Jordan Decomposition Theorem: LinearAlg

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ABSTRACT: Gives a home-grown proof of the Jordan Decomposition Theorem. (Some of the lemmas work in Hilbert space.) The “Partial-form JCF Theorem”, (26), needs to be reworked.

Prolegomenon

Our goal is to prove the “JCF” (Jordan Canonical Form) Theorem for a linear transformation \( T: H \to H \), where \( H \) is a finite-dim’al vectorspace. Formally, we’ll assume that \( H \) is \( F^{\times H} \), where the field \( F \) is either \( \mathbb{R} \) or \( \mathbb{C} \).

For vectorspaces use

\[
\text{vectorspace: } H, A, B, E, V
\]

\[
\text{dimension: } \mathcal{H}, A, B, E, V.
\]

Use sans-serif font for matrices \( A, B, G, I, M \). For square matrices \( A_x \), let \( \text{Diag}(A_1, \ldots, A_x) \) be the partitioned matrix which has \( A_1, \ldots, A_x \) along its diagonal, and zeros elsewhere.

1: Notation. A collection \( \mathcal{C} := \{V_1, V_2, \ldots, V_L\} \) of subspaces of \( H \) is \textit{linearly independent} (abbreviation \textit{lin-indep}) if the only soln to

\[
v_1 + \cdots + v_L = 0, \quad \text{with each } v_\ell \in V_\ell,
\]

is the trivial soln \( v_1 = 0, \ldots, v_L = 0 \).

Recall that a subspace \( V \subset H \) is \( T \)-invariant if \( T(V) \subset V \).

I’ll use \textit{eval}, \textit{evec} and \textit{e-space} for eigenvalue, eigenvector and eigenspace.

\[
2: \textbf{Defn.} \quad \text{W.r.t } T, \text{ a vector } v \text{ is nilpotent if } T^d(v) = 0 \text{ for some posint } d. \text{ Indeed, the } T-\text{depth} \text{ of a vector } v, \text{ written } T-\text{Depth}(v), \text{ is the infimum of all natnums } n \text{ for which } T^n(v) = 0. \text{ The zero-vector has depth } 0. \text{ An evec for eval=0 has depth } 1. (A \text{ non-nilpotent vector has depth } \infty.) \]

Use \( \text{Nil}(T) \) for the \textit{nilspace} of \( T \); it comprises the set of finite-depth vectors. So

\[
\text{Nil}(T) := \bigcup_{n=1}^\infty \text{Ker}(T^n) \supseteq \text{Ker}(T).
\]

Transformation \( T \) is \textbf{nilpotent} if there exists a posint \( D \) such that \( T^D = 0 \). Since \( H \) is finite dimensional, \( \{\text{trn } T \text{ is nilpotent iff } \text{Nil}(T) = H\} \)

\[
3: \textbf{Depth Lemma (preliminary). } \text{Consider a sum} \]

\[
v_1 + v_2 + \cdots + v_L
\]

whose depths satisfy

\[
d_1 > d_2 > \ldots > d_L.
\]

Then the depth of \( (3’) \) is \( d_1. \)

\[
\text{Proof. } \text{Exercise.}\)

A \textit{downtup} (“down tuple”) \( \vec{D} = (D_1, \ldots, D_\mathcal{E}) \) is a sequence of integers with

\[
4: \quad D_1 \geq D_2 \geq \ldots \geq D_\mathcal{E} \geq 1.
\]

The \textbf{size} of \( \vec{D} \) is the sum \( D_1 + \cdots + D_\mathcal{E} \). A posint \( D \) determines a \( D \times D \) \textbf{Jordan Block} matrix

\[
5: \quad \mathcal{J}(D) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\]

with zeros on the diagonal and ones on the first off-diagonal. Every undisplayed position is zero.

6: \textbf{Nilpotent JCF Theorem. } \text{A nilpotent } T:F^{\times H} \text{ has a unique downturn } \vec{D} \text{ so that}

\[
7: \quad M = M(\vec{D}) := \text{Diag} \left( \mathcal{J}(D_1), \ldots, \mathcal{J}(D_\mathcal{E}) \right)
\]

is the matrix of \( T \) w.r.t \textit{some} basis. In particular, \( \text{Size}(\vec{D}) \) equals \( \mathcal{H}. \)

\[
\square
\]
Remark. In general, the above basis is not unique.

The theorem can be restated ITOF of matrices. A nilpotent \( \mathbf{F} \)-matrix \( \mathbf{M}' \) determines a unique downtup \( \overrightarrow{D} \) so that, with \( \mathbf{M} \) from (7),

\[
\mathbf{M}' = \mathbf{G}^{-1} \cdot \mathbf{M}(\overrightarrow{D}) \cdot \mathbf{G},
\]

for some invertible \( \mathbf{F} \)-matrix \( \mathbf{G} \).

Temporarily letting \( c^1, \ldots, c^D \) denote the standard basis, notice that the \( D \times D \) jordan-blk (5) acts on the standard basis by sending \( c^D \to \cdots \to c^1 \to 0 \). Let this motivate our definition of a \textit{chain}:

A sequence \( \mathbf{C} = (c^d)_{d=1}^D \) of vectors, with \( D \geq 1 \), fulfilling

\[
0 \leftarrow c^1 \leftarrow c^2 \leftarrow \cdots \leftarrow c^D.
\]

Furthermore \( c^1 \neq 0 \), i.e., \( c^1 \) is an eigenvector. Equivalently, the depth of each \( c^d \) is \( d \).

Calling the (8)-chain \( \mathbf{C} \), let \( \text{Depth}(\mathbf{C}) := D \).

9: \textbf{Lemma.} Consider a chain \( \mathbf{C} \) as in (8). Then the eigenspace in \( \text{Spn}(\mathbf{C}) \) is just 1-dimensional. Further, \( \mathbf{C} \) is a basis for \( \text{Spn}(\mathbf{C}) \) and so the dimension of \( \text{Spn}(\mathbf{C}) \) is \( D \).

\textbf{Proof.} The Depth Lemma.

A \textit{chain complex} \( \overrightarrow{\mathbf{C}} \) for \( \mathbf{T} \) is a sequence \( \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_\mathcal{E} \) of \( \mathbf{T} \)-chains such that \( \overrightarrow{D} \) is a down tuple, (4), where \( D_e := \text{Depth}(\mathbf{C}_e) \). Furthermore

10: The list \( c^1_1, c^2_1, \ldots, c^1_\mathcal{E} \) of eigenvectors is linearly independent.

The downtup \( \overrightarrow{D} \) is called the \textit{signature} of \( \overrightarrow{\mathbf{C}} \).

By the way, we call \( \overrightarrow{\mathbf{C}} \) a “spanning chain-complex” if \( \bigcup_{e=1}^\mathcal{E} \mathbf{C}_e \) is a basis for \( \mathbf{H} \). Courtesy the next lemma, the chain-complex spans iff \( D_1 + \cdots + D_\mathcal{E} \) equals \( \text{Dim}(\mathbf{H}) \).

11: \textbf{Chain Independence Lemma.} Suppose that \( \mathbf{C}_1, \ldots, \mathbf{C}_\mathcal{E} \) are chains (of possibly different lengths). Then TFAE Equivalent.

\( a \): The list of eigenvectors \( c^1_1, c^2_1, c^3_1, \ldots, c^1_\mathcal{E} \) is linearly independent.

\( b \): The list \( \text{Spn}(\mathbf{C}_1), \text{Spn}(\mathbf{C}_2), \ldots, \text{Spn}(\mathbf{C}_\mathcal{E}) \) of subspaces is linearly independent.

\( c \): The disjoint union \( \biguplus_{e=1}^\mathcal{E} \mathbf{C}_e \) is a lin-indep set.

\textbf{Proof.} That \( (b) \Rightarrow (c) \) follows from Lemma 9. The interesting implication is \( (a) \Rightarrow (b) \).

Were the subspaces dependent, then we could find vectors \( \mathbf{v}_e \in \text{Spn}(\mathbf{C}_e) \), not all \( 0 \), so that

\[
\mathbf{0} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{\mathcal{E}-1} + \mathbf{v}_\mathcal{E}.
\]

Let \( \mathbf{D} \) be the maximum of \( \text{Depth} (\mathbf{v}_e) \), taken over \( e = 1, \ldots, \mathcal{E} \). Replacing each \( \mathbf{v}_e \) by \( \mathbf{T}^{D-1}(\mathbf{v}_e) \) arranges that: Each \( \mathbf{v}_e \) is either an eigenvector or is \( 0 \), and not all are \( 0 \).

Courtesy (9), the eigenspace in each \( \text{Spn}(\mathbf{C}_e) \) is 1-dim’al, so \( \mathbf{v}_e \) is a multiple of \( c^1_e \). But, by hypothesis, these evecs are linearly independent, \( \not\propto \). (I use \( \propto \) for “contradiction”.)

The above proof allows us to jazz up an earlier lemma. For a nilpotent vector \( \mathbf{v} \) of depth \( d \geq 1 \), call \( \mathbf{T}^{d-1}(\mathbf{v}) \) its \textit{penultimate vector}. [This penultimate vector is an evec.]

12: \textbf{Depth Corollary.} Let \( \mathbf{D} \) be the max depth of some list \( \mathbf{v}_1, \ldots, \mathbf{v}_\mathcal{E} \) of vectors. For those vectors of depth \( D \), suppose the set of their penultimate vectors is linearly-independent.

Then \( \text{Depth}(\mathbf{v}_1 + \cdots + \mathbf{v}_\mathcal{E}) = D \).

\textbf{The Construction}

To establish the existence part of \textbf{Nilpotent JCF Theorem}, we fabricate a spanning chain-complex for our nilpotent \( \mathbf{T} \).

Pick a maximum-length chain \( \mathbf{C}_1 \). Look at the lengths of those chains whose evec is \( \not\in \text{Spn}(\mathbf{C}_1) \); among those having the maximum length, take one such chain and call it \( \mathbf{C}_2 \). Pick a maximum-length chain, call it \( \mathbf{C}_3 \), from those whose evec is not in \( \text{Spn}(\mathbf{C}_1 \cup \mathbf{C}_2) \). Continuing, produces a sequence of some \( \mathcal{E} \) many
chains $C_1, \ldots, C_\xi$. By construction, the eigenvectors $c_1^\xi, \ldots, c_\xi^\xi$ are linearly indep., and their lengths satisfy $D_1 \geq D_2 \geq \ldots \geq D_\xi$.

Our goal is to show that $\text{Spn}(C_1 \sqcup \ldots \sqcup C_\xi)$ is all of $H$. [Corollarily, (11)\&(12) will imply that $E$ is the dimension of the eigenspace of $T$.]

**Proof.** If $V := \text{Spn}(C_1, \ldots, C_\xi)$ is not all of $H$, then there exists a “bad” vector $b$ i.e:

13: $b \notin V$, yet $T(b) \in V$.

[This, since $T$ is nilpotent.] Thus we can write $T(b)$ as a lin.comb over the $V$-basis $\bigcup_i^\xi C_i$. Because it will make no difference to the following argument, I will assume, in the expansion of $T(b)$, that each non-zero coeff is 1.

Suppose, for example, that

14: $T(b) = [c_5^5 + c_3^{13} + c_3^{17}] + [c_6^4 + c_6^9] + [c_8^2 + c_8^{33}] + \ldots + [c_7^{14}]$.

Consider the $c_5^5$ term. It has a predecessor on its chain, since $5 < D_3$. (After all, $D_3$ is at least 47.) Hence replacing the bad vector $b$ by $[b - c_6^6]$ preserves (13) and arranges that the $T$-image of this new $b$ has one fewer term in (14). Only the chain-end $c_3^D_e$ of a chain $C_e$ cannot be so removed.

Continue this until there are only chain-ends. For example,\(^{\odot1}\) suppose that the new $b$ maps to

14': $T(b) = c_3^{D_3} + c_8^{D_8} + \ldots + c_7^{D_7}$.

Corollary 12 tells us that vector $T(b)$ has depth\(^{\odot2}\) $\text{Max}(D_3, D_8, \ldots)$ —which is $D_3$. Consequently:

\[
\text{The depth of } b \text{ is } 1 + D_3.
\]

But all the vectors in $\text{RhS}(14')$ were chosen, during “The Construction”, at stages 3 and after. So $D_3$ was not in fact the length of the longest available chain. \(\times\).

\(^{\odot1}\)In this example, (14) and (14') together tell us that $D_3 = 47$, $D_5 > 9$, $D_8 = 31$, and so on.

\(^{\odot2}\)Since $b \notin V$, our $b$ is not an evec; so $T(b)$ is not $0$. Hence $T(b)$ has at least one chain-end.

**Uniqueness of signature.** A spanning chain-complex must have exactly $E = \text{Dim}(E)$ chains, where $E$ is the eigenspace of $T$. Although the spanning chain-complex itself is not unique nonetheless its signature is unique —the Nilpotent JCF Thm asserts this, so I’d better prove it!

15: Lemma. Given a nilpotent $T$, all spanning chain-complexes have the same signature. \(\diamondsuit\)

**Proof.** Consider two spanning chain-complexes

\[
\vec{C} = (C_1, \ldots, C_\xi),
\vec{C}^* = (Q_1, \ldots, Q_\xi),
\]

with different signatures. For specificity, suppose that the two signatures differ in their third term as follows:

\[
\begin{align*}
D_1 &\geq D_2 \geq 9 = D_3 \geq D_4 \geq \ldots \geq D_\xi; \\
D_1 &\geq D_2 \geq 8 = D_3^* \geq D_4^* \geq \ldots \geq D_\xi^*.
\end{align*}
\]

Now write vector $c_1^9$ over the $\vec{C}^*$-basis. This lin-comb must have a vector of depth 9 and none of greater depth. The only depth-9 vectors in $\vec{C}^*$ lie in chains $Q_1$ and $Q_2$. So our lin-comb has form\(^{\dag}\):

\[
c_1^9 = \alpha_1 q_1^9 + \alpha_2 q_2^9 + u,
\]

for some scalars $\alpha_1, \alpha_2$ and some vector $u$ in $\text{Spn}(\vec{C}^*)$ whose depth is at most 8; this, courtesy (12). Applying $T^8$ to (\(\dag\)) thus tells us that\(^{\ddag}\):

\[
c_1^1 \in \text{Spn}(q_1^1, q_2^1).
\]

Repeating the argument twice more gives

\[
\{c_1^1, c_2^1, c_3^1\} \subset \text{Spn}(q_1^1, q_2^1).
\]

But a 3-dim’al space won’t fit inside a 2-dim’al space. \(\ddag\)

Having proved JCF in the nilpotent case, (6), we now develop the tools for the general case.
§2 Algebraic information

The characteristic poly of an $M \times M$ matrix $M$ is

$$\varphi_M(x) := \operatorname{Det}(xI - M).$$

So $\varphi_M$ is a monic deg-$M$ poly.

16: Lemma. For a $B \times B$ matrix $B$,

$$\operatorname{Ker}(B) \text{ is trivial } \iff \varphi_B(0) \neq 0,$$

i.e, IFF $\varphi_B()$ has a [non-zero] constant term. ♦

Suppose $A,B \subset H$ is a lin-indep pair of subspaces, which jointly span $H$. Indicate this by writing

$$A \oplus B = H.$$  

Let $\operatorname{Proj}_B^A$ be projection, parallel to $A$, from $H$ onto $B$. Said differently, put an inner-product on $H$ making $A \perp B$. Then $\operatorname{Proj}_B^A$ is simply the orthogonal projection $\operatorname{Proj}_B$. An arbitrary linear trans $T:H \circlearrowleft$ gives a composition

$\ast$:  

$$B \xrightarrow{\operatorname{Proj}_B^A} H \xleftarrow{T} H$$

Let “$T_B^A$” denote the restriction of $(\ast)$ to $B$, i.e., the mapping $B\circlearrowleft$ by

$$T_B^A := [\operatorname{Proj}_B^A \circ T]|B.$$  

17: Block-UT-matrix Lemma. Consider an upper-triangular partitioned matrix

18:

$$M = \begin{bmatrix} A_{A \times A} & G_{A \times B} \\ 0_{B \times A} & B_{B \times B} \end{bmatrix}$$

Then $\operatorname{Det}(M) = \operatorname{Det}(A) \cdot \operatorname{Det}(B)$. In consequence, the char-poly $\varphi_M$ factors as

18':  

$$\varphi_M = \varphi_A \cdot \varphi_B.$$  

Restated, suppose $T:H \circlearrowleft$ has subspaces $A, B$ st.

19:  

$$H = A \oplus B.$$  

If subspace $A$ is $T$-invariant then

19':  

$$\varphi_T = \varphi_{T|A} \cdot \varphi_{T|B}.  \quad \diamond$$

Proof of $\operatorname{Det}(M) = \operatorname{Det}(A) \cdot \operatorname{Det}(B)$. Since $\operatorname{Det}(M)$ is a sum of products taken over all transversals of $M$, ISTS that a transversal straying from the $A,B$ blocks necessarily has product zero.

WLOG this misguided transversal hits $G$. It therefore misses some row of $A$ hence (since $A$ is square) some column of $A$. In this column, then, the transversal must hit the $0_{B \times A}$ block.

Exer: Why do the signs of the permutations work out correctly?

Proof of (19'). Let $B = (a_1, \ldots, a_A, b_1, \ldots, b_B)$ be a basis for $H$, with each $a_i \in A$ and $b_j \in B$. Then $M$, the $B$-matrix of $T$, has form (18). Furthermore, the $(a_1, \ldots, a_A)$-matrix of $T|A$ is $A$ and the $(b_1, \ldots, b_B)$-matrix of $T_B^A$ is $B$. Hence (18')$\Rightarrow$(19').

Recall from (2) the defn of nilpotent and Nil($T$).

20: Lemma. Consider a nilpotent $S:F^A \circlearrowleft$. Then

$$\varphi_S(x) = [x - 0]^A.$$  

Pf. The char-poly of a eval$=0$ Jordan-Block (5) is $x^D$. By the Block-UT-matrix Lemma, the char-poly of $\operatorname{Diag}(J_B(D_1), \ldots, J_B(D_\epsilon))$ is the product $x^{D_1} \cdots x^{D_\epsilon}$, i.e $x^{\sum_{i=1}^\epsilon D_i}$. ♦

21: Multiplicity Theorem. Let $A := \operatorname{Nil}(T)$. Then $A := \operatorname{Dim}(A)$ is the multiplicity of 0 in the characteristic poly $\varphi_T$, i.e,

$$\varphi_T(x) = [x - 0]^A \cdot g(x),$$

where $g$ is a poly with a constant term. ♦

Proof. Let $B$ be a complementary subspace $B \oplus A = H$. Then (19') and (20) tell us that

$$\varphi_T(x) = [x - 0]^A \cdot \varphi_{T_B^A}(x).$$

Consequently, courtesy (16), ISTProve that $T_B^A$ has no kernel. So fix a $v \in B$ sent to 0 by $T_B^A$.

Decompose its image as $T(v) = b + a$, with $b \in B$ and $a \in A$. Then $0 = T_B^A(v) \not= b$. Hence $T(v) = a$. So $T(v)$ is nilpotent. Thus $v$ too is nilpotent. So $v \in A \cap B$ and is therefore 0. ♦
§3 Using all the eigenvalues

For \( \lambda \in \mathbb{C} \), we now return to using “\( \lambda \)-evec” to mean an eigenvector with eigenvalue \( \lambda \), and we extend our defs to other evals.

An \( \lambda \)-Jordan Block is a \( D \times D \) matrix

\[
\lambda \text{-JB}(D) := \begin{bmatrix}
\lambda & 1 \\
& \lambda & 1 \\
& & \ddots & \ddots \\
& & & \lambda & 1 \\
& & & & \lambda
\end{bmatrix}.
\]

Generalizing, a downtup \( D = (D_1, \ldots, D_\ell) \) engenders an \( \lambda, D \)-Jordan Block

\[
\lambda \text{-JB}(D) := \text{Diag}(\lambda \text{-JB}(D_1), \ldots, \lambda \text{-JB}(D_\ell))
\]

24: Jordan Canonical Form Theorem. Suppose that \( T : \mathbb{F}^{\times H} \) has all of its eigenvalues \( \lambda_1, \ldots, \lambda_L \) in \( \mathbb{F} \). Then there is a unique list of downtups, \( \overrightarrow{D} = (D_1, \ldots, D_\ell) \) so that

\[
\text{Diag}(\lambda_1 \text{-JB}(D_1), \ldots, \lambda_L \text{-JB}(D_\ell))
\]

is the matrix of \( T \) relative to some basis.

In particular, \( \text{Size}(\overrightarrow{D}) = S_\ell \), where

\[
\varphi_T(x) = (x - \lambda_1)^{S_1} \cdot (x - \lambda_2)^{S_2} \cdots (x - \lambda_L)^{S_L}.
\]

is the \( \mathbb{F} \)-factorization of the char-poly of \( T \).

25: Partial-form JCF Theorem. Given linear \( T : \mathbb{F}^{\times H} \), factor its char-poly over \( \mathbb{F} \) as

\[
\varphi_T(x) = \prod_{\lambda} (x - \lambda)^{S_\lambda} \cdot g(x),
\]

where \( g \) is an \( \mathbb{F} \)-poly with no roots in \( \mathbb{F} \). (And \( \lambda_1, \ldots, \lambda_L \in \mathbb{F} \) are distinct.) Then there is a unique list of downtups, \( \overrightarrow{D} = (D_1, \ldots, D_\ell) \), Unfinished: as of 19Apr2019

For \( \alpha \in \mathbb{F} \), let \( T_\alpha \) abbreviate the \( T - \alpha I \) transformation, and let \( E_\alpha^{(d)} \) comprise the vectors of \( T_\alpha \)-depth at most \( d \). Evidently

\[
E_\alpha^{(d)} = \ker(T_\alpha^{\circ d})
\]
is a subspace, and \( E_\alpha^{(1)} \) is the eigenspace (when \( \alpha \) is an eigenvalue). Certainly

\[
\{0\} = E_\alpha^{(0)} \subset E_\alpha^{(1)} \subset E_\alpha^{(2)} \subset E_\alpha^{(3)} \subset \cdots \subset L_\alpha,
\]

where \( L_\alpha := \cup_{d=0}^{\infty} E_\alpha^{(d)} \) is the nilspace.

28: Lemma. Fix \( \alpha, \beta \in \mathbb{C} \). For each \( d = 0, 1, \ldots \), the subspace \( E_\alpha^{(d)} \) is forward-invariant under \( T_\beta \). Therefore \( L_\alpha \) is \( T_\beta \)-forward-invariant.

Proof. WLOG \( \alpha = 0 \) (replace \( T \) by \( T - \alpha I \) and \( \beta \) by \( \beta - \alpha \)). Fix an order \( d \), say \( d = 3 \), and fix a vector \( v \in E^{(3)}(\beta) \). Automatically \( T(v) \in E_\alpha^{(2)} \subset E_\alpha^{(3)} \). Hence \( [T - \beta I]v = (T - \beta I)v \in E^{(3)}(\beta) \) as desired. ♦

29: Lemma. Consider distinct scalars \( \alpha \neq \beta \). For \( d = 0, 1, 2, \ldots \), the restricted operator

\[
[T - \beta I] \mid E^{(d)}_\alpha
\]
has trivial kernel and so is a (linear) automorphism of \( E^{(d)}_\alpha \) (since \( E^{(d)}_\alpha \) is finite-dim). Taking a union, then, \( [T - \beta I] \mid L_\alpha \) is an automorphism of \( L_\alpha \).

Proof. WLOG \( \alpha = 0 \); so \( \beta \neq 0 \). By the preceding lemma, \( T_\beta \) maps \( E^{(d)}(\beta) \) into \( E^{(d)}(\beta) \). So FTSO with contradiction we may suppose that there is a non-zero \( v \in E^{(d)}(\beta) \) which is sent to \( 0 \) by \( T_\beta \).

Evidently \( T \in T_\beta \). For \( j = 0, 1, 2, \ldots \), consequently, the vector \( T^j(v) \) is also in \( \ker(T_\beta) \). Consider the value of \( j \) for which \( T^j(v) \) is in \( E^{(1)}(\beta) \setminus \{0\} \).

Redefining \( v \) to be this \( T^j(v) \), we now have that

\( v \) is a non-zero vector simultaneously in \( \ker(T) \) and \( \ker(T_\beta) \).

But then \( 0 = T_\beta(v) = T(v) - \beta v = -\beta v \). And this latter is not zero, since \( \beta \neq 0 \).

30: Prop’n. Let \( \lambda_1, \ldots, \lambda_L \) be the distinct eigenvalues of \( T \). Then the collection \( L_{\lambda_1}, \ldots, L_{\lambda_L} \) of nilspaces is linearly independent.

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Proof. Consider a sum \( \mathbf{v}_1 + \cdots + \mathbf{v}_L = \mathbf{0} \), with each \( \mathbf{v}_\ell \in \mathbf{L}_\lambda_\ell \). ISTShow that \( \mathbf{v}_1 = \mathbf{0} \). So ISTConstruct a linear \( \Lambda: \mathcal{H} \circlearrowleft \) sending each of \( \mathbf{v}_2, \ldots, \mathbf{v}_L \) to \( \mathbf{0} \), so that \( \Lambda|\mathbf{L}_{\lambda_1} \) is an automorphism of \( \mathbf{L}_{\lambda_1} \).

To this end, pick a number \( D \) large enough that

\[
[T_{\lambda_\ell}]^{\circ D}(\mathbf{v}_\ell) = \mathbf{0}, \text{ for each } \ell = 2, 3, \ldots, L.
\]

Since all the operators \( (T_a)_{\alpha \in \mathcal{C}} \) commute, it follows that the composition

\[
\Lambda := [T_{\lambda_2} \circ T_{\lambda_3} \circ \cdots \circ T_{\lambda_L}]^{\circ D}
\]

sends each of \( \mathbf{v}_2, \ldots, \mathbf{v}_L \) to \( \mathbf{0} \). And Lemma 29 assures us that \( \Lambda \) is an automorphism of \( \mathbf{L}_{\lambda_1} \). ♦

Proof of JCF, (24). Apply the Nilpotent JCF to \( T_{\lambda_\ell} \) on \( \mathbf{L}_{\lambda_\ell} \) to get a basis \( \mathcal{B}_\ell \) for \( \mathbf{L}_{\lambda_\ell} \) against which \( T \) has a matrix-block of form \( \lambda_\ell T_{\mathcal{B}}(\mathcal{D}_\ell) \). Then \( \bigcup_{\ell=1}^L \mathcal{B}_\ell \) is a basis against which \( T \) looks like (25). That the downtup sequence is unique follows from the uniqueness in Nilpotent thm and that \( T \) uniquely determines its nilspaces. ♦

End Notes

See cayley_hamilton.latex for several applications of JCF, and the minimum polynomial of a matrix.

Transposes. Let \( J \) be a jordan block. Its transpose \( J^\mathsf{T} \) is conjugate to \( J \) simply by reversing the order of the vectors in the basis. It follows that

\[
JCF(T^\mathsf{T}) = JCF(T)
\]

for an arbitrary square matrix \( T \). □

Remark. Fix a square matrix \( T \). Given a scalar \( \lambda \), let \( \mathcal{D}_T(\lambda) \) be the corresponding downtup in \( JCF(T) \); if \( \lambda \) is not a \( T \)-eval, then the downtup is empty.

The complex-conjugate of a JCF is a JCF. So \( JCF(T) \approx \bar{J} \). This gives the \((\Rightarrow)\) direction below.

31: JCF-of-real Theorem. A complex JCF \( B \) is the JCF of some real matrix IFF

\[
\mathcal{D}_B(\lambda) = \overline{\mathcal{D}_B(\lambda)},
\]

for each complex number \( \lambda \). ♦

Proof of \((\Leftarrow)\). ISTProve this when \( B \) consists of a jordan block and its complex-conjugate. For specificity suppose that each jordan block has dimension \( D = 3 \).

Fix reals \( c \) and \( s \), and let

\[
\lambda^+ := c + is \quad \text{and} \quad \lambda^- := c - is.
\]

We show that \( B := \diag(\lambda^+, \lambda^-, \lambda^+ \mathcal{B}(3), \lambda^- \mathcal{B}(3)) \) is conjugate to a real matrix by producing a basis \( \{u_j, w_j\}_{j=1}^3 \) for \( \mathbb{C}^{\times 6} \), against which \( B \) acts using only real coefficients.

For each choice of \( "^+" \) and \( "^-" \), let \( \{e_j^\pm\}_{j=1}^3 \) be the std basis for \( \lambda^\pm \mathcal{B}(3) \). Thus for \( j \in [1 \ldots 3] \),

32:

\[
B(e_j^+) = \lambda^+ e_j^+ + 1 \cdot e_{j-1}^+,
\]

where \( e_0^\pm \) are two other names for \( 0 \).

Define new vectors

\[
u_j := 1 \cdot e_j^+ + i \cdot e_j^-; \quad w_j := i \cdot e_j^+ + 1 \cdot e_j^-;
\]

so \( u_0 \) and \( w_0 \) are each \( 0 \). Check that

\[
\frac{1}{2} [u_j \pm iw_j] = e_j^+,
\]

so (32) spans all the \( e \)'s. Thus (32) indeed is a basis for \( \mathbb{C}^{\times 6} \).

The \( B \)-images of vectors. Verify that

\[
cu_j + sw_j = \lambda^+ e_j^+ + i \lambda^- e_j^- \quad \text{and} \quad -su_j + cw_j = i \lambda^+ e_j^+ + \lambda^- e_j^-.
\]
From (33) we compute:

\[ B(u_j) = 1 \cdot \left[ \lambda^+ \cdot e_j^+ + 1 \cdot e_{j-1}^+ \right] + 
            i \cdot \left[ \lambda^- \cdot e_j^- + 1 \cdot e_{j-1}^- \right]. \]

Grouping terms by subscript, our \( B(u_j) \) equals

\[ \left[ \lambda^+ e_j^+ + i \lambda^- e_j^- \right] + \left[ 1 \cdot e_{j-1}^+ + i \cdot e_{j-1}^- \right]. \]

This, together with similar elbow grease, yields

\[
B(u_j) = [cu_j + sw_j] + u_{j-1}; \\
B(w_j) = [-su_j + cw_j] + w_{j-1}.
\]

Since all the coefficients are real, we get that \( B \) is conjugate to a real matrix. ♦

Cyclic decompositions. The (forward) cyclic subspace generated by \( v \) is

\[ \text{Spn}(v, Tv, T^2v, T^3v, \ldots). \]

And \( T \) is a cyclic operator if there is a \( v \) whose cyclic subspace is all of \( V \).

Easily, a jordan-block is a cyclic operator on its space. So the jordan decomposition of \( T \) yields a \( T \)-cyclic decomposition of the vectorspace.

Yo! Look in source file, here. □