

Intermediate-value Theorem

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Bernard Bolzano (1781–1848) proved the following form of the Intermediate-value Theorem.

1: IVT. Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous, with $f(a)$ and $f(b)$ non-zero and having different signs. Then there exists a point $c \in (a, b)$ which is a zero of f , i.e. $f(c) = 0$. \diamond

Proof. WLOGenerality, $f(a) < 0$ and $f(b) > 0$; otherwise, simply replace f by $-f$ (which preserves continuity) and note that a zero of $-f$ is a zero of f .

Let $L_0 := a$ and $R_0 := b$. For stage $n = 1, 2, \dots$, either up to some integer K , or out to ∞ , I will produce numbers L_n and R_n such that:

- i[n]: $a \leq L_{n-1} \leq L_n < R_n \leq R_{n-1} \leq b$;
- ii[n]: $R_n - L_n = \frac{1}{2}[R_{n-1} - L_{n-1}]$;
- iii[n]: $f(L_n) < 0 < f(R_n)$.

Stage- n construction. Let M be the midpoint of interval $[L_{n-1}, R_{n-1}]$, i.e. $M := \frac{1}{2}[L_{n-1} + R_{n-1}]$.

CASE: If $f(M)$ is zero, then STOP Set $K := n-1$. By (i[K]), note that M is strictly between a and b . So $c := M$ fulfills the conclusion of the theorem.

CASE: Otherwise, $f(M) \neq 0$. If $f(M)$ negative then let $L_n := M$ & $R_n := R_{n-1}$. If $f(M)$ positive then let $L_n := L_{n-1}$ & $R_n := M$. In either case, conditions (i,ii,iii[n]), automatically hold.

Last step. WLOGenerality, we may assume that our construction never STOPped. So we have two sequences, $\vec{L} := (L_n)_{n=0}^\infty$ and $\vec{R} := (R_n)_{n=0}^\infty$.

By (i), \vec{L} is increasing and is bounded above by b . Since a bounded monotone seq must converge, $L_\infty := \lim_{n \rightarrow \infty} L_n$ exists; it is in interval $[a, b]$, courtesy (i).

Thus f is defined –hence continuous– at L_∞ , so $f(L_\infty)$ equals $\lim_n f(L_n)$. And $f(L_\infty) \stackrel{\text{must}}{\leq} 0$ since each $f(L_n) \leq 0$.

Analogously, $f(R_\infty) := \lim_{n \rightarrow \infty} f(R_n)$ exists, and is non-negative. Furthermore

$$\begin{aligned} R_\infty - L_\infty &= \lim_{n \rightarrow \infty} [R_n - L_n], \quad \text{by what thm?}, \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2}\right]^n \cdot [b - a], \quad \text{by (iii) and induction,} \\ &= 0. \end{aligned}$$

Thus R_∞ and L_∞ equal a common value, call it c , in interval $[a, b]$. The preceding paragraphs tell us that $f(c) \leq 0$ and $f(c) \geq 0$; so $f(c)$ must be zero. Hence $c \notin \{a, b\}$. \diamond

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