Lagrange Multipliers
Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
Webpage http://people.clas.ufl.edu/squash/
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Cubic-surface Problem
On \( \mathbb{R}^3 \), find all extreme points \( P = (x, y, z) \) of function
\[
\psi(P) := x^2 + y^2 + z^2,
\]
subject to the two constraints that
\[
C_f: \quad x \cdot y \cdot z = 54; \quad \text{(a cubic surface)}
\]
\[
C_g: \quad x + y + z = 12. \quad \text{(a plane)}
\]

A picture. Note that the objective function and the two constraints are symmetric under all permutations of \( x, y, z \). Thus our set of Lagrange points must have a six-fold (6 = 3!) symmetry.

A good picture suggests that the only Lagrange pts have some two of \( x, y, z \) equal. In the case that \( x = z \), say, a picture suggests that there are 3 solutions; two solns where the common value is positive and one soln with it negative. Thus the \( z = y, y = x \) and \( x = z \) solutions would give us 9 Lagrange pts all together, but only 3 types of Lagrange-pt.

A solution
Define specifier functions
\[
f(P) := x \cdot y \cdot z \quad \text{and} \quad g(P) := x + y + z,
\]
so that locus \((C_f)\) is some level-set of \( f \), and locus \((C_g)\) is some level-set of \( g \).

The Lagrange eqns. Using variables \( \alpha, \beta \) to be the Lagrange multipliers for \( f \) and \( g \), respectively, the Lagrange eqns become:
\[
L_x: \quad \psi_x = \alpha f_x + \beta g_x, \quad \text{i.e.,} \quad 2x = \alpha \cdot yz + \beta \cdot 1.
\]
\[
L_y: \quad \psi_y = \alpha f_y + \beta g_y, \quad \text{i.e.,} \quad 2y = \alpha \cdot xz + \beta \cdot 1.
\]
\[
L_z: \quad \psi_z = \alpha f_z + \beta g_z, \quad \text{i.e.,} \quad 2z = \alpha \cdot xy + \beta \cdot 1.
\]

We have 5 unknowns: The spatial variables \( x, y, z \) and the Lagr.vars \( \alpha, \beta \). We have 5 eqns: The constraints eqns \((C_f), (C_g)\), and the Lagr.eqns \((L_x), (L_y), (L_z)\).

Solving this system of eqns. If possible, we try to eliminate the Lagrange-vars first. We can subtract \((L_y)\) from \((L_z)\) to obtain
\[
2[z - y] = \alpha xy - \alpha zx
\]
\[
\text{note} \quad \alpha x y [y - z].
\]

We now use symmetry to solve the system.

Case \([x = y = z]\). Equation \((C_g)\) yields that \( x = y = z = 4 \). So their product, \( x \cdot y \cdot z \), is 64. This contradicts \((C_f)\). Thus: No solutions, here.

Case \([x, y, z \ \text{are distinct}]\). Then \( y - z \neq 0 \) so we may divide equation \((L_y) - (L_z)\) to conclude that \(-2 = \alpha x\). But \(-2\) is not zero, so \( \alpha \) must be non-zero and so \( x = \frac{-2}{\alpha} \).

By symmetry \([z - x \neq 0, \text{etc}\] ), we may conclude that \( y, \text{too, equals } \frac{-2}{\alpha} \). Alas, this contradicts that \( x \neq y \). So: No solutions in this case.

Case \([\text{Exactly two of } x, y, z \ \text{are equal}]\). Use \( t \) for the value which occurs twice among \(\{x, y, z\}\), and use \( u \) for the value which occurs uniquely. Eqn \((C_g)\) tells us then that \( t + t + u = 12 \), i.e., \( u = 12 - 2t \). From \((C_f)\), then,
\[
54 = t^2 u \quad \text{note} \quad t^2 [12 - 2t].
\]
Multiplying out, then,
\[
0 = q(t), \quad \text{where } q(t) := t^3 - 6t^2 + 27.
\]
Evidently \( q(3) \) is zero, so we divide \( t - 3 \) into \( t^3 - 6t^2 + 27 \) to obtain this factorization:
\[
q(t) = [t - 3][t^2 - 3t + 9].
\]
The QF (Quadratic Formula) tells us that the zeros of \( t^2 - 3t + 9 = \frac{3}{2}[1 \pm \sqrt{5}] \). And since \( u = 12 - 2t \), we obtain this table of Lagrange pairs \((t,u)\). In the table, the three values of \( t \) decrease from left to right.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \text{Soln 1} )</th>
<th>( \text{Soln 2} )</th>
<th>( \text{Soln 3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{2}[1 + \sqrt{5}] )</td>
<td>3</td>
<td>( \frac{3}{2}[1 - \sqrt{5}] )</td>
<td></td>
</tr>
<tr>
<td>( 3[3 - \sqrt{5}] )</td>
<td>6</td>
<td>( 3[3 + \sqrt{5}] )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2}\psi(t,u) )</td>
<td>17 - 5( \sqrt{5} )</td>
<td>6</td>
<td>17 + 5( \sqrt{5} )</td>
</tr>
<tr>
<td>Size</td>
<td>closest</td>
<td>med.</td>
<td>farthest</td>
</tr>
</tbody>
</table>

### The Method in General

Suppose we have \( K \) constraints in \( N \)-dim space; use \( Q = (x_1,\ldots,x_N) \) for a general point in \( N \)-space. We wish to find all constrained extrema of a function \( \psi: \mathbb{R}^N \to \mathbb{R} \).

#### Step I. Write down a formula for \( \psi \), the objective function, and label the constraint equations \((C_1),\ldots,(C_K)\).

#### Step II. Write down definitions \((=)\) of specifier functions \( f_1,\ldots,f_K \) so that each \((C_j)\) is a level-set of the corresponding \( f_j \).

Introduce unknowns \( \alpha_1,\ldots,\alpha_K \) to be the Lagrange multipliers of \( f_1,\ldots,f_K \).

#### Step III. For \( m = 1,2,\ldots,N \), write down the Lagrange equations

\[
L_m: \quad \frac{\partial \psi}{\partial x_m} = \sum_{j=1}^{K} \left[ \alpha_j \cdot \frac{\partial f_j}{\partial x_m} \right].
\]

We now have a system of \( K+N \) eqns \((C_1),\ldots,(C_K)\), \((L_1),\ldots,(L_N)\) in \( N+K \) unknowns \( x_1,\ldots,x_N,\alpha_1,\ldots,\alpha_K \).

#### Step IV. A soln to the above SoE (System of Eqns) has form \((\vec{x},\vec{\alpha})\), where \( \vec{x} \) abbreviates \( x_1,\ldots,x_N \) and \( \vec{\alpha} \) abbreviates \( \alpha_1,\ldots,\alpha_K \).

Find all solutions to this system. Each tuple \( P := (x_1,\ldots,x_N) \) is a Lagrange point.

#### Step V. Evaluate \( \psi \) at each Lagrange pt and find those which give largest and smallest values.

### Will we examine the comet?

Our astronomy community has informed us that a comet, in parabolic orbit about the Sun, is going to arrive in about 2 years.

Unfinished.

### Problems

**LM1:** Find all points \( P \) on the intersection of the sphere \( x^2 + y^2 + z^2 = 1^2 \) and plane \( 3x + 2y = z \) which extremize \( \psi \), where \( \psi(Q) := y + z \).

**LM2:** Consider light rays emitted from point \( Q_1 := (0,1) \), bounce once off the \( xy=1 \) hyperbola –let \( P \) denote the point where it hits– and then arrive at the origin \( Q_2 \).

Carefully perform steps I, II, III en route to finding all points \( P \) which minimize the length of the polygonal path \( Q_1PQ_2 \). (Do not solve the system of eqns; just set it up.)

**LM3:** Set up the above light-ray problem but in 3-space, with \( Q_1 := (-79,0,0) \) and \( Q_2 := (3,23,29) \), using an ellipsoidal mirror

\[
\frac{x^2}{3^2} + \frac{y^2}{5^2} + \frac{z^2}{7^2} = 1^2.
\]

(Just set up the SoE; do not solve it.)