Zoids

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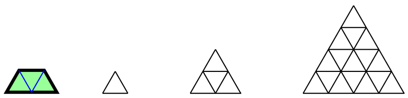
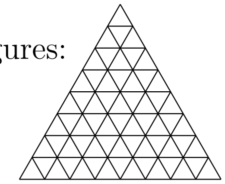
Tessellations. *Voici* example figures:

$\mathbf{T}$ $\quad\mathbf{E}_0$ $\quad\mathbf{E}_1{=}\Delta_2$ $\quad\mathbf{E}_2{=}\Delta_4$ $\quad\mathbf{E}_3{=}\Delta_8$

Notation. Use ***zilable*** for "zoid-tilable". Use ***little-triangle*** to mean a copy of $\mathbf{E}_0$.

For F a figure, have $|F|{=}5$ mean that F comprises 5 little-triangles. So $|\mathbf{T}| = 3$ and $|\mathbf{E}_k| = 4^k$ and $|\Delta_k| = k^2$.

Use $\mathbf{T}_n$ for the bottom 2^n rows of $\mathbf{E}_{n+1}$. Thus $\mathbf{T}_n = \mathbf{E}_{n+1} \smallsetminus \mathbf{E}_n$, whence $|\mathbf{T}_n| = 4^n{\cdot}3$. And $\mathbf{T}_0{=}\mathbf{T}$. More generally, the "$n$-***band*** of height H" is

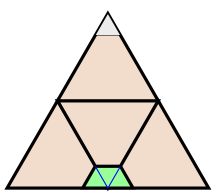
$$ {}_{\mathrm{H}}\mathbf{B}_n \;:=\; \Delta_{\mathrm{H}+n} \smallsetminus \Delta_n. \qquad \textit{Thus,} $$
$$ |{}_{\mathrm{H}}\mathbf{B}_n| \;=\; [\mathrm{H}{+}n]^2 - n^2 \;=\; 2n\mathrm{H} + \mathrm{H}^2\,. $$

The widths of the top/bottom edges of ${}_{\mathrm{H}}\mathbf{B}_n$ are n and $\mathrm{H}{+}n$, respectively.

1: Theorem. *Each $\widetilde{\mathbf{E}_n}$ is zilable.* ♢

Pf of (1). Base case: Since $\widetilde{\mathbf{E}_0}$ is empty, it admits the empty tiling.

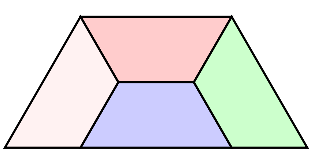
Our $\widetilde{\mathbf{E}_{n+1}}$ has an $\widetilde{\mathbf{E}_n}$ upstairs,



and three copies of $\mathbf{E}_n$ downstairs. The bottom row has a central zoid [since the row-length, 2^{n+1}, is even]. Removing it punctures the three downstairs figures, giving four $\widetilde{\mathbf{E}_n}$ total, each zilable. ♦

2: Trap Lemma. *Each $\mathbf{T}_n$ is zilable.* ♢

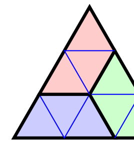
Proof. Base case: By defn, $\mathbf{T}_0$ is zilable. This recursive picture,

$\mathbf{T}_{n+1} \;=$ 

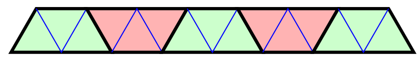
shows that 4 copies of $\mathbf{T}_n$ tile $\mathbf{T}_{n+1}$. Consequently, $[\mathbf{T}_n \text{ zilable}] \implies [\mathbf{T}_{n+1} \text{ zilable}]$. ♦

Alt Pf of (1). Observe that $\widetilde{\mathbf{E}_{n+1}}$ is a $\widetilde{\mathbf{E}_n}$ on top of a $\mathbf{T}_n$. The Trap Lemma asserts that $\mathbf{T}_n$ is zilable. ♦

3: Band Lemma. *Figures Δ_3 and ${}_1\mathbf{B}_{1+3n}$ and ${}_2\mathbf{B}_{2+3n}$ are zilable.* ♢

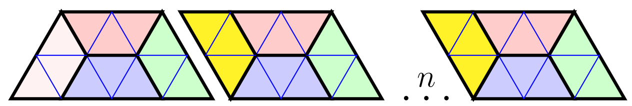
Proof. A zoid-tiling of Δ_3 is  . ♦

Pf. Band ${}_1\mathbf{B}_{1+3n}$ can be constructed by interleaving n many ***down***-zoids [their short edge is down], with $n{+}1$ ***up***-zoids. The band's top edge thus has

An $n{=}2$ example: 

length $[n{\cdot}2] + [n{+}1]{\cdot}1 \overset{\text{note}}{=\!=} 1 + 3n$. ♦

Pf. A zoid-tiling of ${}_2\mathbf{B}_{2+3n}$ is



one trapezoid followed by n parallelograms. ♦

Note. Below, $\equiv$ means "*mod-3 congruent to*". □

4: Thm. *Triangle Δ_k is zilable IFF $k \equiv 0$.* ♢

Pf. Necessarily, $k^2 = |\Delta_k| \equiv 0$. The primeness of 3 forces $k \equiv 0$. CONVERSELY: In Δ_n replace each little-triangle by the zilable Δ_3, to produce Δ_{3n}. ♦

5: Thm. *Punct. $\widetilde{\Delta_k}$ is zilable IFF $k \equiv {+}1$ or $k \equiv {-}1$.* ♢

Proof. Necessarily, $k^2 - 1 = \left|\widetilde{\Delta_k}\right| \equiv 0$, so $k \equiv \pm 1$.

CONVERSELY: Empty $\widetilde{\Delta_1}$ is zilable. Arguing inductively, suppose $\widetilde{\Delta_{1+3n}}$ has been shown zilable. Placing it atop band ${}_1\mathbf{B}_{1+3n}$, shows $\widetilde{\Delta_{2+3n}}$ zilable, courtesy our beloved Band Lemma.

Placing $\widetilde{\Delta_{2+3n}}$ atop ${}_2\mathbf{B}_{2+3n}$ proves (Band Lemma) that $\widetilde{\Delta_{4+3n}}$ is zilable . And $4 + 3n = 1 + 3[n{+}1]$, so we've inducted from n to $n{+}1$. ♦