The Laplace Transform: Calculus

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/

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Abstract: This first gives a defn of *exponential order* which is better adapted to convolution. Following, is a discussion of the "tapping on a bell" problem; one text called this "soldiers marching on a bridge". Both interpretations need a grain-of-salt...

Prelims. On a (possibly infinite) interval $J \subset \mathbb{R}$, a function $f: J \to \mathbb{C}$ is ***locally-integrable*** if, for each bounded subinterval $[a,b] \subset J$, integral $\int_a^b f$ exists and is finite. A *sufficient* (but not necessary) condition is that on each $[a,b]$, our f is bounded with only finitely-many discontinuities.

For a fnc $f: [0,\infty) \to \mathbb{C}$ and complex number s we define the **"**Laplace transform of f**"**, evaluated at s,

$$\text{1:} \qquad \widehat{f}(s) \;=\; [\mathcal{L}(f)](s) \;:=\; \int_0^\infty \mathsf{e}^{-st} \cdot f(t) \cdot \mathrm{d}t\,,$$

for those values of s where this integral exists.

For complex numbers α and β, let $\alpha \succ \beta$ mean $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$. For a real $\boldsymbol{\mu}$, say that **"**f has ***exponential order*** $\boldsymbol{\mu}$**"**, written $\boxed{f \in \mathrm{Ord}(\boldsymbol{\mu})}$, if $f: [0,\infty) \to \mathbb{C}$ and f is locally-integrable *and*

$$\dagger: \qquad \forall Q > \boldsymbol{\mu}: \quad \lim_{t\to\infty} |f(t)| / \mathsf{e}^{Q \cdot t} \;=\; 0\,.$$

One can replace (†) by the seemingly weaker

$$\ddagger: \qquad \forall Q > \boldsymbol{\mu}: \quad \limsup_{t\to\infty} |f(t)| / \mathsf{e}^{Q \cdot t} \;<\; \infty\,,$$

but they are [exercise] equivalent.

2: Lemma. *Consider an $f \in \mathrm{Ord}(\boldsymbol{\mu})$. Then $\widehat{f}(s)$ exists for each s with $\mathrm{Re}(s) > \boldsymbol{\mu}$. Indeed the integrand in (1) is absolutely integrable.* ♢

Proof. Fix s with $x := \mathrm{Re}(s) > \boldsymbol{\mu}$, then pick Q with $x > Q > \boldsymbol{\mu}$. Our t is real, so $|\mathsf{e}^{-st}| = \mathsf{e}^{-xt}$. Hence the integrand in (1) is eventually bounded,

$$\left|\mathsf{e}^{-st} \cdot f(t)\right| \;<\; \mathsf{e}^{-xt} \cdot \mathsf{e}^{Qt} \;\stackrel{\text{note}}{=\!=}\; \mathsf{e}^{-[x-Q]t}\,,$$

once t is large enough. Since $x - Q$ is positive, this last is integrable over $t \in [0,\infty)$. ♦

Convolution. Recall that the (one-sided) ***convolution*** of two (locally-integrable) functions $f, g: [0,\infty) \to \mathbb{C}$, is the function

$$\text{3a:} \qquad [f \circledast g](t) \;:=\; \int_0^t f(t-v) \cdot g(v)\,\mathrm{d}v\,.$$

3b: Lemma. *Suppose $f, g \in \mathrm{Ord}(\boldsymbol{\mu})$. Then the convolution $f \circledast g \in \mathrm{Ord}(\boldsymbol{\mu})$.* ♢

[Proof omitted.]

We need the following standard tool, to prove a uniqueness result.

4a: Identically-zero lemma. *On a closed bounded interval $J := [b,c] \subset \mathbb{R}$, consider continuous functions $h, G: J \to \mathbb{C}$.*

*i: Suppose $h \geqslant 0$. If $\int_J h = 0$, then $h \equiv 0$, i.e, h is **identically-zero**.* [Exercise:]

ii: If G is real-valued and $\int_J [G \cdot G] = 0$, then $G \equiv 0$. [Exercise:]

iii: Suppose

$$\text{4b:} \qquad \forall n \in \mathbb{N}: \; \int_J x^n \cdot G(x)\,\mathrm{d}x \;=\; 0\,.$$

Then G is identically-zero on J. ♢

Proof of (iii). Splitting G into real and imaginary parts, WLOG G is real-valued. Stmt (4b) implies, for each polynomial p, that $\int_J [p \cdot G]$ is zero.

By the Weierstrass Approximation Thm, there is a sequence of (real) polynomials, so that $p_k \xrightarrow{k\to\infty} G$ *uniformly*. Consequently,

$$\int_J [G \cdot G] \;\stackrel{\text{Exer.}}{=\!=}\; \lim_{k\to\infty} \int_J [p_k \cdot G] \;=\; \lim_{k\to\infty} 0 \;=\; 0\,. \;♦$$

4c: Uniqueness Thm. *Consider fncs $g, \gamma \in \mathrm{Ord}(\boldsymbol{\mu})$. Suppose their Laplace transforms, $\widehat{g}$ and $\widehat{\gamma}$, agree on some real interval $\mathbf{I} := [s_0, \infty)$. If g and γ are continuous, then $g = \gamma$.* ♢

Proof. The difference fnc $f := g - \gamma$ is cts and in $\operatorname{Ord}(\boldsymbol{\mu})$, with $\widehat{f}$ identically-zero on $\mathbf{I}$. So our goal is to show f identically-zero on $[0, \infty)$.

WLOG $s_0 > \boldsymbol{\mu}$. Define $G(u) := u^{s_0} \cdot f(\text{-}\log(u))$, and $G(0) := 0$. CoV $t = \text{-}\log(u)$ shows that

$$\lim_{u \searrow 0} G(u) \;=\; \lim_{t \nearrow \infty} \mathsf{e}^{\text{-}s_0 \cdot t} f(t)$$

equals 0, since $s_0 > \boldsymbol{\mu}$. Hence $\boxed{G \text{ is cts on } [0,1]}$.

CoV $t = \text{-}\log(u)$ makes $\mathrm{d}t = \text{-}\,\mathrm{d}u/u$ and

$$\mathsf{e}^{\text{-}st} \;=\; [\mathsf{e}^{\text{-}t}]^s \;=\; u^s .$$

Evaluating at $s{+}1$, the CoV rewrites (1) as

$$\begin{aligned} \widehat{f}(s{+}1) \;&=\; \int_1^0 u^{s+1} \cdot f(\text{-}\log(u)) \cdot \frac{\text{-}\,\mathrm{d}u}{u} \\ &=\; \int_0^1 u^s \cdot f(\text{-}\log(u))\,\mathrm{d}u \\ &=\; \int_0^1 u^{s-s_0} \cdot G(u)\,\mathrm{d}u . \end{aligned}$$

For $n = 0, 1, 2, \ldots$, let $s := s_0 + n$ and note $s{+}1$ is in $\mathbf{I}$. Hence $\widehat{f}(s{+}1)$ is zero, i.e $\int_0^1 u^n G(u)\,\mathrm{d}u$ is zero. By (4a), then, $G \equiv 0$ on $[0,1]$.

So on half-open $(0,1]$, map $[u \mapsto f(\text{-}\log(u))]$ is zero. Hence $f \equiv 0$ on $[0, \infty)$. ♦

5a: TL-Diff. *Fix a natnum N. Taking the N^{th} derivative,*

$$\begin{aligned} \left[\frac{\mathrm{d}}{\mathrm{d}s}\right]^N \widehat{f}(s) \;&=\; \mathcal{L}\Big([\text{-}t]^N \cdot f(t)\Big)(s) \\ &\underset{\text{note}}{=}\; [\text{-}1]^N \cdot \mathcal{L}\Big(t^N \cdot f(t)\Big)(s) . \end{aligned}$$ ♢

5b: LT-Diff. *Suppose f differentiable. Then*

$$\widehat{f'}(s) \;=\; \Big[s \cdot \widehat{f}(s)\Big] - f(0) .$$

For $N \in \mathbb{N}$, suppose that an N-times differentiable f, has $f, f', \ldots, f^{(N-1)} \in \operatorname{Ord}(\boldsymbol{\mu})$. Then

$$\widehat{f^{(N)}}(s) \;=\; \Big[s^N \cdot \widehat{f}(s)\Big] - \sum_{\substack{j+k=N-1 \\ j,k \in \mathbb{N}}} s^j \cdot \Big[f^{(k)}(0)\Big] ,$$

where the sum is over all ordered-pairs (j,k) of natnums. ♢

5c: LT-Exp. *Fix $B \in \mathbb{C}$. Then*

$$\mathcal{L}\Big(\mathsf{e}^{Bt} \cdot f(t)\Big)(s) \;=\; \widehat{f}(s - B) ,$$

whenever $s \succ B + \boldsymbol{\mu}$. ♢

5d: LT-Convolve. $\Big[\mathcal{L}(f \circledast g)\Big](s) \;=\; \widehat{f}(s) \cdot \widehat{g}(s).$ ♢

5e: TL-Integrate. *For real s with $s > \boldsymbol{\mu}$,*

$$\int_s^\infty \widehat{f}(u)\,\mathrm{d}u \;=\; \mathcal{L}\Big(\frac{f(t)}{t}\Big)(s)$$ ♢

Proof of (5e). For t positive, note

$$*: \quad \begin{aligned} \int_s^\infty \mathsf{e}^{-ut}\,\mathrm{d}u \;&=\; \tfrac{1}{t} \cdot \mathsf{e}^{-ut}\Big|_{u=\infty}^{u=s} \\ &=\; \tfrac{1}{t} \cdot [\mathsf{e}^{-st} - \mathsf{e}^{-\infty}] \;=\; \tfrac{1}{t} \cdot \mathsf{e}^{-st} . \end{aligned}$$

Applying the definition, $\int_s^\infty \widehat{f}(u)\,\mathrm{d}u$ equals

$$\begin{aligned} \int_s^\infty \Big[\int_0^\infty \mathsf{e}^{-ut} \cdot f(t) \cdot \mathrm{d}t\Big]\,\mathrm{d}u \;&=\; \int_0^\infty f(t) \int_s^\infty \mathsf{e}^{-ut}\,\mathrm{d}u\,\mathrm{d}t \\ &\underset{\text{by } (*)}{=}\; \int_0^\infty f(t) \cdot \tfrac{1}{t} \cdot \mathsf{e}^{-st}\,\mathrm{d}t . \end{aligned}$$

And this is the defn of $\mathcal{L}\Big(\frac{f(t)}{t}\Big)(s)$. ♦

Pf of (5a), *TL-Diff.* With appropriate conditions on f, we can differentiate under the integral sign in (1), applying $\frac{\mathrm{d}}{\mathrm{d}s}$, to get the $N{=}1$ case of (5a). Now induct on N. ♦

Pf LT-Diff. Our $\widehat{f'}(s)$ is the limit, as $M \nearrow \infty$, of

$$\int_0^M \underbrace{\mathsf{e}^{\text{-}st}}_{u} \cdot \underbrace{f'(t)\,\mathrm{d}t}_{\mathrm{d}v} .$$

Integrating by parts, this equals

$$*: \quad \begin{aligned} &\underbrace{\mathsf{e}^{\text{-}st}}_{u} \cdot \underbrace{f(t)}_{v}\Big|_{t=0}^{t=M} - \int_0^M \underbrace{f(t)}_{v} \cdot \underbrace{[\text{-}s]\mathsf{e}^{\text{-}st}\,\mathrm{d}t}_{\mathrm{d}u} \\ &= \Big[s \cdot \int_0^M f(t) \cdot \mathsf{e}^{\text{-}st}\,\mathrm{d}t\Big] - \frac{f(t)}{\mathsf{e}^{st}}\Big|_{t=M}^{t=0} . \end{aligned}$$

This last term equals $\frac{f(0)}{1} - \frac{f(M)}{\mathrm{e}^{sM}} = f(0) - \frac{f(M)}{\mathrm{e}^{sM}}$. If $\mathrm{Re}(s) > \boldsymbol{\mu}$, then $\frac{f(M)}{\mathrm{e}^{sM}} \to 0$. as $M \nearrow \infty$. Sending $M \nearrow \infty$ thus sends $(*)$ to $\left[s \cdot \widehat{f}(s)\right] - f(0)$, as desired.

Finally, the formula for $\widehat{f^{(N)}}$ follows by induction on N. ♦

Pf LT-Exp. By defn, $\mathcal{L}\left(\mathrm{e}^{Bt} \cdot f(t)\right)(s)$ equals

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{Bt} f(t)\,\mathrm{d}t \;=\; \int_0^\infty \mathrm{e}^{-[s-B]t} f(t)\,\mathrm{d}t\,.$$

This last integral converges once $\mathrm{Re}(s-B) > \boldsymbol{\mu}$, i.e, once $s \succ B + \boldsymbol{\mu}$. ♦

Proof of LT-Convolve. Define $\mathbf{1}(\mathsf{true}) := 1$ and $\mathbf{1}(\mathsf{false}) := 0$. We can now write RhS(3*a*) as

$$\int_0^\infty \mathbf{1}(v \leqslant t) \cdot f(t-v) \cdot g(v)\,\mathrm{d}v\,.$$

Hence $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty \mathrm{e}^{-st} \int_0^\infty \mathbf{1}(v \leqslant t) \cdot f(t-v) \cdot g(v)\,\mathrm{d}v\,\mathrm{d}t\,.$$

Under mild conditions on f and g, we can reverse the integrals, giving that $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty g(v) \int_0^\infty \mathbf{1}(v \leqslant t) \cdot \mathrm{e}^{-st} \cdot f(t-v)\,\mathrm{d}t\,\mathrm{d}v\,.$$

The inner integral can be written as

$$\int_v^\infty \mathrm{e}^{-st} \cdot f(t-v)\,\mathrm{d}t\,.$$

The change of variable $x = t - v$ gives that $\mathrm{d}x = \mathrm{d}t$. So this integral becomes

$$\int_0^\infty \mathrm{e}^{-s[x+v]} \cdot f(x)\,\mathrm{d}x \;=\; \mathrm{e}^{-sv} \int_0^\infty \mathrm{e}^{-sx} \cdot f(x)\,\mathrm{d}x$$
$$=\; \mathrm{e}^{-sv} \cdot \widehat{f}(s)\,.$$

Thus $\left[\mathcal{L}(f \circledast g)\right](s)$ equals

$$\int_0^\infty g(v) \cdot \mathrm{e}^{-sv} \cdot \widehat{f}(s)\,\mathrm{d}v \;\stackrel{\text{note}}{=\!=}\; \widehat{f}(s) \cdot \widehat{g}(s)\,.$$ ♦

Examples

Below, $B \in \mathbb{C}$ and $F \in \mathbb{R}$. Easily $[\mathcal{L}(1)](s)$ equals $1/s$. From TL-Diff, then, $[-1]^N \cdot \mathcal{L}\left(t^N\right)(s)$ equals $[\frac{\mathrm{d}}{\mathrm{d}s}]^N\left(\widehat{1}(s)\right) = [\frac{\mathrm{d}}{\mathrm{d}s}]^N\left(\frac{1}{s}\right)$. Thus

6a: $$\mathcal{L}\left(t^N\right)(s) \;=\; \frac{N!}{s^{N+1}}\,.$$

From LT-Exp, note,

$*$: $$[\mathcal{L}(\mathrm{e}^{Bt})](s) \;=\; \frac{1}{s-B}\,.$$

Recall $\cos(Ft) = \frac{1}{2}[\mathrm{e}^{\boldsymbol{i}Ft} + \mathrm{e}^{-\boldsymbol{i}Ft}]$. So $(*)$ gives

$$2 \cdot \mathcal{L}\left(\cos(Ft)\right)(s) \;=\; \tfrac{1}{s-\boldsymbol{i}F} + \tfrac{1}{s+\boldsymbol{i}F} \;=\; \frac{2s}{s^2+F^2}\,.$$

Dividing both sides by 2,

6b: $$\mathcal{L}\left(\cos(Ft)\right)(s) \;=\; \frac{s}{s^2+F^2}\,, \quad \text{and similarly}$$
$$\mathcal{L}\left(\sin Ft\right)(s) \;=\; \frac{F}{s^2+F^2}\,.$$

As $\sin()$ is bounded, it has a Laplace transform on $\mathbb{R}_+$. Since $\widehat{\sin}(u) = \frac{1}{u^2+1} = \arctan'(u)$, it follows that

$$\int_0^\infty \widehat{\sin}(u)\,\mathrm{d}u \;=\; \arctan(\infty) - \arctan(0) \;=\; \frac{\boldsymbol{\pi}}{2} - 0\,.$$

Skipping the effort it takes to *justify* applying (5e) at $s{=}0$, we get that

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \;= \mathcal{L}\left(\frac{\sin(t)}{t}\right)(0) \;\stackrel{\text{by (5e)}}{=\!=}\; \int_0^\infty \widehat{\sin}\,.$$

This, together with the previous line, gives

6c: $$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \;=\; \frac{\boldsymbol{\pi}}{2}\,.$$

Preliminaries

The ***Heaviside fnc*** $\mathbf{H}:\mathbb{R}\to\{0,1\}$ is 0 on $(-\infty,0)$ and is 1 on $[0,+\infty)$. Thus[♡1]

$$\sum_{K=1}^{\infty} \mathbf{H}(x-K) \;=\; \lfloor x \rfloor\,, \qquad \text{for each } x \geqslant 0.$$

Let $\boldsymbol{\delta}$ denote the ***Dirac delta "function"***.[♡2] Write $\boldsymbol{\delta}_5$ for its translate $x \mapsto \boldsymbol{\delta}(x-5)$. So for P a posreal, and f *continuous* at P:

$$\int_0^\infty f \cdot \boldsymbol{\delta}_{\mathrm{P}} \;=\; f(\mathrm{P})\,.$$

In particular, for each f which is continuous:

7a: $$[f \circledast \boldsymbol{\delta}_5](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t \in [0,5) \\ f(t-5) & \text{, if } t \in [5,\infty) \end{matrix}\right\}.$$

I.e. $$f \circledast \boldsymbol{\delta}_5 \;=\; \mathbf{T}_5\big(\mathbf{H}\cdot f\big)\,.$$

Periodicity. A posreal number P is a "***period*** of fnc f" if

$$\forall x: \quad f(x+\mathrm{P}) \;=\; f(x)\,.$$

Typically, there is a *smallest* such period, which is called the "***least-period*** of f".

For a posreal P, let $f_{\langle \mathrm{P}\rangle} := f\cdot \mathbf{1}_{[0,\mathrm{P})}$ abbreviate what I will call "f clipped at P". That is, we restrict f to interval $[0,\mathrm{P})$. Thus

$$\widehat{f_{\langle \mathrm{P}\rangle}}(s) \;=\; \int_0^{\mathrm{P}} \mathsf{e}^{-st} f(t)\,\mathrm{d}t\,.$$

7b: Periodicity Theorem. *Suppose $f,g \in \mathrm{Ord}(\boldsymbol{\mu})$ and P is a posreal. Then*

1: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s) = \mathsf{e}^{-s\mathrm{P}}\cdot \widehat{f}(s)$.

2: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})g(t)\Big)(s) = \mathsf{e}^{-s\mathrm{P}}\cdot \mathcal{L}\Big(g(t+\mathrm{P})\Big)(s)$.

3: Suppose now that P is a period of f. Then

$$\widehat{f}(s) \;=\; \widehat{f_{\langle \mathrm{P}\rangle}}(s)\Big/\big[1-\mathsf{e}^{-s\mathrm{P}}\big]\,. \qquad \diamondsuit$$

Proof of (7b.1). The LhS equals $\int_{\mathrm{P}}^\infty \mathsf{e}^{-st} f(t-\mathrm{P})\,\mathrm{d}t$. CoV $x = t-\mathrm{P}$ gives $\mathrm{d}x = \mathrm{d}t$, and thus

$$\mathrm{LhS}(7b.1) = \int_0^\infty \mathsf{e}^{-s\,[x+\mathrm{P}]} f(x)\,\mathrm{d}x \overset{\text{note}}{=\!=\!=} \mathrm{RhS}(7b.1)\,. \blacklozenge$$

Pf (7b.3). Because our "clipped" $f_{\langle \mathrm{P}\rangle}(t)$ equals $[\mathbf{H}(t)-\mathbf{H}(t-\mathrm{P})]\cdot f(t)$, it follows that

$$*: \quad \begin{aligned} \mathcal{L}(f_{\langle \mathrm{P}\rangle}) &= \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t)\Big) \\ &\overset{\text{P an } f\text{-period}}{=\!=\!=\!=\!=} \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)\,. \end{aligned}$$

Our (7b.1) says that $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s)$ equals $\mathsf{e}^{-s\mathrm{P}}\cdot \widehat{f}(s)$. And, always, $\mathcal{L}(\mathbf{H}\cdot f) = \mathcal{L}(f)$, since the Laplace integral is over $[0,\infty)$. So we can re-write $(*)$ as $\widehat{f_{\langle \mathrm{P}\rangle}}(s) = [1-\mathsf{e}^{-s\mathrm{P}}]\cdot \widehat{f}(s)$. $\blacklozenge$

Using periodicity. Function sin() is periodic, with period $2\boldsymbol{\pi}$. So for each posint K:

$$\Big[\sin \circledast \boldsymbol{\delta}_{2\boldsymbol{\pi}K}\Big](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t \in [0,2\boldsymbol{\pi}K) \\ \sin(t) & \text{, if } t \in [2\boldsymbol{\pi}K,\infty) \end{matrix}\right\}.$$

For $N \in \{1,2,3,\ldots\}\cup\{\infty\}$, define the sum

$$R_N \;:=\; \sum_{K=1}^{N} \boldsymbol{\delta}_{2\boldsymbol{\pi}K}\,.$$

For N finite, then,

$$\Big[\sin \circledast R_N\Big](t) \;=\; \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)$$

holds when $0 \leqslant t < 2\boldsymbol{\pi}[N+1]$. There are no convergence problems as $N \nearrow \infty$. So $\forall t \in [0\,..\,\infty)$:

7c: $$\Big[\sin \circledast R_\infty\Big](t) \;=\; \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)\,.$$

Moreover, for each finite N:

$7c_N$: $$\Big[\sin \circledast R_N\Big](t) \;=\; \mathrm{Min}\Big(N\,,\, \lfloor \tfrac{t}{2\boldsymbol{\pi}} \rfloor\Big)\cdot \sin(t).$$

[♡1] For a real x, the expression $\lfloor x \rfloor$ is called the ***floor*** of x; it is the largest integer less-equal x. So $\lfloor \boldsymbol{\pi} \rfloor$ is 3.

Use DE to mean '*differential eqn*'.

[♡2] It is actually a *Schwartzian distribution*, named after Laurant Schwartz. As distributions, $\mathbf{H}' = \boldsymbol{\delta}$.

Hammering an undamped spring

(One of the textbooks called this *Soldiers marching in cadence*, but the interpretation is less clear, since soldiers already on the bridge are still marching.) With y the unknown-fnc, let us examine DE

$$\mathfrak{D}: \qquad y'' + y = \sum_{K=1}^{\infty} \boldsymbol{\delta}_{2\pi K}\,.$$

Given two complex numbers α, β, let $(\mathfrak{D}_{\alpha,\beta})$ mean the DE together with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$.

Consider the corresponding ZeroTar[♡3] IVP

$$z'' + z = 0\,, \qquad \textit{with } z(0) = \alpha \textit{ and } z'(0) = \beta\,.$$

Its solution is

$$\text{7d:} \qquad z = [\alpha \cdot \cos] + [\beta \cdot \sin]\,.$$

So the soln, y, to $(\mathfrak{D}_{0,0})$ will, when added to z, give the solution to $(\mathfrak{D}_{\alpha,\beta})$.

Computing,

$$\begin{aligned}\mathcal{L}\Big(y'' + y\Big)(s) &= [s^2+1] \cdot \widehat{y}(s) - [s \cdot y(0) + y'(0)] \\ &= [s^2+1] \cdot \widehat{y}(s)\,.\end{aligned}$$

Now $(\mathfrak{D})$ says that $\mathcal{L}\Big(y''+y\Big) = \mathcal{L}(R_\infty)$. Dividing,

$$\widehat{y}(s) = \tfrac{1}{s^2+1} \cdot \widehat{R_\infty}(s)\,.$$

The RhS is a product, so its inv-lap-xform is a convolution.

$$y = \sin \circledast R_\infty\,.$$

In other words, the function

$$\text{7e:} \quad y_{\alpha,\beta}(t) := \alpha\cos(t) + \beta\sin(t) + \left\lfloor \frac{t}{2\boldsymbol{\pi}} \right\rfloor \cdot \sin(t)$$

is the solution to IVP $(\mathfrak{D}_{\alpha,\beta})$.

At this point, it would be good to have a careful sketch of $y_{\alpha,\beta}()$. One could also analyze, at each time t, the potential and kinetic energy in the spring. If the spring is damped, is there a net absorption, or loss, of energy?

Suppose the tapping on the spring was <u>not</u> aligned with the resonant frequency of the spring —would the spring nonetheless absorb energy?

[♡3] Some textbooks call this (yuck!) "homogeneous".

Square Wave

For a period $\mathrm{P}>0$ and a ***duty cycle*** $\alpha \in [0,1]$, define the ***square-wave fnc*** $\mathbb{SW}_{\mathrm{P},\alpha}()$ to be *the* P-periodic fnc mapping $\mathbb{R}\to\mathbb{R}$, which agrees with $\mathbf{1}_{[0,\,\alpha\mathrm{P})}$ on interval $[0, \mathrm{P})$.

So $\mathbb{SW}_{\mathrm{P},0} \equiv 0$ and $\mathbb{SW}_{\mathrm{P},1} \equiv 1$. Thus we expect

$$*: \qquad \widehat{\mathbb{SW}_{\mathrm{P},0}} \equiv 0 \quad \textit{and} \quad \widehat{\mathbb{SW}_{\mathrm{P},1}}(s) = 1/s\,.$$

Setting $f := \mathbb{SW}_{\mathrm{P},\alpha}$, our Periodicity Thm says that $f_{\langle\mathrm{P}\rangle}(t)$ equals

$$\begin{aligned}\int_0^{\mathrm{P}} \mathsf{e}^{-st} \cdot \mathbf{1}_{[0,\,\alpha\mathrm{P})}(t)\,\mathrm{d}t &\overset{\text{since } \alpha \leqslant 1}{=\!=} \int_0^{\alpha\mathrm{P}} \mathsf{e}^{-st}\,\mathrm{d}t \\ &= \frac{1}{s} \cdot [1 - \mathsf{e}^{-s\cdot\alpha\mathrm{P}}]\,.\end{aligned}$$

Consequently,

$$\dagger 1: \qquad \widehat{\mathbb{SW}_{\mathrm{P},\alpha}}(s) = \frac{1}{s} \cdot M_\alpha(s)\,,$$

where this multiplier fnc M_α is

$$\dagger 2: \qquad M_\alpha(s) = \frac{1 - \mathsf{e}^{-\alpha\mathrm{P}\cdot s}}{1 - \mathsf{e}^{-\mathrm{P}\cdot s}} \overset{\text{note}}{=\!=} \frac{\mathsf{e}^{\mathrm{P}\cdot s} - \mathsf{e}^{\beta\mathrm{P}\cdot s}}{\mathsf{e}^{\mathrm{P}\cdot s} - 1}\,;$$

here, $\beta := 1 - \alpha$ represents the "off" part of the cycle. Happily, (†1,†2) is consistent with $(*)$.

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