

Lagrange Multipliers

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Cubic-surface Problem

On \mathbb{R}^3 , find all extreme points $P = (x, y, z)$ of function

$$\psi(P) := x^2 + y^2 + z^2,$$

subject to the *two* constraints that

$$\begin{aligned} C_f: & \quad x \cdot y \cdot z = 54; \quad (\text{a cubic surface}) \\ C_g: & \quad x + y + z = 12. \quad (\text{a plane}) \end{aligned}$$

A picture. Note that the objective fnc *and* the two constraints are symmetric under all permutations of x, y, z . Thus our set of Lagrange points must have a six-fold ($6 = 3!$) symmetry.

A good picture suggests that the only Lagrange pts have some two of x, y, z equal. In the case that $x = z$, say, a picture suggests that there are 3 solutions; two solns where the common value is positive and one soln with it negative. Thus the $z=y$, $y=x$ and $x=z$ solutions would give us 9 Lagrange-pts all together, but only 3 *types* of Lagrange-pt.

A solution

Define *specifier functions*

$$\begin{aligned} f(P) &:= x \cdot y \cdot z \quad \text{and} \\ g(P) &:= x + y + z, \end{aligned}$$

so that locus (C_f) is some level-set of f , and locus (C_g) is some level-set of g .

The Lagrange eqns. Using variables α, β to be the Lagrange multipliers for f and g , respectively, the Lagrange eqns become:

$$\begin{aligned} L_x: & \quad \psi_x = \alpha f_x + \beta g_x, \text{ i.e. } 2x = \alpha \cdot yz + \beta \cdot 1. \\ L_y: & \quad \psi_y = \alpha f_y + \beta g_y, \text{ i.e. } 2y = \alpha \cdot zx + \beta \cdot 1. \\ L_z: & \quad \psi_z = \alpha f_z + \beta g_z, \text{ i.e. } 2z = \alpha \cdot xy + \beta \cdot 1. \end{aligned}$$

We have 5 unknowns: The spatial variables x, y, z and the Lagr.vars α, β . We have 5 eqns: The constraints eqns (C_f), (C_g), and the Lagr.eqns (L_x), (L_y), (L_z).

Solving this system of eqns. If possible, we try to eliminate the Lagrange-vars first. We can subtract (L_y) from (L_z) to obtain

$$(L_z)-(L_y): \quad 2[z - y] = \alpha xy - \alpha zx \quad \stackrel{\text{note}}{=} \alpha x[y - z].$$

We now use symmetry to solve the system.

Case $x = y = z$. Equation (C_g) yields that $x = y = z = 4$. So their product, $x \cdot y \cdot z$, is 64. This contradicts (C_f). **THUS: No solutions**, here.

Case x, y, z are distinct. Then $y - z \neq 0$ so we may divide equation (L_y) - (L_z) to conclude that $-2 = \alpha x$. But -2 is not zero, so α must be non-zero and so $x = \frac{-2}{\alpha}$.

By symmetry [$z - x \neq 0$, etc], we may conclude that y , too, equals $\frac{-2}{\alpha}$. Alas, this contradicts that $x \neq y$. **SO: No solutions** in this case.

Case Exactly two of x, y, z are equal. Use **t** for the value which occurs twice among $\{x, y, z\}$, and use **u** for the value which occurs uniquely. Eqn (C_g) tells us then that $\mathbf{t} + \mathbf{t} + \mathbf{u} = 12$, i.e. $\mathbf{u} = 12 - 2\mathbf{t}$. From (C_f), then, $54 = \mathbf{t}^2 \mathbf{u} \stackrel{\text{note}}{=} \mathbf{t}^2 [12 - 2\mathbf{t}]$. Multiplying out, then,

$$0 = q(\mathbf{t}), \quad \text{where } q(t) := t^3 - 6t^2 + 27.$$

Evidently $q(3)$ is zero, so we divide $t - 3$ into $t^3 - 6t^2 + 27$ to obtain this factorization:

$$q(t) = [t - 3][t^2 - 3t + 9].$$

The QF (Quadratic Formula) tells us that the zeros of $t^2 - 3t + 9$ are $\frac{3}{2}[1 \pm \sqrt{5}]$. And since $\mathbf{u} = 12 - 2\mathbf{t}$, we obtain this table of Lagrange pairs (\mathbf{t}, \mathbf{u}) . In the table, the three values of \mathbf{t} decrease from left to right.

	Soln 1	Soln 2	Soln 3
\mathbf{t}	$\frac{3}{2}[1 + \sqrt{5}]$	3	$\frac{3}{2}[1 - \sqrt{5}]$
\mathbf{u}	$3[3 - \sqrt{5}]$	6	$3[3 + \sqrt{5}]$
$\frac{1}{9}\psi(\mathbf{t}, \mathbf{t}, \mathbf{u})$	$17 - 5\sqrt{5}$	6	$17 + 5\sqrt{5}$
Size	<i>closest</i>	<i>med.</i>	<i>farthest</i>

The Method in General

Suppose we have (or have derived) K constraints in N -dim space; use $Q = (x_1, \dots, x_N)$ for a general point in N -space. We wish to find all constrained extrema of a function $\psi: \mathbb{R}^N \rightarrow \mathbb{R}$.

Step I. Write down a formula for ψ , the *objective function*, and label the constraint equations $(C_1), \dots, (C_K)$.

Step II. Write down definitions ($:=$) of specifier functions f_1, \dots, f_K so that each (C_j) is a level-set of the corresponding f_j .

Introduce unknowns $\alpha_1, \dots, \alpha_K$ to be the *Lagrange multipliers* of f_1, \dots, f_K .

Step III. For $m = 1, 2, \dots, N$, write down the *Lagrange equations*

$$L_m: \quad \frac{\partial \psi}{\partial x_m} = \sum_{j=1}^K \left[\alpha_j \cdot \frac{\partial f_j}{\partial x_m} \right].$$

We now have a system of $K+N$ eqns $(C_1), \dots, (C_K), (L_1), \dots, (L_N)$ in $N+K$ unknowns $x_1, \dots, x_N, \alpha_1, \dots, \alpha_K$.

Step IV. A soln to the above SoE (System of Eqns) has form $(\vec{x}, \vec{\alpha})$, where \vec{x} abbreviates x_1, \dots, x_N and $\vec{\alpha}$ abbreviates $\alpha_1, \dots, \alpha_K$.

Find all *solutions* to this system. Each tuple $P := (x_1, \dots, x_N)$ is a *Lagrange point*.

Step V. Evaluate ψ at each Lagrange pt and find those which give largest and smallest values.

Will we examine the comet?

Our astronomy community has informed us that a comet, in parabolic orbit about the Sun, is going to arrive in about 2 years.

Unfinished.

Problems

LM1: Find all points P on the intersection of the sphere $x^2 + y^2 + z^2 = 1^2$ and plane $3x + 2y = z$ which extremize ψ , where $\psi(Q) := y + z$.

LM2: Consider light rays emitted from point $Q_1 := (0, 1)$, bounce once off the $xy=1$ hyperbola –let P denote the point where it hits– and then arrive at the origin Q_2 .

Carefully perform steps **I, II, III** *en route* to finding all points P which minimize the length of the polygonal path $Q_1 P Q_2$. (Do not solve the system of eqns; just set it up.)

LM3: Set up the above light-ray problem but in 3-space, with $Q_1 := (-79, 0, 0)$ and $Q_2 := (3, 23, 29)$, using an ellipsoidal mirror

$$\frac{x^2}{3^2} + \frac{y^2}{5^2} + \frac{z^2}{7^2} = 1^2.$$

(Just set up the SoE; do not solve it.)

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