The Laplace Transform: Calculus

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ABSTRACT: This first gives a defn of *exponential order* which is better adapted to convolution. Following, is a discussion of the "tapping on a bell" problem; one text called this "soldiers marching on a bridge". Both interpretations need a grain-of-salt...

Prelims. On a (possibly infinite) interval $J \subset \mathbb{R}$, a function $f: J \to \mathbb{C}$ is ***locally-integrable*** if, for each bounded subinterval $[a, b] \subset J$, integral $\int_a^b f$ exists and is finite. A *sufficient* (but not necessary) condition is that on each $[a, b]$, our f is bounded with only finitely-many discontinuities.

For a fnc $f: [0, \infty) \to \mathbb{C}$ and complex number s we define the "Laplace transform of f", evaluated at s,

$$1: \qquad \widehat{f}(s) = [\mathcal{L}(f)](s) := \int_0^\infty \mathrm{e}^{-st} \cdot f(t) \cdot \mathrm{d}t ,$$

for those values of s where this integral exists.

For complex numbers α and β, let $\alpha \succ \beta$ mean $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$. For a real $\boldsymbol{\mu}$, say that "f has ***exponential order*** $\boldsymbol{\mu}$", written $\boxed{f \in \mathrm{Ord}(\boldsymbol{\mu})}$, if $f: [0, \infty) \to \mathbb{C}$ and f is locally-integrable *and*

$$\dagger: \qquad \forall Q > \boldsymbol{\mu}: \quad \lim_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} = 0 .$$

One can replace (†) by the seemingly weaker

$$\ddagger: \qquad \forall Q > \boldsymbol{\mu}: \quad \limsup_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} < \infty ,$$

but they are [exercise] equivalent.

2: Lemma. *Consider an $f \in \mathrm{Ord}(\boldsymbol{\mu})$. Then $\widehat{f}(s)$ exists for each s with $\mathrm{Re}(s) > \boldsymbol{\mu}$. Indeed the integrand in (1) is absolutely integrable.* ♢

Proof. Fix s with $x := \mathrm{Re}(s) > \boldsymbol{\mu}$, then pick Q with $x > Q > \boldsymbol{\mu}$. Our t is real, so $|\mathrm{e}^{-st}| = \mathrm{e}^{-xt}$. Hence the integrand in (1) is eventually bounded,

$$\left| \mathrm{e}^{-st} \cdot f(t) \right| < \mathrm{e}^{-xt} \cdot \mathrm{e}^{Qt} \stackrel{\text{note}}{=\!=} \mathrm{e}^{-[x-Q]t} ,$$

once t is large enough. Since $x - Q$ is positive, this last is integrable over $t \in [0, \infty)$. ♦

We need the following standard tool, to prove a uniqueness result.

3a: Identically-zero lemma. *On a closed bounded interval $J := [b, c] \subset \mathbb{R}$, consider continuous functions $h, G: J \to \mathbb{C}$.*

*i: Suppose $h \geqslant 0$. If $\int_J h = 0$, then $h \equiv 0$, i.e, h is **identically-zero**.* [Exer:]

ii: If G is real-valued and $\int_J [G \cdot G] = 0$, then $G \equiv 0$. [Exer: Set $h := G \cdot G$, etc.]

iii: Suppose

$$3b: \qquad \forall n \in \mathbb{N}: \int_J x^n \cdot G(x) \, \mathrm{d}x = 0 .$$

Then G is identically-zero on J. ♢

Proof of (iii). Splitting G into real and imaginary parts, WLOG G is real-valued. Stmt (3b) implies, for each polynomial p, that $\int_J [p \cdot G]$ is zero.

By the Weierstrass Approximation Thm, there is a sequence of (real) polynomials, so that $p_k \xrightarrow{k \to \infty} G$ *uniformly*. Consequently,

$$\int_J [G \cdot G] \stackrel{\text{Exer.}}{=\!=} \lim_{k \to \infty} \int_J [p_k \cdot G] = \lim_{k \to \infty} 0 = 0 . \; ♦$$

3c: Uniqueness Thm. *Consider fncs $g, \gamma \in \mathrm{Ord}(\boldsymbol{\mu})$. Suppose their Laplace transforms, $\widehat{g}$ and $\widehat{\gamma}$, agree on some real interval $\mathbf{I} := [s_0, \infty)$. If g and γ are continuous, then $g = \gamma$.* ♢

[Proof is in notes, commented out.]

Convolution. Recall that the (one-sided) ***convolution*** of two (locally-integrable) functions $f, g\colon [0,\infty) \to \mathbb{C}$, is the function

4a: $$[f \circledast g](t) := \int_0^t f(t-v)\cdot g(v)\,\mathrm{d}v\,.$$

4b: Lemma. *Suppose $f, g \in \mathrm{Ord}(\boldsymbol{\mu})$. Then the convolution $f \circledast g \in \mathrm{Ord}(\boldsymbol{\mu})$.* ♢

[Proof is in notes, commented out.]

4c: Lap-of-Convolve. $\Big[\mathcal{L}(f \circledast g)\Big](s) = \widehat{f}(s)\cdot\widehat{g}(s)$. ♢

Proof. Define $\mathbf{1}(\mathsf{true}) := 1$ and $\mathbf{1}(\mathsf{false}) := 0$. We can now write RhS(4*a*) as

$$\int_0^\infty \mathbf{1}(v \leqslant t)\cdot f(t-v)\cdot g(v)\,\mathrm{d}v\,.$$

Hence $\Big[\mathcal{L}(f \circledast g)\Big](s)$ equals

$$\int_0^\infty \mathsf{e}^{-st}\int_0^\infty \mathbf{1}(v \leqslant t)\cdot f(t-v)\cdot g(v)\,\mathrm{d}v\,\mathrm{d}t\,.$$

Under mild conditions[♡1] on f and g, we can reverse the integrals, giving that $\widehat{[f \circledast g]}(s)$ equals

$$\int_0^\infty g(v)\int_0^\infty \mathbf{1}(v \leqslant t)\cdot \mathsf{e}^{-st}\cdot f(t-v)\,\mathrm{d}t\,\mathrm{d}v\,.$$

The inner integral can be written as

¥: $$\int_v^\infty \mathsf{e}^{-st}\cdot f(t-v)\,\mathrm{d}t\,.$$

CoV $x = t-v$ traverses $v-v \nearrow t-v \nearrow \infty - v = \infty$, and $\mathrm{d}x = \mathrm{d}t$. So (¥) equals

$$\begin{aligned}\int_0^\infty \mathsf{e}^{-s[x+v]}\cdot f(x)\,\mathrm{d}x &= \mathsf{e}^{-sv}\int_0^\infty \mathsf{e}^{-sx}\cdot f(x)\,\mathrm{d}x \\ &= \mathsf{e}^{-sv}\cdot \widehat{f}(s)\,.\end{aligned}$$

Thus $\Big[\mathcal{L}(f \circledast g)\Big](s)$ equals

$$\int_0^\infty g(v)\cdot \mathsf{e}^{-sv}\cdot\widehat{f}(s)\,\mathrm{d}v \stackrel{\text{note}}{=\!=} \widehat{f}(s)\cdot\widehat{g}(s)\,. \quad ♦$$

[♡1] See Fubini–Tonelli theorem.

5a: Deriv-of-Lap. *Fix a natnum N. Taking the N^{th} derivative,*

$$\left[\frac{d}{ds}\right]^N \widehat{f}(s) \;=\; \mathcal{L}\Big([-t]^N \cdot f(t)\Big)(s) \;\stackrel{\text{note}}{=\!=}\; [-1]^N \cdot \mathcal{L}\Big(t^N \cdot f(t)\Big)(s)\,. \qquad \Diamond$$

5b: Integral-of-Lap. *For real s with $s > \boldsymbol{\mu}$,*

$$\int_s^\infty \widehat{f}(u)\,du \;=\; \mathcal{L}\left(\frac{f(t)}{t}\right)(s)\,. \qquad \Diamond$$

5c: Lap-of-Deriv. *Suppose f differentiable. Then*

$$\widehat{f'}(s) \;=\; \left[s \cdot \widehat{f}(s)\right] - f(0)\,.$$

For $N \in \mathbb{N}$, suppose that an N-times differentiable f, has $f, f', \dots, f^{(N-1)} \in \mathrm{Ord}(\boldsymbol{\mu})$. Then

$$\widehat{f^{(N)}}(s) \;=\; \left[s^N \cdot \widehat{f}(s)\right] - \sum_{\substack{j+k=N-1\\ j,k\in\mathbb{N}}} s^j \cdot \left[f^{(k)}(0)\right],$$

where the sum is over all ordered-pairs (j,k) of natnums. $\Diamond$

Pf of (5a), Deriv-of-Lap. With appropriate conditions on f, we can differentiate under the integral sign in (1), applying $\frac{d}{ds}$, to get the $N{=}1$ case of (5a). Now induct on N. ♦

Pf of (5b), Integral-of-Lap. For t positive, note

$$*: \quad \int_s^\infty e^{-ut}\,du \;=\; \frac{1}{t} \cdot e^{-ut}\Big|_{u=\infty}^{u=s} \;=\; \frac{1}{t} \cdot [e^{-st} - e^{-\infty}] \;=\; \frac{1}{t} \cdot e^{-st}\,.$$

Applying the definition, $\int_s^\infty \widehat{f}(u)\,du$ equals

$$\int_s^\infty \left[\int_0^\infty e^{-ut} \cdot f(t) \cdot dt\right] du \;=\; \int_0^\infty f(t) \int_s^\infty e^{-ut}\,du\,dt \;\stackrel{\text{by } (*)}{=\!=}\; \int_0^\infty f(t) \cdot \frac{1}{t} \cdot e^{-st}\,dt\,.$$

And this is the defn of $\mathcal{L}\left(\frac{f(t)}{t}\right)(s)$. ♦

Pf of (5c), Lap-of-Deriv. Our $\widehat{f'}(s)$ is the limit, as $M \nearrow \infty$, of

$$\int_0^M \underbrace{e^{-st}}_{u} \cdot \underbrace{f'(t)\,dt}_{dv}\,.$$

Integrating by parts, this equals

$$*: \quad \underbrace{e^{-st}}_{u} \cdot \underbrace{f(t)}_{v}\Big|_{t=0}^{t=M} - \int_0^M \underbrace{f(t)}_{v} \cdot \underbrace{[-s]e^{-st}\,dt}_{du} \;=\; \left[s \cdot \int_0^M f(t) \cdot e^{-st}\,dt\right] - \frac{f(t)}{e^{st}}\Big|_{t=M}^{t=0}\,.$$

This last term equals $\frac{f(0)}{1} - \frac{f(M)}{e^{sM}} = f(0) - \frac{f(M)}{e^{sM}}$. If $\mathrm{Re}(s) > \boldsymbol{\mu}$, then $\frac{f(M)}{e^{sM}} \to 0$. as $M \nearrow \infty$. Sending $M \nearrow \infty$ thus sends $(*)$ to $\left[s \cdot \widehat{f}(s)\right] - f(0)$, as desired.

Finally, the formula for $\widehat{f^{(N)}}$ follows by induction on N. ♦

6: Lap-of-Exp-prod. *Fix $B \in \mathbb{C}$. Then*

$$\mathcal{L}\Big(e^{Bt} \cdot f(t)\Big)(s) \;=\; \widehat{f}(s - B)\,,$$

whenever $s \succ B + \boldsymbol{\mu}$. $\Diamond$

Proof. By defn, $\mathcal{L}\Big(e^{Bt} \cdot f(t)\Big)(s)$ equals

$$\int_0^\infty e^{-st} e^{Bt} f(t)\,dt \;=\; \int_0^\infty e^{-[s-B]t} f(t)\,dt\,.$$

This last integral converges once $\mathrm{Re}(s - B) > \boldsymbol{\mu}$, i.e, once $s \succ B + \boldsymbol{\mu}$. ♦

Examples

Below, $B \in \mathbb{C}$ and $F \in \mathbb{R}$. Easily $[\mathcal{L}(1)](s)$ equals $1/s$. From Deriv-of-Lap, then, $[\text{-}1]^N \cdot \mathcal{L}\Big(t^N\Big)(s)$ equals $[\frac{\mathrm{d}}{\mathrm{d}s}]^N\Big(\widehat{1}(s)\Big) = [\frac{\mathrm{d}}{\mathrm{d}s}]^N\Big(\frac{1}{s}\Big)$. Thus

7a: $$\mathcal{L}\Big(t^N\Big)(s) \;=\; \frac{N!}{s^{N+1}}\,.$$

From Lap-of-Exp-prod, note,

$*$: $$[\mathcal{L}(\mathsf{e}^{Bt})](s) \;=\; \frac{1}{s-B}\,.$$

Recall $\cos(Ft) = \frac{1}{2}[\mathsf{e}^{\boldsymbol{i}Ft} + \mathsf{e}^{\text{-}\boldsymbol{i}Ft}]$. So $(*)$ gives

$$2 \cdot \mathcal{L}\Big(\cos(Ft)\Big)(s) \;=\; \tfrac{1}{s-\boldsymbol{i}F} + \tfrac{1}{s+\boldsymbol{i}F} \;=\; \frac{2s}{s^2+F^2}\,.$$

Dividing both sides by 2,

7b: $$\mathcal{L}\Big(\cos(Ft)\Big)(s) \;=\; \frac{s}{s^2+F^2}\,, \qquad \textit{and similarly}$$
$$\mathcal{L}\Big(\sin Ft\Big)(s) \;=\; \frac{F}{s^2+F^2}\,.$$

As sin() is bounded, it has a Laplace transform on $\mathbb{R}_+$. Since $\widehat{\sin}(u) = \frac{1}{u^2+1} = \arctan'(u)$, it follows that

$$\int_0^\infty \widehat{\sin}\,(u)\,\mathrm{d}u \;=\; \arctan(\infty) - \arctan(0) \;=\; \frac{\boldsymbol{\pi}}{2} - 0\,.$$

Skipping the effort it takes to *justify* applying (5b) at $s{=}0$, we get that

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \;=\mathcal{L}\Big(\frac{\sin(t)}{t}\Big)(0) \;\underset{\text{by (5b)}}{=\!=}\; \int_0^\infty \widehat{\sin}\,.$$

This, together with the previous line, gives

7c: $$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \quad=\quad \frac{\boldsymbol{\pi}}{2}\,,$$

in an appropriate sense.

Preliminaries

The ***Heaviside fnc*** $\mathbf{H}:\mathbb{R}\to\{0,1\}$ is 0 on $(-\infty,0)$ and is 1 on $[0,+\infty)$. Thus[♡2]

$$\sum_{K=1}^{\infty}\mathbf{H}(x-K) \;=\; \lfloor x\rfloor\,, \qquad \text{for each } x\geqslant 0.$$

Let $\boldsymbol{\delta}$ denote the ***Dirac delta "function"***.[♡3] Write $\boldsymbol{\delta}_5$ for its translate $x\mapsto\boldsymbol{\delta}(x-5)$. So for P a posreal, and f *continuous* at P:

$$\int_0^\infty f\cdot\boldsymbol{\delta}_{\mathrm{P}} \;=\; f(\mathrm{P})\,.$$

In particular, for each f which is continuous:

8a:
$$[f\circledast\boldsymbol{\delta}_5](t) \;=\; \begin{cases}0 & ,\text{if } t\in[0,5)\\ f(t-5) & ,\text{if } t\in[5,\infty)\end{cases}\,.$$
I.e. $\quad f\circledast\boldsymbol{\delta}_5 \;=\; \mathbf{T}_5\big(\mathbf{H}\cdot f\big)$.

Periodicity. A posreal number P is a "***period*** of fnc f" if

$$\forall x:\quad f(x+\mathrm{P}) \;=\; f(x)\,.$$

Typically, there is a *smallest* such period, which is called the "***least-period*** of f".

For a posreal P, let $f_{\langle\mathrm{P}\rangle} := f\cdot\mathbf{1}_{[0,\mathrm{P})}$ abbreviate what I will call "f clipped at P". That is, we restrict f to interval $[0,\mathrm{P})$. Thus

$$\widehat{f_{\langle\mathrm{P}\rangle}}(s) \;=\; \int_0^{\mathrm{P}} \mathrm{e}^{-st}f(t)\,\mathrm{d}t\,.$$

8b: Periodicity Theorem. *Suppose $f,g\in\mathrm{Ord}(\boldsymbol{\mu})$ and P is a posreal. Then*

1: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s) = \mathrm{e}^{-s\mathrm{P}}\cdot\widehat{f}(s)$.

2: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})g(t)\Big)(s) = \mathrm{e}^{-s\mathrm{P}}\cdot\mathcal{L}\Big(g(t+\mathrm{P})\Big)(s)$.

3: Suppose now that P is a period of f. Then

$$\widehat{f}(s) \;=\; \widehat{f_{\langle\mathrm{P}\rangle}}(s)\Big/\big[1-\mathrm{e}^{-s\mathrm{P}}\big]\,. \qquad\diamondsuit$$

Proof of (8b.1). The LhS equals $\int_{\mathrm{P}}^{\infty}\mathrm{e}^{-st}f(t-\mathrm{P})\,\mathrm{d}t$. CoV $x=t-\mathrm{P}$ gives $\mathrm{d}x=\mathrm{d}t$, and thus

$$\mathrm{LhS}(8b.1) = \int_0^\infty \mathrm{e}^{-s[x+\mathrm{P}]}f(x)\,\mathrm{d}x \stackrel{\text{note}}{=\!=} \mathrm{RhS}(8b.1)\,.\blacklozenge$$

Pf (8b.3). Because our "clipped" $f_{\langle\mathrm{P}\rangle}(t)$ equals $[\mathbf{H}(t)-\mathbf{H}(t-\mathrm{P})]\cdot f(t)$, it follows that

$$\begin{aligned} &\mathcal{L}(f_{\langle\mathrm{P}\rangle}) = \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t)\Big)\\ *:\quad &\underset{}{\stackrel{\text{P an } f\text{-period}}{=\!=\!=\!=}} \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)\,.\end{aligned}$$

Our (8b.1) says that $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s)$ equals $\mathrm{e}^{-s\mathrm{P}}\cdot\widehat{f}(s)$. And, always, $\mathcal{L}(\mathbf{H}\cdot f)=\mathcal{L}(f)$, since the Laplace integral is over $[0,\infty)$. So we can re-write $(*)$ as $\widehat{f_{\langle\mathrm{P}\rangle}}(s) = [1-\mathrm{e}^{-s\mathrm{P}}]\cdot\widehat{f}(s)$. $\blacklozenge$

[♡2] For a real x, the expression $\lfloor x\rfloor$ is called the ***floor*** of x; it is the largest integer less-equal x. So $\lfloor\boldsymbol{\pi}\rfloor$ is 3.

Use DE to mean '*differential eqn*'.

[♡3] It is actually a *Schwartzian distribution*, named after Laurant Schwartz. As distributions, $\mathbf{H}'=\boldsymbol{\delta}$.

Using periodicity. Function sin() is periodic, with period $2\boldsymbol{\pi}$. So for each posint K:

$$\Big[\sin \circledast \boldsymbol{\delta}_{2\pi K}\Big](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t \in [0, 2\boldsymbol{\pi}K) \\ \sin(t) & \text{, if } t \in [2\boldsymbol{\pi}K, \infty) \end{matrix}\right\}.$$

For $N \in \{1,2,3,\ldots\} \cup \{\infty\}$, define the sum

$$R_N \;:=\; \sum_{K=1}^{N} \boldsymbol{\delta}_{2\pi K}\,.$$

For N finite, then,

$$\Big[\sin \circledast R_N\Big](t) \;=\; \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t)$$

holds when $0 \leqslant t < 2\boldsymbol{\pi}[N{+}1]$. There are no convergence problems as $N \nearrow \infty$. So $\forall t \in [0 \,..\, \infty)$:

$$\text{8c:} \qquad \Big[\sin \circledast R_\infty\Big](t) \;=\; \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t)\,.$$

Moreover, for each finite N:

$$\text{8c}_N\text{:} \quad \Big[\sin \circledast R_N\Big](t) \;=\; \operatorname{Min}\Big(N\,,\, \lfloor \tfrac{t}{2\pi} \rfloor\Big) \cdot \sin(t).$$

Hammering an undamped spring

(One of the textbooks called this *Soldiers marching in cadence*, but the interpretation is less clear, since soldiers already on the bridge are still marching.) With y the unknown-fnc, let us examine DE

$$\mathfrak{D}\text{:} \qquad y'' + y \;=\; \sum_{K=1}^{\infty} \boldsymbol{\delta}_{2\pi K}\,.$$

Given two complex numbers α, β, let $(\mathfrak{D}_{\alpha,\beta})$ mean the DE together with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$.

Consider the corresponding ZeroTar[4] IVP

$$z'' + z \;=\; 0\,, \qquad \textit{with } z(0) = \alpha \textit{ and } z'(0) = \beta\,.$$

Its solution is

$$\text{8d:} \qquad z \;=\; [\alpha \cdot \cos] + [\beta \cdot \sin]\,.$$

So the soln, y, to $(\mathfrak{D}_{0,0})$ will, when added to z, give the solution to $(\mathfrak{D}_{\alpha,\beta})$.

Computing,

$$\begin{aligned} \mathcal{L}\Big(y'' + y\Big)(s) &= [s^2 + 1] \cdot \widehat{y}(s) - [s \cdot y(0) + y'(0)] \\ &= [s^2 + 1] \cdot \widehat{y}(s)\,. \end{aligned}$$

Now $(\mathfrak{D})$ says that $\mathcal{L}\Big(y'' + y\Big) = \mathcal{L}(R_\infty)$. Dividing,

$$\widehat{y}(s) \;=\; \tfrac{1}{s^2+1} \cdot \widehat{R_\infty}(s)\,.$$

The RhS is a product, so its inv-lap-xform is a convolution.

$$y \;=\; \sin \circledast R_\infty\,.$$

In other words, the function

$$\text{8e:} \quad y_{\alpha,\beta}(t) \;:=\; \alpha\cos(t) + \beta\sin(t) + \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t)$$

is the general solution to IVP $(\mathfrak{D}_{\alpha,\beta})$.

At this point, it would be good to have a careful sketch of $y_{\alpha,\beta}()$. One could also analyze, at each time t, the potential and kinetic energy in the spring. If the spring is damped, is there a net absorption, or loss, of energy?

Suppose the tapping on the spring was not aligned with the resonant frequency of the spring —would the spring nonetheless absorb energy?

Random hammering. Times $0{<}G_1{<}G_2{<}\ldots$ define

$$R_{\mathbf{G}} \;:=\; \sum_{K=1}^{\infty} \boldsymbol{\delta}_{G_K}\,.$$

The above reasoning applied to DE $\boxed{y'' + y = R_{\mathbf{G}}}$ yields $y = \sin \circledast R_{\mathbf{G}}$, as before. Computing the convolution gives

$$\text{8f:} \qquad y(t) \;=\; \sum_{K=1}^{\infty} \mathbf{H}(t - G_K) \cdot \sin(t - G_K)\,.$$

Adding $[\alpha\cos(t) + \beta\sin(t)]$ hands us the general solution $y_{\alpha,\beta}$.

[4] Some textbooks call this (yuck!) "homogeneous".

Square Wave

For a period P>0 and a ***dUty cycle*** $U \in [0,1]$, define the ***square-wave fnc*** $\mathbb{SW}_{P,U}()$ to be *the* P-periodic fnc mapping $\mathbb{R} \to \mathbb{R}$, which agrees with $\mathbf{1}_{[0,UP)}$ on interval $[0,P)$; the ***pulse width*** is UP.

So $\mathbb{SW}_{P,0} \equiv 0$ and $\mathbb{SW}_{P,1} \equiv 1$. Thus we expect

$$*: \qquad \widehat{\mathbb{SW}_{P,0}} \equiv 0 \quad \textit{and} \quad \widehat{\mathbb{SW}_{P,1}}(s) = 1/s\,.$$

Setting $f := \mathbb{SW}_{P,U}$, our Periodicity Theorem says that $\widehat{f_{\langle P\rangle}}(s)$ equals

$$\int_0^P e^{-st} \cdot \mathbf{1}_{[0,UP)}(t)\,dt \;\overset{\text{since } U \leqslant 1}{=\!=\!=}\; \int_0^{UP} e^{-st}\,dt = \frac{1}{s} \cdot [1 - e^{-s \cdot UP}]\,.$$

Consequently,

$$\dagger1: \qquad \widehat{\mathbb{SW}_{P,U}}(s) = \frac{1}{s} \cdot M_{P,U}(s)\,,$$

where this multiplier fnc $M_{P,U}$ is

$$\dagger2: \quad M_{P,U}(s) = \frac{1 - e^{-UP \cdot s}}{1 - e^{-P \cdot s}} \overset{\text{note}}{=\!=} \frac{e^{P \cdot s} - e^{FP \cdot s}}{e^{P \cdot s} - 1}\,;$$

here, $F := 1 - U$ represents the "off" part of the cycle. Happily, $(\dagger1, \dagger2)$ is consistent with $(*)$.

Square-Wave into Spring. Laplace transforming $\boxed{y'' + y = \mathbb{SW}_{P,U}}$ yields

$$¥: \qquad [s^2+1] \cdot \widehat{y}(s) = \widehat{\mathbb{SW}_{P,U}}(s)\,.$$

As before, then,

$$9: \qquad y = \sin \circledast \mathbb{SW}_{P,U}\,.$$

Alternatively, write (¥) as

$$[s^2+1] \cdot \widehat{y}(s) = \tfrac{1}{s} \cdot M_{P,U}(s)\,. \quad \textit{So}$$

$$\widehat{y}(s) = \frac{1}{s^2+1} \cdot \frac{1}{s} \cdot M_{P,U}(s)\,. \quad \textit{Thus}$$

$$9': \qquad y = \sin \circledast \mathbf{1} \circledast \mathcal{L}^{-1}(M_{P,U}) = [1 - \cos] \circledast \mathcal{L}^{-1}(M_{P,U})\,.$$