

Isomorphism

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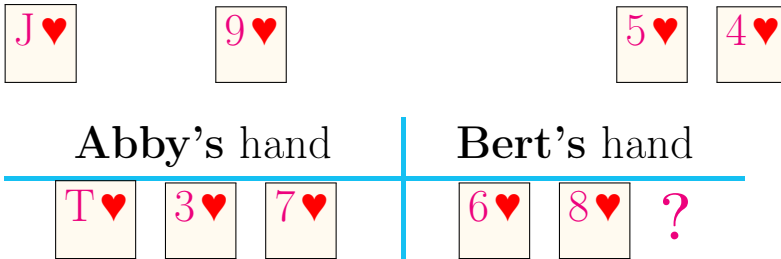
Gambling

Let's play BLACKJACK with these cards:



Blackjack (=21) is the goal. Abby and Bert alternate taking a card from the list and putting the card in their hand.

After 5 turns, perhaps the position is this:



A player wins if, after adjoining a card to his hand, he now has some three cards summing to *Blackjack*. If *all* the cards are in player's hands, yet nobody has won, then the game is drawn.

Have You played this game ?

Thinking Isomorphically. Suppose we subtract 1 from each card-value. We're now playing with this deck:



Since we need 3 cards to win, our new target is $21 - 3$, i.e. 18. This gives us a game which is isomorphic to BLACKJACK.

What happens if we follow this line of thought to its Logical Conclusion ?

Logical Conclusion. ... we iterate the subtract-1 idea. Since $\frac{21}{3} = 7$, subtracting 7 from each card gives symmetry



with 0, now, as the target sum.

But... so what ?

Ringing the Bell. The symmetry suggests something *Magic!* that we've seen before...

-3	+4	-1
+2	0	-2
+1	-4	+3

All **eight** TTTs (tic-tac-toes) [**three** vertical, **three** horizontal, and **two** diagonal] sum to **0**. And the remaining $\binom{9}{3} - 8 = 84 - 8 = 76$ '**bad**' (non-TTT) triples, do *not* sum to **zero**. Thus:

*: BLACKJACK is game-isomorphic to TTT.

[Exer: Even though there are 76 bad triples, why do we only need to check 2 of them?]

Soln is temporarily hidden.

Here's why. We *need-not* check bad triples that have two cells on the same row/column; because the number that *would* sum to **zero**, is on that row/column.

There are only $3! = 6$ triples on distinct rows/columns, –and **2** of them are the diagonals. This leaves $6 - 2 = 4$ triples to check: *Each corner cell together with its two cells a knight's-move-away*. This gives us only **4** triples to check.

But *wait a second!* The board is centrally symmetric when we change sign, so we only need check *one* corner from each diagonal; say, lower-left and lower-right. Their sums are $3 \neq 0$ and $9 \neq 0$, respectively. ♦

Automorphisms. 3×3 -TTT has 8 *automorphisms* [self-isomorphisms]. One can check that the only permutations of the 9 cells that preserve TTTs, are the geometric symmetries of a square. This group of 8 is called the *4th-dihedral group*. It comprises the 4 rotations of the board, together with flipping the board over, then rotating.

[Exer: What is the TTT-automorphism-group of the 4×4 board?]

We carry these automorphisms back to BLACKJACK, thusly:

4♥	J♥	6♥
9♥	7♥	5♥
8♥	3♥	T♥

BLACKJACK *is suddenly easier to win.*
(Las Vegas, *here I come!*)