Isomorphism

JLF King

Gambling

Let's play BLACKJACK with these cards:

J♥ T♥ 9♥ 8♥ 7♥ 6♥ 5♥ 4♥ 3♥

Blackjack (= *21*) is the goal. Abby and Bert alternate taking a card from the list and putting the card in their hand.

After 5 turns, perhaps the position is this:

J♥ 9♥ 5♥ 4♥

Abby's hand	**Bert's** hand
T♥ 3♥ 7♥	6♥ 8♥ ?

A player wins if, after adjoining a card to his hand, he now has some three cards summing to *Blackjack*. If *all* the cards are in player's hands, yet nobody has won, then the game is drawn.

Have You played this game?

Thinking Isomorphically. Suppose we subtract 1 from each card-value. We're now playing with this deck:

T♥ 9♥ 8♥ 7♥ 6♥ 5♥ 4♥ 3♥ 2♥

Since we need 3 cards to win, our new target is $21 - 3$, i.e *18*. This gives us a game which is isomorphic to BLACKJACK.

What happens if we follow this line of thought to its Logical Conclusion ?

Logical Conclusion. ... we iterate the subtract-1 idea. Since $\frac{21}{3} = 7$, subtracting 7 from each card gives symmetry

+4♥	+3♥	+2♥	+1♥	0♥	−1♥	−2♥	−3♥	−4♥

with *0*, now, as the target sum.

But... so what ?

Ringing the Bell. The symmetry suggests something ***Magic!*** that we've seen before...

−3	**+4**	**−1**
+2	**0**	**−2**
+1	**−4**	**+3**

All eight TTTs (tic-tac-toes) [three vertical, three horizontal, and two diagonal] sum to *0*. And the remaining $\binom{9}{3} - 8 = 84 - 8 = 76$ '***bad***' (non-TTT) triples, do *not* sum to *zero*. Thus:

∗: Blackjack *is game-isomorphic to* TTT.

[Exer: Even though there are 76 bad triples, why do we only need to check 2 of them?]

Soln is temporarily hidden.

Here's why. We *need-not* check bad triples that have two cells on the same row/column; because the number that *would* sum to *zero*, is <u>on</u> that row/column.

There are only $3! = 6$ triples on distinct rows/columns, –and 2 of them are the diagonals. This leaves $6 - 2 = 4$ triples to check: *Each corner cell together with its two cells a knight's-move-away.* This gives us only 4 triples to check.

But *wait a second!* The board is centrally symmetric when we change sign, so we only need check *one* corner from each diagonal; say, lower-left and lower-right. Their sums are $3 \neq 0$ and $9 \neq 0$, respectively. ♦

Automorphisms. 3×3-TTT has 8 ***automorphisms*** [self-isomorphisms]. One can check that the only permutations of the 9 cells that preserve TTTs, are the geometric symmetries of a square. This group of 8 is called the *4^{th}-**dihedral** group*. It comprises the 4 rotations of the board, together with flipping the board over, then rotating.

[Exer: What is the TTT-automorphism-group of the 4×4 board?]

We carry these automorphisms back to BLACKJACK, thusly:

4♥	J♥	6♥
9♥	7♥	5♥
8♥	3♥	T♥

BLACKJACK *is suddenly easier to win.*
(Las Vegas, *here I come!*)