = \mathrm{e}^{-xt}$. Hence the integrand in (1) is eventually bounded,

$$\left|\mathrm{e}^{-st} \cdot f(t)\right| < \mathrm{e}^{-xt} \cdot \mathrm{e}^{Qt} \stackrel{\text{note}}{=\!=} \mathrm{e}^{-[x-Q]t}\,,$$

once t is large enough. Since $x - Q$ is positive, this last is integrable over $t \in [0, \infty)$. ♦

Convolution. Recall that the (one-sided) ***convolution*** of two (locally-integrable) functions $f, g: [0, \infty) \to \mathbb{C}$, is the function

3a: $$[f \circledast g](t) := \int_0^t f(t - v) \cdot g(v)\, \mathrm{d}v\,.$$

3b: Lemma. *Suppose $f, g \in \mathrm{Ord}(\boldsymbol{\mu})$. Then the convolution $f \circledast g \in \mathrm{Ord}(\boldsymbol{\mu})$.* ♢

[Proof omitted.]

We need the following standard tool, to prove a uniqueness result.

4a: Identically-zero lemma. *On a closed bounded interval $J := [b, c] \subset \mathbb{R}$, consider continuous functions $h, G: J \to \mathbb{C}$.*

*i: Suppose $h \geqslant 0$. If $\int_J h = 0$, then $h \equiv 0$, i.e, h is **identically-zero**.* [Exer:]

ii: If G is real-valued and $\int_J [G \cdot G] = 0$, then $G \equiv 0$. [Exer:]

iii: Suppose

4b: $$\forall n \in \mathbb{N}: \int_J x^n \cdot G(x)\, \mathrm{d}x = 0\,.$$

Then G is identically-zero on J. ♢

Proof of (iii). Splitting G into real and imaginary parts, WLOG G is real-valued. Stmt (4b) implies, for each polynomial p, that $\int_J [p \cdot G]$ is zero.

By the Weierstrass Approximation Thm, there is a sequence of (real) polynomials, so that $p_k \xrightarrow{k \to \infty} G$ *uniformly*. Consequently,

$$\int_J [G \cdot G] \stackrel{\text{Exer.}}{=\!=} \lim_{k \to \infty} \int_J [p_k \cdot G] = \lim_{k \to \infty} 0 = 0\,. \blacklozenge$$

4c: Uniqueness Thm. *Consider fncs $g, \gamma \in \mathrm{Ord}(\boldsymbol{\mu})$. Suppose their Laplace transforms, $\widehat{g}$ and $\widehat{\gamma}$, agree on some real interval $\mathbf{I} := [s_0, \infty)$. If g and γ are continuous, then $g = \gamma$.* ♢

Proof. The difference fnc $f := g - \gamma$ is cts and in $\mathrm{Ord}(\boldsymbol{\mu})$, with $\widehat{f}$ identically-zero on $\mathbf{I}$. So our goal is to show f identically-zero on $[0, \infty)$.

WLOG $s_0 > \boldsymbol{\mu}$. Define $G(u) := u^{s_0} \cdot f(\text{-}\log(u))$, and $G(0) := 0$. CoV $t = \text{-}\log(u)$ shows that

$$\lim_{u \searrow 0} G(u) = \lim_{t \nearrow \infty} \mathrm{e}^{-s_0 \cdot t} f(t)$$

equals 0, since $s_0 > \boldsymbol{\mu}$. Hence $\boxed{G \text{ is cts on } [0,1]}$.

CoV $t = \text{-}\log(u)$ makes $\mathrm{d}t = -\mathrm{d}u/u$ and

$$\mathrm{e}^{-st} = [\mathrm{e}^{-t}]^s = u^s .$$

Evaluating at $s{+}1$, the CoV rewrites (1) as

$$\begin{aligned}\widehat{f}(s{+}1) &= \int_1^0 u^{s+1} \cdot f(\text{-}\log(u)) \cdot \frac{-\mathrm{d}u}{u} \\ &= \int_0^1 u^s \cdot f(\text{-}\log(u))\,\mathrm{d}u \\ &= \int_0^1 u^{s-s_0} \cdot G(u)\,\mathrm{d}u .\end{aligned}$$

For $n = 0, 1, 2, \ldots$, let $s := s_0 + n$ and note $s{+}1$ is in $\mathbf{I}$. Hence $\widehat{f}(s{+}1)$ is zero, i.e $\int_0^1 u^n G(u)\,\mathrm{d}u$ is zero. By (4a), then, $G \equiv 0$ on $[0, 1]$.

So on half-open $(0, 1]$, map $[u \mapsto f(\text{-}\log(u))]$ is zero. Hence $f \equiv 0$ on $[0, \infty)$. ♦

5a: TL-Diff. *Fix a natnum N. Taking the N^{th} derivative,*

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}\right]^N \widehat{f}(s) = \mathcal{L}\Big([\text{-}t]^N \cdot f(t)\Big)(s) \overset{\text{note}}{=\!=} [\text{-}1]^N \cdot \mathcal{L}\Big(t^N \cdot f(t)\Big)(s) . \quad \diamondsuit$$

5b: LT-Diff. *Suppose f differentiable. Then*

$$\widehat{f'}(s) = \Big[s \cdot \widehat{f}(s)\Big] - f(0) .$$

For $N \in \mathbb{N}$, suppose that an N-times differentiable f, has $f, f', \ldots, f^{(N-1)} \in \mathrm{Ord}(\boldsymbol{\mu})$. Then

$$\widehat{f^{(N)}}(s) = \Big[s^N \cdot \widehat{f}(s)\Big] - \sum_{\substack{j+k = N-1 \\ j,k \in \mathbb{N}}} s^j \cdot \Big[f^{(k)}(0)\Big] ,$$

where the sum is over all ordered-pairs (j, k) of natnums. ♢

5c: LT-Exp. *Fix $B \in \mathbb{C}$. Then*

$$\mathcal{L}\Big(\mathrm{e}^{Bt} \cdot f(t)\Big)(s) = \widehat{f}(s - B) ,$$

whenever $s \succ B + \boldsymbol{\mu}$. ♢

5d: LT-Convolve. $\Big[\mathcal{L}(f \circledast g)\Big](s) = \widehat{f}(s) \cdot \widehat{g}(s)$. ♢

5e: TL-Integrate. *For real s with $s > \boldsymbol{\mu}$,*

$$\int_s^\infty \widehat{f}(u)\,\mathrm{d}u = \mathcal{L}\Big(\frac{f(t)}{t}\Big)(s) \quad \diamondsuit$$

Proof of (5e). For t positive, note

$$*: \quad \begin{aligned}\int_s^\infty \mathrm{e}^{-ut}\,\mathrm{d}u &= \tfrac{1}{t} \cdot \mathrm{e}^{-ut}\Big|_{u=\infty}^{u=s} \\ &= \tfrac{1}{t} \cdot [\mathrm{e}^{-st} - \mathrm{e}^{-\infty}] = \tfrac{1}{t} \cdot \mathrm{e}^{-st} .\end{aligned}$$

Applying the definition, $\int_s^\infty \widehat{f}(u)\,\mathrm{d}u$ equals

$$\begin{aligned}\int_s^\infty \Big[\int_0^\infty \mathrm{e}^{-ut} \cdot f(t) \cdot \mathrm{d}t\Big]\,\mathrm{d}u &= \int_0^\infty f(t) \int_s^\infty \mathrm{e}^{-ut}\,\mathrm{d}u\,\mathrm{d}t \\ &\overset{\text{by } (*)}{=\!=} \int_0^\infty f(t) \cdot \tfrac{1}{t} \cdot \mathrm{e}^{-st}\,\mathrm{d}t .\end{aligned}$$

And this is the defn of $\mathcal{L}\Big(\frac{f(t)}{t}\Big)(s)$. ♦

Pf of (5a), TL-Diff. With appropriate conditions on f, we can differentiate under the integral sign in (1), applying $\frac{\mathrm{d}}{\mathrm{d}s}$, to get the $N{=}1$ case of (5a). Now induct on N. ♦

Pf LT-Diff. Our $\widehat{f'}(s)$ is the limit, as $M \nearrow \infty$, of

$$\int_0^M \underbrace{\mathrm{e}^{-st}}_{u} \cdot \underbrace{f'(t)\,\mathrm{d}t}_{\mathrm{d}v} .$$

Integrating by parts, this equals

$$*: \quad \begin{aligned}&\underbrace{\mathrm{e}^{-st}}_{u} \cdot \underbrace{f(t)}_{v}\Big|_{t=0}^{t=M} - \int_0^M \underbrace{f(t)}_{v} \cdot \underbrace{[\text{-}s]\mathrm{e}^{-st}\,\mathrm{d}t}_{\mathrm{d}u} \\ &= \Big[s \cdot \int_0^M f(t) \cdot \mathrm{e}^{-st}\,\mathrm{d}t\Big] - \frac{f(t)}{\mathrm{e}^{st}}\Big|_{t=M}^{t=0} .\end{aligned}$$

This last term equals $\frac{f(0)}{1} - \frac{f(M)}{\mathrm{e}^{sM}} = f(0) - \frac{f(M)}{\mathrm{e}^{sM}}$. If $\mathrm{Re}(s) > \boldsymbol{\mu}$, then $\frac{f(M)}{\mathrm{e}^{sM}} \to 0$. as $M \nearrow \infty$. Sending $M \nearrow \infty$ thus sends $(*)$ to $\left[s \cdot \widehat{f}(s)\right] - f(0)$, as desired.

Finally, the formula for $\widehat{f^{(N)}}$ follows by induction on N. ♦

Pf LT-Exp. By defn, $\mathcal{L}\left(\mathrm{e}^{Bt} \cdot f(t)\right)(s)$ equals

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{Bt} f(t)\,\mathrm{d}t \;=\; \int_0^\infty \mathrm{e}^{-[s-B]t} f(t)\,\mathrm{d}t\,.$$

This last integral converges once $\mathrm{Re}(s - B) > \boldsymbol{\mu}$, i.e, once $s \succ B + \boldsymbol{\mu}$. ♦

Proof of LT-Convolve. Define $\mathbf{1}(\mathsf{true}) := 1$ and $\mathbf{1}(\mathsf{false}) := 0$. We can now write RhS(3$a$) as

$$\int_0^\infty \mathbf{1}(v \leqslant t) \cdot f(t-v) \cdot g(v)\,\mathrm{d}v\,.$$

Hence $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty \mathrm{e}^{-st} \int_0^\infty \mathbf{1}(v \leqslant t) \cdot f(t-v) \cdot g(v)\,\mathrm{d}v\,\mathrm{d}t\,.$$

Under mild conditions on f and g, we can reverse the integrals, giving that $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty g(v) \int_0^\infty \mathbf{1}(v \leqslant t) \cdot \mathrm{e}^{-st} \cdot f(t-v)\,\mathrm{d}t\,\mathrm{d}v\,.$$

The inner integral can be written as

$$\int_v^\infty \mathrm{e}^{-st} \cdot f(t-v)\,\mathrm{d}t\,.$$

The change of variable $x = t - v$ gives that $\mathrm{d}x = \mathrm{d}t$. So this integral becomes

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-s[x+v]} \cdot f(x)\,\mathrm{d}x \;&=\; \mathrm{e}^{-sv} \int_0^\infty \mathrm{e}^{-sx} \cdot f(x)\,\mathrm{d}x \\ &=\; \mathrm{e}^{-sv} \cdot \widehat{f}(s)\,.\end{aligned}$$

Thus $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty g(v) \cdot \mathrm{e}^{-sv} \cdot \widehat{f}(s)\,\mathrm{d}v \;\stackrel{\text{note}}{=\!=}\; \widehat{f}(s) \cdot \widehat{g}(s)\,. \quad ♦$$

Examples

Below, $B \in \mathbb{C}$ and $F \in \mathbb{R}$. Easily $[\mathcal{L}(1)](s)$ equals $1/s$. From TL-Diff, then, $[-1]^N \cdot \mathcal{L}\left(t^N\right)(s)$ equals $[\frac{\mathrm{d}}{\mathrm{d}s}]^N\left(\widehat{1}(s)\right) = [\frac{\mathrm{d}}{\mathrm{d}s}]^N\left(\frac{1}{s}\right)$. Thus

6a:
$$\mathcal{L}\left(t^N\right)(s) \;=\; \frac{N!}{s^{N+1}}\,.$$

From LT-Exp, note,

$*$:
$$[\mathcal{L}(\mathrm{e}^{Bt})](s) \;=\; \frac{1}{s-B}\,.$$

Recall $\cos(Ft) = \frac{1}{2}[\mathrm{e}^{\boldsymbol{i}Ft} + \mathrm{e}^{-\boldsymbol{i}Ft}]$. So $(*)$ gives

$$2 \cdot \mathcal{L}\left(\cos(Ft)\right)(s) \;=\; \frac{1}{s - \boldsymbol{i}F} + \frac{1}{s + \boldsymbol{i}F} \;=\; \frac{2s}{s^2 + F^2}\,.$$

Dividing both sides by 2,

6b:
$$\mathcal{L}\left(\cos(Ft)\right)(s) \;=\; \frac{s}{s^2 + F^2}\,, \quad \text{and similarly}$$
$$\mathcal{L}\left(\sin Ft\right)(s) \;=\; \frac{F}{s^2 + F^2}\,.$$

As sin() is bounded, it has a Laplace transform on $\mathbb{R}_+$. Since $\widehat{\sin}(u) = \frac{1}{u^2+1} = \arctan'(u)$, it follows that

$$\int_0^\infty \widehat{\sin}(u)\,\mathrm{d}u \;=\; \arctan(\infty) - \arctan(0) \;=\; \frac{\boldsymbol{\pi}}{2} - 0\,.$$

Skipping the effort it takes to *justify* applying (5e) at s=0, we get that

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \;= \mathcal{L}\left(\frac{\sin(t)}{t}\right)(0) \;\stackrel{\text{by (5e)}}{=\!=}\; \int_0^\infty \widehat{\sin}\,.$$

This, together with the previous line, gives

6c:
$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \;=\; \frac{\boldsymbol{\pi}}{2}\,.$$

Preliminaries

The ***Heaviside fnc*** $\mathbf{H}:\mathbb{R}\to\{0,1\}$ is 0 on $(-\infty,0)$ and is 1 on $[0,+\infty)$. Thus[♡1]

$$\sum_{K=1}^{\infty} \mathbf{H}(x-K) \;=\; \lfloor x \rfloor\,, \qquad \text{for each } x\geqslant 0.$$

Let $\boldsymbol{\delta}$ denote the ***Dirac delta "function"***.[♡2] Write $\boldsymbol{\delta}_5$ for its translate $x \mapsto \boldsymbol{\delta}(x-5)$. So for P a posreal, and f *continuous* at P:

$$\int_0^\infty f\cdot \boldsymbol{\delta}_{\mathrm{P}} \;=\; f(\mathrm{P})\,.$$

In particular, for each f which is continuous:

7a: $$[f \circledast \boldsymbol{\delta}_5](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t\in[0,5) \\ f(t-5) & \text{, if } t\in[5,\infty)\end{matrix}\right\}.$$

I.e. $f \circledast \boldsymbol{\delta}_5 \;=\; \mathbf{T}_5\big(\mathbf{H}\cdot f\big)$.

Periodicity. A posreal number P is a "***period*** of fnc f" if

$$\forall x: \quad f(x+\mathrm{P}) \;=\; f(x)\,.$$

Typically, there is a *smallest* such period, which is called the "***least-period*** of f".

For a posreal P, let $f_{\langle \mathrm{P}\rangle} := f\cdot \mathbf{1}_{[0,\mathrm{P})}$ abbreviate what I will call "f clipped at P". That is, we restrict f to interval $[0,\mathrm{P})$. Thus

$$\widehat{f_{\langle \mathrm{P}\rangle}}(s) \;=\; \int_0^{\mathrm{P}} \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,.$$

7b: Periodicity Theorem. *Suppose $f,g \in \mathrm{Ord}(\boldsymbol{\mu})$ and P is a posreal. Then*

1: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s) = \mathrm{e}^{-s\mathrm{P}}\cdot \widehat{f}(s)$.

2: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})g(t)\Big)(s) = \mathrm{e}^{-s\mathrm{P}}\cdot \mathcal{L}\Big(g(t+\mathrm{P})\Big)(s)$.

3: Suppose now that P is a period of f. Then

$$\widehat{f}(s) \;=\; \widehat{f_{\langle \mathrm{P}\rangle}}(s)\Big/\big[1-\mathrm{e}^{-s\mathrm{P}}\big]\,. \qquad \diamond$$

Proof of (7b.1). The LhS equals $\int_{\mathrm{P}}^\infty \mathrm{e}^{-st} f(t-\mathrm{P})\,\mathrm{d}t$. CoV $x=t-\mathrm{P}$ gives $\mathrm{d}x=\mathrm{d}t$, and thus

$$\mathrm{LhS(7b.1)} = \int_0^\infty \mathrm{e}^{-s\,[x+\mathrm{P}]} f(x)\,\mathrm{d}x \overset{\text{note}}{=\!=} \mathrm{RhS(7b.1)}\,. \blacklozenge$$

Pf (7b.3). Because our "clipped" $f_{\langle \mathrm{P}\rangle}(t)$ equals $[\mathbf{H}(t)-\mathbf{H}(t-\mathrm{P})]\cdot f(t)$, it follows that

$$*: \quad \begin{aligned} \mathcal{L}(f_{\langle \mathrm{P}\rangle}) &= \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t)\Big) \\ &\overset{\text{P an } f\text{-period}}{=\!=\!=} \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)\,. \end{aligned}$$

Our (7b.1) says that $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s)$ equals $\mathrm{e}^{-s\mathrm{P}}\cdot\widehat{f}(s)$. And, always, $\mathcal{L}(\mathbf{H}\cdot f)=\mathcal{L}(f)$, since the Laplace integral is over $[0,\infty)$. So we can re-write $(*)$ as $\widehat{f_{\langle \mathrm{P}\rangle}}(s) = [1-\mathrm{e}^{-s\mathrm{P}}]\cdot\widehat{f}(s)$. ♦

[♡1] For a real x, the expression $\lfloor x\rfloor$ is called the ***floor*** of x; it is the largest integer less-equal x. So $\lfloor \boldsymbol{\pi} \rfloor$ is 3.

Use DE to mean '*differential eqn*'.

[♡2] It is actually a *Schwartzian distribution*, named after Laurant Schwartz. As distributions, $\mathbf{H}' = \boldsymbol{\delta}$.

Using periodicity. Function sin() is periodic, with period $2\boldsymbol{\pi}$. So for each posint K:

$$\left[\sin \circledast \boldsymbol{\delta}_{2\pi K}\right](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t \in [0, 2\boldsymbol{\pi}K) \\ \sin(t) & \text{, if } t \in [2\boldsymbol{\pi}K, \infty) \end{matrix}\right\}.$$

For $N \in \{1, 2, 3, \dots\} \cup \{\infty\}$, define the sum

$$R_N \;:=\; \sum\nolimits_{K=1}^{N} \boldsymbol{\delta}_{2\pi K}\,.$$

For N finite, then,

$$\left[\sin \circledast R_N\right](t) \;=\; \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)$$

holds when $0 \leqslant t < 2\boldsymbol{\pi}[N{+}1]$. There are no convergence problems as $N \nearrow \infty$. So $\forall t \in [0 \mathinner{..} \infty)$:

7c: $$\left[\sin \circledast R_\infty\right](t) \;=\; \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)\,.$$

Moreover, for each finite N:

$7c_N$: $$\left[\sin \circledast R_N\right](t) \;=\; \operatorname{Min}\left(N\,,\, \lfloor \tfrac{t}{2\pi} \rfloor\right) \cdot \sin(t).$$

Hammering an undamped spring

(One of the textbooks called this *Soldiers marching in cadence*, but the interpretation is less clear, since soldiers already on the bridge are still marching.) With y the unknown-fnc, let us examine DE

$\mathfrak{D}$: $$y'' + y \;=\; \sum_{K=1}^{\infty} \boldsymbol{\delta}_{2\pi K}\,.$$

Given two complex numbers α, β, let $(\mathfrak{D}_{\alpha,\beta})$ mean the DE together with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$.

Consider the corresponding ZeroTar[♡3] IVP

$$z'' + z \;=\; 0\,, \qquad \textit{with } z(0) = \alpha \textit{ and } z'(0) = \beta\,.$$

Its solution is

7d: $$z \;=\; [\alpha \cdot \cos] + [\beta \cdot \sin]\,.$$

[♡3] Some textbooks call this (yuck!) "homogeneous".

So the soln, y, to $(\mathfrak{D}_{0,0})$ will, when added to z, give the solution to $(\mathfrak{D}_{\alpha,\beta})$.

Computing,

$$\begin{aligned} \mathcal{L}\left(y'' + y\right)(s) \;&=\; [s^2 + 1] \cdot \widehat{y}(s) \;-\; [s \cdot y(0) + y'(0)] \\ &=\; [s^2 + 1] \cdot \widehat{y}(s)\,. \end{aligned}$$

Now $(\mathfrak{D})$ says that $\mathcal{L}\left(y'' + y\right) = \mathcal{L}(R_\infty)$. Dividing,

$$\widehat{y}(s) \;=\; \tfrac{1}{s^2+1} \cdot \widehat{R_\infty}(s)\,.$$

The RhS is a product, so its inv-lap-xform is a convolution.

$$y \;=\; \sin \circledast R_\infty\,.$$

In other words, the function

7e: $$y_{\alpha,\beta}(t) \;:=\; \alpha\cos(t) + \beta\sin(t) + \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)$$

is the general solution to IVP $(\mathfrak{D}_{\alpha,\beta})$.

At this point, it would be good to have a careful sketch of $y_{\alpha,\beta}()$. One could also analyze, at each time t, the potential and kinetic energy in the spring. If the spring is damped, is there a net absorption, or loss, of energy?

Suppose the tapping on the spring was <u>not</u> aligned with the resonant frequency of the spring —would the spring nonetheless absorb energy?

Random hammering. Times $0{<}G_1{<}G_2{<}\dots$ define

$$R_{\mathbf{G}} \;:=\; \sum\nolimits_{K=1}^{\infty} \boldsymbol{\delta}_{G_K}\,.$$

The above reasoning applied to DE $\boxed{y'' + y = R_{\mathbf{G}}}$ yields $y = \sin \circledast R_{\mathbf{G}}$, as before. Computing the convolution gives

7f: $$y(t) \;=\; \sum_{K=1}^{\infty} \mathbf{H}(t - G_K) \cdot \sin(t - G_K)\,.$$

Adding $[\alpha \cos(t) + \beta \sin(t)]$ hands us the general solution $y_{\alpha,\beta}$.

Square Wave

For a period P>0 and a ***dUty cycle*** $U \in [0,1]$, define the ***square-wave fnc*** $\mathbb{SW}_{\mathrm{P},U}()$ to be *the* P-periodic fnc mapping $\mathbb{R}\to\mathbb{R}$, which agrees with $\mathbf{1}_{[0,\,U\mathrm{P})}$ on interval $[0,\mathrm{P})$; the ***pulse width*** is $U\mathrm{P}$.

So $\mathbb{SW}_{\mathrm{P},0} \equiv 0$ and $\mathbb{SW}_{\mathrm{P},1} \equiv 1$. Thus we expect

$$*\colon \qquad \widehat{\mathbb{SW}_{\mathrm{P},0}} \;\equiv\; 0 \quad \textit{and} \quad \widehat{\mathbb{SW}_{\mathrm{P},1}}(s) \;=\; 1/s\,.$$

Setting $f := \mathbb{SW}_{\mathrm{P},U}$, our Periodicity Thm says that $\widehat{f_{\langle \mathrm{P}\rangle}}(s)$ equals

$$\int_0^{\mathrm{P}} \mathrm{e}^{-st}\cdot \mathbf{1}_{[0,\,U\mathrm{P})}(t)\,\mathrm{d}t \;\overset{\text{since } U\leqslant 1}{=\!=\!=}\; \int_0^{U\mathrm{P}} \mathrm{e}^{-st}\,\mathrm{d}t$$
$$= \frac{1}{s}\cdot\left[1 - \mathrm{e}^{-s\cdot U\mathrm{P}}\right].$$

Consequently,

$$\dagger 1\colon \qquad \widehat{\mathbb{SW}_{\mathrm{P},U}}(s) \;=\; \frac{1}{s}\cdot M_{\mathrm{P},U}(s)\,,$$

where this multiplier fnc $M_{\mathrm{P},U}$ is

$$\dagger 2\colon \quad M_{\mathrm{P},U}(s) \;=\; \frac{1-\mathrm{e}^{-U\mathrm{P}\cdot s}}{1-\mathrm{e}^{-\mathrm{P}\cdot s}} \;\overset{\text{note}}{=\!=}\; \frac{\mathrm{e}^{\mathrm{P}\cdot s} - \mathrm{e}^{F\mathrm{P}\cdot s}}{\mathrm{e}^{\mathrm{P}\cdot s} - 1}\,;$$

here, $F := 1-U$ represents the "off" part of the cycle. Happily, $(\dagger 1, \dagger 2)$ is consistent with $(*)$.

Square-Wave into Spring. Laplace transforming $\boxed{y''+y = \mathbb{SW}_{\mathrm{P},U}}$ yields

$$¥\colon \qquad [s^2+1]\cdot\widehat{y}(s) \;=\; \widehat{\mathbb{SW}_{\mathrm{P},U}}(s)\,.$$

As before, then,

$$8\colon \qquad y \;=\; \sin \circledast \mathbb{SW}_{\mathrm{P},U}\,.$$

Alternatively, write (¥) as

$$[s^2+1]\cdot\widehat{y}(s) \;=\; \tfrac{1}{s}\cdot M_{\mathrm{P},U}(s)\,. \quad \textit{So}$$
$$\widehat{y}(s) \;=\; \frac{1}{s^2+1}\cdot\frac{1}{s}\cdot M_{\mathrm{P},U}(s)\,. \quad \textit{Thus}$$

$$8'\colon \qquad y \;=\; \sin \circledast \mathbf{1} \circledast \mathcal{L}^{-1}(M_{\mathrm{P},U})$$
$$= [1-\cos] \circledast \mathcal{L}^{-1}(M_{\mathrm{P},U})\,.$$