Examples of mathematical induction, PHP, invariance, extremal arguments . . . . . . and *Thinking*

J.L.F. King

In these notes, USAMO is '*United States of America Mathematical Olympiad*'. And HMMT is '*Harvard-MIT Mathematics Tournament*'. Problems from these, and from the Putnam competition, are labeled as such.

Each class has had an Amanuensis, Problem Czar, Royal Scribe, whom I thank. Some were: *Knight Max Redmond, Sir Alexander Widom, Prime Minister James Cherry, Lady Lindsey Grigsby.*

§Outline

Pigeon-hole Principle

See Wikipedia for various versions.

1: N-friends Problem. *In each set of $N \geqslant 2$ people, some two of them have the same number of friends. (View friendship as an anti-reflexive, symmetric relation.)* ◊

SOLVED BY: *Jeremy S.*, 2011t. *Caleb S.*, 2014g. *Patrick B.* & *Isaac K.*, 2017g.

2.1: Points-in-a-square. *In square $\mathbf{C} := [0,1]\times[0,1]$, there are 10 "special" points. Prove that some two of them are no-further-apart than $\sqrt{2}/3$.* ♢

SOLVED BY: *Diego R.*, 2014g. *Yifei L.*, 2017g.

Soln: Points-in-a-square. Like a tic-tac-toe board, cut $\mathbf{C}$ into 9 little "squarettes", each of sidelength $\frac{1}{3}$. (Each squarette is closed, and they may share boundary-points.) By PHP, some two of the specials lie in the same squarette. And a squarette's diameter is $\frac{\sqrt{2}}{3}$. ♦

2.2: ⁇ Generalizing Points-in-a-square. *Fix an $N \geqslant 2$.*

For an N-tuple $\mathcal{T} := (p_1, \ldots, p_N)$ of points in $\mathbf{C}$, let $f(\mathcal{T})$ be the minimum of $\mathrm{Dist}(p_i, p_j)$ taken over all $\binom{N}{2}$ many pairs of indices $i < j$ in $[1\,..\,N]$.

*The N^{th} **MinDist** (minimum distance) number, μ_N, satisfies:*

a: *Each N-tuple $\mathcal{T}$ satisfies $f(\mathcal{T}) \leqslant \mu_N$.*

b: *There exists an N-tuple $\mathcal{T}'$ with $f(\mathcal{T}') = \mu_N$.*

[Such a number μ_N exists, because $\mathbf{C}$ is compact.]

*Here, an "N-**cover** of $\mathbf{C}$" is a length $N-1$ list*

$$\mathcal{L} \;:=\; (S_1, S_2, S_3, \ldots, S_{N-1})$$

of subsets of $\mathbf{C}$, whose union is all of $\mathbf{C}$. Moreover, here I'll require that each S_k be a closed[♡1] *set. Use $\widehat{f}(\mathcal{L})$ for the maximum diameter of the $\mathcal{L}$-patches. [Each S_k is a "**patch**" of $\mathcal{L}$. The foregoing "squarettes" were patches.] I.e*

$$\widehat{f}(\mathcal{L}) \;:=\; \mathrm{Max}\,\{\mathrm{Diam}(S_k) \mid k \in [1\,..\,N)\}$$

Finally, let $\widehat{\mu}_N$ be the minimum of $\widehat{f}(\mathcal{L})$, taken over all N-covers $\mathcal{L}$.

*Our **PHP** argument from class shows, for each $\mathcal{T}$ and $\mathcal{L}$, that $f(\mathcal{T}) \leqslant \widehat{f}(\mathcal{L})$. Consequently,*

$$*: \qquad \mu_N \;\leqslant\; \widehat{\mu}_N\,.$$

Evidently,

$$\mu_2 \;=\; \sqrt{2} \;=\; \widehat{\mu}_2\,.$$

Are the $(*)$-quantities equal for $N = 3$? What are the $(*)$-values, for higher N. In particular, what happens at $N{=}10$? ♢

[♡1] This, since the closure of a set has the same diameter as the original set.

Partial result. For $N = 3$ …

3.1: $2N$-Subset-Prob. *For $N \in \mathbb{Z}_+$, let $J_N := [1\,..\,2N]$. For each subset $S \subset J_N$ having $|S| \geqslant N{+}1$, prove:*

Appetizer: There exist <u>distinct</u> numbers $x, y \in S$ with $x \perp y$.

*Entrée: There exist <u>distinct</u> numbers $u, d \in S$ with u ⦁ d. [Such a (u,d) is a **divisibility-pair**.]* ♢

SOLVED BY: *Hannah P.* & *Patrick W.*, 2011t. *Zach N.*, 2012t. *Morgan W.*, 2014g. Appetizer by *CJ* [*Charles F.*]; Entrée by *Jessie C.*, 2017g.

Divisibility proof. Stronger, we'll prove the existance of such a pair, with u/d a power-of-two.

Define an equivalence relation $\approx$ on J_N by:

$$x \approx y \;\text{ if } \exists K \in \mathbb{Z} \text{ with } x = 2^K y.$$

Given a point $z \in [1\,..\,N]$, keep doubling it until it first leaves $[1\,..\,N]$. The departure value necessarily exceeds N, yet is less-equal $2{\cdot}N$. Thus:

Each $\approx$-equivalence-class owns a member of "half-open" interval-of-integers $\mathbf{I} := (N\,..\,2N]$.

But $|\mathbf{I}| = N$, so there are at most N equiv-classes. (There are *exactly* N many, since no two members of $\mathbf{I}$ are $\approx$-equivalent. But we don't need this.) Hence some two members $d{<}u$ of S are in the same $\approx$-equiv-class. ♦

3.2: ⁇ *Generalized $2N$-Subset-Prob.* If $|S| \geqslant N{+}2$, must S have at least *two* divisor-pairs? How does the-above result generalize? □

4: Monochromatic rectangle (USAMO 1976.1).

a: *Suppose that each cell of a 4×7 chessboard is colored either black or white. Prove, for each such coloring, that the board must contain a rectangle [formed by the horizontal and vertical lines of the board] whose four distinct corner-cells are all of the same color; a **monochromatic rectangle**.*

b: *Exhibit a black-white coloring of the 4×6 board with <u>no</u> monochromatic rectangle.* ♢

SOLVED BY: *James C.* & *Caleb S.*, 2014g. *Ken D.*, 2017g.

Pf of (a). Each row has 4 cells. FTSOC, suppose we have:

$*$: *A 4×7-coloring with no mono-rectangle.*

A row with 2 whitecells and 2 blackcells will be called ***balanced***. Note that

¥: *No two balanced rows are identical.*

For otherwise, those two rows would engender a monochromatic rectangle; indeed, four mono-rects.

Lower-bound on balanced rows. Imagine that some two rows each had $\geqslant 3$ whitecells; then these two rows admit a white mono-rect —since, together, they have $\leqslant 2$ blackcells, so at most two columns are excluded. Consequently:

†: *There is at most one $\geqslant 3$ white row, and at most one $\geqslant 3$ black row. So there are at least* $7 - 2 = \mathbf{5}$ *balanced rows.*

Upper-bound on balanced rows. Suppose *some* row has $\geqslant 3$ whitecells; pick three of them, and call their columns the "W-columns". There are $\binom{3}{2} = 3$ ways of picking two columns from the W-columns. Hence there are only

$$\binom{4}{2} - \binom{3}{2} = 6 - 3 = \mathbf{3}$$

ways of picking two columns from the four columns, which *avoid* sharing two columns with the W-columns. So *at most* 3 rows are balanced, courtesy (¥). But that contradicts (†). In consequence,

No row has $\geqslant 3$ white cells.

Similarly, no row has $\geqslant 3$ **black** cells. Thus

‡: *Every row is a balanced-row.*

No row-pattern can occur twice, by (¥).

Applying PHP. There are only $\binom{4}{2} = 6$ balanced patterns, and yet there are 7 rows total. *Contradiction.* Hence $(*)$ must not exist. ♦

Pf of (b). For each of the $\binom{4}{2} = 6$ patterns for a balanced row, make that one of the 6 rows. ♦

Rooks

Let 7×7 denote the 7×7 chessboard, viewed as a set of 49 cells. A subset $S \subset 7{\times}7$ is ***friendly*** if its elements lie in distinct rows, and in distinct columns. [I.e, no rook in S could capture another S-rook.]

5.1: Non-attacking rooks Thm. *Say a subset $\Gamma \subset 7{\times}7$ is **large** if $|\Gamma| = 22$. Then: Each large Γ admits a friendly 4-subset.* ◊

SOLVED BY: *Alisa M.*, 2015g.

Defn. A *partition* P of 7×7 is ***peaceful*** if each P-atom is friendly *and* P has no more than 7 atoms. By PHP, this forces P to have exactly 7 atoms, each comprised of exactly 7 cells.

Index the rows of the board by $\mathbb{Z}_7 = [0..7)$, and also index the columns by $\mathbb{Z}_7$. Let $\oplus, \ominus$ denote addition,subtraction mod 7. Use $\equiv$ for $\equiv_7$.

The number of friendly 7-sets is $7! = 5040$, since I have 7 choices for which cell I pick in row-0, then 6 choices for the cell in row-1, etc. A friendly 7-set S engenders a peaceful ptn P_S with atoms $A_0, \ldots, A_6$, simply by "rotating S around the cylinder", i.e, defining

$$A_j \quad := \quad \Big\{ (x \oplus j,\, y) \;\Big|\; (x, y) \in S \Big\}.$$

There are peaceful ptns not of this form, so there are *more than* 5040 peaceful ptns.

As a convenience, use ***clump*** for "friendly 4-set", and ***double-clump*** for "disjoint-pair of clumps". □

5.2: ⁇ Disjoint-pair non-attacking rooks Thm. *A subset $\Gamma \subset 7{\times}7$ is **big** if $|\Gamma| = 23$. Then: Each big Γ includes a disjoint-pair of friendly 4-sets.* ◊

Counting in Two Ways (CiTWa)

Here we count a (usually finite) set in two different ways, to arrive at an identity.

6: Candy-store identity. *The number of ways of picking 5 candies from 4 types equals… the number of ways of picking 3 candies from 6 types.* *Proof.* Stars–and–bars. ◊

SOLVED BY: *Samantha-S*, 2017g. *Ken D.*, 2017g.

7: Binomal-product-PoT Lemma. *Consider natnums $N \geqslant E$. Then*

7a: $$\sum_{k \in [E\,..\,N]} \binom{N}{k}\binom{k}{E} \;=\; 2^{N-E} \cdot \binom{N}{E}. \qquad \Diamond$$

SOLVED BY: *Mike C.*, 2014g. *Ross P.*, 2015g. *Ken D.*, 2017g.

Soln. We have N people. Here, a "committee" has some number, k, of people, and exactly E of them are in the *Executive (sub)committee*. CLAIM: *Each side of* $(*)$ *is the number of such committees.*

To obtain a committee, pick E people; $\binom{N}{E}$. Each of the remaining $N-E$ many people has a Yes/No choice to join the committee, leading to 2^{N-E} many choice-tuples. Hence RhS$(*)$ is the number of committees.

In contrast, for each $k \in [E\,..\,N]$, how many size-k committees are there? There are $\binom{N}{k}$ many groups of k people. Within each group, we have $\binom{k}{E}$ many ways of picking our Executive subcommittee. ♦

Induction

8: ⁇ Modsum-zero Problem. *Given a posint V (initial Value), define a sequence $\vec{\mathbf{b}}$ by $b_1 := V$ and, for each $n \in [2\,..\,\infty)$, let b_n be the unique value in $[0\,..\,n)$ for which sum*

$$S_n \;:=\; b_1 + b_2 + \ldots + b_n$$

is divisible by n. Prove that $\vec{\mathbf{b}}$ is eventually-constant.

E.g.

b_n:	31	1	1	3	4	2	0	6	6 ⋯
n:	1	2	3	4	5	6	7	8	9 ⋯

◊

Defn. For numbers, recall that A^{B^C} means $A^{[B^C]}$.

The n^{th} ***Fermat number*** is $F_n := 2^{2^n} + 1$. E.g $F_0 = 3$ and $F_3 = 1 + 2^{2^3} = 1 + 2^8 = 257$. □

9: ⁇ Coprime Fermat. *For each pair $K < N$ of natnums, Fermat numbers F_K and F_N are coprime. (Coro: There are infinitely many prime numbers. [How does this follow?])* ◊

Hint. How is $G_n := F_n - 2$ related to F_n? □

Caveat: The Wikipedia page has a proof.
SOLVED BY: *2013t and 2015g classes, on a takehome.*

10: ⁇ Coloring subsets (USAMO 2002.1). *An element of*

$$J \;:=\; \{1, 2, 3, \ldots, 2013\}$$

*is a **token**. A set-of-tokens is a **blip**. A **"coloring over J"** is a map, $\mathcal{C}$, which assigns to each blip either "black" or "white" such that*

†: *The union of each two white blips is white, and the union of each two black blips is black.*

Let $W(\mathcal{C})$ denote the number of white blips. Prove:

‡: *$\forall n \in [0\,..\,2^{2013}]$, there exists an n-**coloring**, $\mathcal{C}$, i.e, a coloring with $W(\mathcal{C}) = n$.* ◊

Defn. For each natural number M, let $J_M := [1\,..\,M]$.

Use ***TAP*** for "3-*term arithmetic progression*"; a triple $(\boldsymbol{\tau},\, \boldsymbol{\tau}+G,\, \boldsymbol{\tau}+2G)$ of numbers, with $G > 0$. □

11.1: ⁇⁇ 3Term integer AP (precursor of USAMO 1980.2). *Compute $f(M)$, the number of TAPs in J_M.* ◊

[*Suggestion:* Inclusion-exclusion. Induction.]

11.2: ⁇⁇ 3Term real AP (USAMO 1980.2). *Determine $g(M)$, the maximum number of three-term arithmetic progressions which can be chosen from a sequence of M real numbers [which we'll call **tokens**]*

$*$: $$\boldsymbol{\tau}_1 < \boldsymbol{\tau}_2 < \cdots < \boldsymbol{\tau}_M\,.$$

[I.e, $g(M)$ is the max taken over all M-sequences of tokens.] ◊

[*Suggestion:* Induction.]

12: ⁇ Stable-table Conundrum (USAMO 2005.4). *Legs L_1, L_2, L_3, L_4 of a square table each have length n, where $n \in \mathbb{N}$. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of natnums can we cut a piece of length k_i from the end of leg L_i, and still have a stable table? Let A_n denote this number. (The table is **stable** if it can be placed so that all four of the leg-ends touch the floor. Note that a cut leg of length 0 is permitted.)* ◊

Infinite descent

Break a stick into a long piece, length L, and a short piece, S. Suppose we have that ratios $\frac{\text{Tatal len}}{\text{long}}$ and $\frac{\text{long}}{\text{short}}$ are equal, i.e $\frac{L+S}{L} = \frac{L}{S}$. The common ratio is called the ***golden ratio***, $\boldsymbol{\lambda}$. A ***golden rectangle*** is a $W \times H$ rectangle where $\frac{\text{long side}}{\text{short side}}$ is $\boldsymbol{\lambda}$.

13: ⁇ Football Prob. (*Research possibility: Tug-of-war*). *A tuple* $\vec{\mathbf{w}} = (w_1, w_2, \ldots, w_{23})$ *represents the* [real number] *weights of football players. Tuple* $\vec{\mathbf{w}}$ *is a* ***football tuple*** *if: No matter whom is chosen as referee, there exists a partitioning of the remaining players into two equal-cardinality, equal total-weight teams.*

Prove that the only football tuples are the constant tuples. ◊

Solved by: *Forrest* for integral weights. [But what about real weights?]

14.1: ⁇ Favorite-Toy Problem (HMMT2013.7). *There is a set* $\mathbf{K}$ *of* n *kids, and a set* Ω *of* n *toys. Each child has a* (strict) *preference ordering on the toys. A* ***distribution*** *of the toys, is a bijection* $f{:}\mathbf{K} \hookrightarrow\!\!\!\!\rightarrow \Omega$*; it indicates that child* $\mathbf{c}$ *gets toy* $f(\mathbf{c})$*. A distribution is* ***disappointing*** *if no child gets his favorite toy.*

Distribution h ***dominates*** f*, written* $h \succcurlyeq f$*, if each child likes his* h*-toy at least as much as his* f*-toy.* [*Write* $h \succ f$ *if* $h \succcurlyeq f$ *and* $h \neq f$*.*] The goal is to prove:

‡[n]: *Suppose* f *is a disappointing* n*-distribution. Then there exists an* h *with* $h \succ f$*.* ◊

15.1: Coalescing Robots. *Consider an $K \times L$ chessboard, which we'll think of as the $K \cdot L$ many rooms of a building. Initially, the walls are all the edges of rooms. Remove some of the interior walls, so the building is* ***connected****; it is possible to walk from any room to any other room. Call a connected building a* ***house****.*

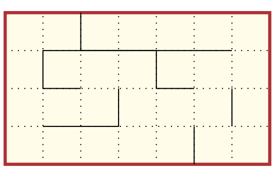
Put a robot mouse in each room. You can radio commands N,E,S,W [North, East, South, West] to all the robots. If you radio N, then each robot with a room to his north and no wall between, rolls to that room; otherwise, he doesn't move. [Now, some rooms might contain two robots; a room can hold any number of robots.]

A house is ***coalesceable*** *if there exists a finite instruction sequence [e.g NNES…WWN] after which all $K \cdot L$ robots are in a single room.*

An (K, L) pair is ***good*** *if every house on $K \times L$ is coalesceable. With proof, which (K, L) pairs are good?* ◊

SOLVED BY: *Isaac K., 2017g.*

EXAMPLE:
A 7×4 house.



15.2: *Ans.* Each (K, L) is good.

Tool: In a house, define the ***distance*** between two rooms A, B as the length of a minimum length walking-path [which need not be unique] between them. For example, $\text{Dist}(A, A) = 0$ and, for $A = (3, 5)$ and $B = (3, 6)$: *If there is no wall between these rooms, then* $\text{Dist}(A, B) = 1$, *else* $\text{Dist}(A, B) \geqslant 2$. □

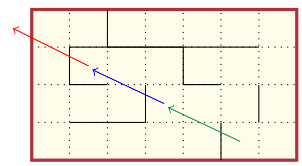
Proof. [Exer: A (finite) house is coalesceable IFF it is ***pair-coalesceable***; each two robots (i.e, rooms) can be coalesced.]

FTSOC, fix a non-coalesceable house. This must have a non-coalesceable pair of rooms $A, B \in K \times L$ in that, placing a robot in each,, *no* sequence of instructions can coalesce those two robots.

Fix a non-coalesceable pair A, B at minimum distance among all non-coalesceable pairs. WELOG $\text{Dist}(A, B) = 7$. Take a length-7 path between them, e.g $\pi :=$WNNE…S. The W instruction moves Alice [the robot currently in A] along this shortest-path π; so if Bob [the robot currently in B] didn't move, then the distance between them would be no larger than 6. But they were a *minimum*-distance non-coalesceable pair, so Bob was able to move W.[♡2]

Continuing, playing out the entire sequence π, our Alice moves from room A to room B. I.e, Alice moves from A to $A + \mathbf{v}$, where $\mathbf{v}$ is the difference vector $B - A$. Since Bob also moves at each instruction, he traversed the $\mathbf{v}$-translate of π, moving from B to $B + \mathbf{v}$, ie to $A + 2\mathbf{v}$. Moving *Alice* along this translated path carries her from $A + \mathbf{v}$ to $A + 2\mathbf{v}$, and carries Bob from $A + 2\mathbf{v}$ to $A + 3\mathbf{v}$. This can continue forever, yield the contradiction that the house is infinite. ♦

Iterating the NWW word.



15.3: *Defn.* Consider two robots (i.e, rooms) A, B in a $K \times L$ house $\mathbf{H}$. Their ***Pair-coalescence Time***, $\text{PT}_{\mathbf{H}}(A, B)$, is the *minimum* time it takes to coalesce A with B. The worst-case pair-coalescence is thus

$$\widehat{\mathbf{H}} := \text{Max}\{\text{PT}_{\mathbf{H}}(A, B) \mid A, B \in \mathbf{H}\}. \quad \text{So}$$

$$\widehat{K \times L} := \text{Max}\{\widehat{\mathbf{H}} \mid \mathbf{H} \text{ a } K \times L \text{ house}\}.$$

is the worst-case over houses with a given footprint. □

Every $\mathbf{H}$ has $\widehat{\mathbf{H}} \geqslant [K-1] + [L-1]$, since that many horizontal+vertical commands are need to unite antipodal corners. In a house with a single path visiting every room, its end rooms are distance $\text{Area}(\mathbf{H}) - 1$ apart. So the minimum time to coalesce those two robots is at least

$$\left\lceil \frac{KL-1}{2} \right\rceil \overset{\text{hence}}{\leqslant} \widehat{K \times L}.$$

15.4: ⁇ **Pair-coalescence Time.** *What are interesting upper and lower bounds for $\widehat{K \times L}$?* ◊

15.5: *Defn.* For a N,E,S,W-word π, let $B \cdot \pi$ be the room where π would bring a robot from room B. Say that rooms A, B are ***exchangeable*** if $\exists \pi$ st. $A \cdot \pi = B$ and $B \cdot \pi = A$. House $\mathbf{H}$ is ***universally exchangeable*** if *every* pair A, B is exchangeable. □

[♡2] As Horace Greeley said, "*Go* W*, young man, go* W."

15.6: ??? Exchangeable Robots. *Which $K \times L$ admit a house with an exchangeable pair $A \neq B$? Which $K \times L$ admit a universally exchangeable house?* ♢

15.7: *Defn.* An ***Infinity two-house*** has $\mathbb{Z}\times\mathbb{Z}$ for rooms, is connected, and *each room has at least two walls.* □

15.8: ??? Robots in Infinity-House. *Does there exist an ∞-2-house which is pair-coalescable?* ♢

Tiling questions

16: ?? IFF domino-tiling criterion. *Consider a 8×8 chess board with 1 black cell and 1 white cell removed. We seek an IFF-condition, on the removed-pair, for the board to be domino-tilable (by $\frac{62}{2} = 31$ many dominos), under the assumption that the board is:*

a: *Toroidal: The top-and-bottom edges connect, and the left-and-right edges connect.*

b: *Cylindrical: Just the the left-and-right edges connect.*

c: *Normal: No edges connect.* ♢

17: ?? 4mino-tilable rectangles. *A **four-mino** is a 1×4 tile. Which $2N \times 2K$ boards admit a four-mino tiling?* ♢

SOLVED BY: *Keven H., 2013t.*

18: ?? Lmino puncture-tilable. *An **Lmino** (pron. "ell-mino") comprises three squares in an "L" shape (all four orientations are allowed).*

*A board is **"Lmino puncture-tilable"** if: No matter which cell is removed, the resulting puntured-board is Lmino tilable.*

Which posint pairs N, K, with $NK \equiv_3 1$, are such that the $N \times K$ board is Lmino puncture-tilable? ♢

19: ?? Multi-dimensional Lminos. *In class we showed, for each $n \in \mathbb{N}$, that the $2^n \times 2^n$ board is Lmino puncture-tilable.*

Generalize this to a D-dimensional board, $2^n \times 2^n \times \overset{D}{\ldots} \times 2^n$. You will first need to decide what your D-dimensional generalization of an Lmino should be. Are there several reasonable possibilities? ♢

Invariants

The key underlying certain proofs, is that some *quantity* or some *relation* is preserved under the relevant operations.

***Eg:* Invariant quantity.** Have B be the 8×8 chessboard, but with the lower-right and upper-left cells removed; so $|B| = 62$. We start laying down dominos. Can we cover the board with 31 dominos? *No!*

Why? Initially, the uncovered part of the board (i.e, all of B) has 32 black cells and 30 white cells. These numbers are *not* invariant under placing a domino. But the ***discrepancy***, this difference

$$\dagger: \qquad \#\left\{\begin{matrix}\text{Uncovered}\\ \text{black cells}\end{matrix}\right\} - \#\left\{\begin{matrix}\text{Uncovered}\\ \text{white cells}\end{matrix}\right\},$$

is unaltered by placing a domino —it is invariant. Since the discrepancy is 2 initially, it will *always* be 2, no matter how many dominos we place. But a *covered* board would have a discrepancy of 0, not 2.

***Eg:* Invariant relation.** Our *Lightning bolt alg.* chose "seeds" for the s- and t- columns, so that

$$\ddagger: \qquad r_n = s_n \cdot r_0 + t_n \cdot r_1 \,,$$

for $n = 0{,}1$. [The n^{th}: *remainder*, *quotient*, and *Bézout multipliers* are called r_n, q_n, s_n, t_n.] The LBolt update rule *preserved* relation (‡), in building row n from rows $n-2$ and $n-1$. When we found the index K where $r_K = \text{Gcd}(r_0, r_1)$, this invariance handed us the GCD as a linear-combination of r_0 and r_1.

Boomerangs cannot tile a convex polygon

(Problem from David Gale.) A ***boomerang*** is a non-convex quadrilat(eral); call its [$>\pi$] interior-angle **"thick"**. Conversely, a quadrilat with each angle $\leqslant\pi$ (a **"thin"** angle) is a ***kite***. [So a polygon is convex IFF all its angles are thin.] A dissection of a polygon **P** into *finitely many* quadrilats is a **"*quadritiling* of P"**. [The tiles *need not* be congruent to each other.]

21.1: ??? Boom-Kite Theorem. *Each quadritiling of a convex polygon* **P** *must use a kite.* ◊

21.2: *Fails with "Quad" replaced by "Penta".* Let **P** be the square with vertices $(\pm 2, \pm 2)$. Cut **P** with a polygonal path going from/to

$$(2,2) \to (-1,1) \to (1,-1) \to (-2,-2)\,.$$

This cuts **P** [which is convex] into two non-convex pentagons [which are congruent to each other].

Exer: Each polygon **Q**, convex or not, admits a (finite) tiling by non-convex *penta*gons. □

22.1: Coloring a 99-gon (USAMO1994.2). *Let R,B,Y denote the colors red, blue, yellow, respectively.*

The sides of a 99-gon are initially colored so that, traveling CW (clockwise), consecutive sides are

†: $R,\ B,\ R,\ B,\ \ldots,\ R,\ B,\ R,\ B,\ Y\,.$

Is it possible, still traveling CW, to obtain

‡: $R,\ B,\ R,\ B,\ \ldots,\ R,\ B,\ R,\ Y,\ B$

by a sequence of modifications? A ***modification*** *changes the color of one side (to one of R,B,Y) under the constraint that at no time may two adjacent sides have the same color.* ◊

SOLVED BY: *Tyler A.*, 2014g. *Christopher P.*, *Nate G.*, 2012g. *Ken D.*, 2017g.

The PRATT-NATE INVARIANT is the sum $\langle R,B\rangle + \langle B,Y\rangle + \langle Y,R\rangle$ of the below invariants; i.e, it uses a cyclic ordering of the colors.

The TYLER INVARIANT uses a linear ordering $R<B<Y$; it is the quantity $\langle R,B\rangle + \langle B,Y\rangle - \langle Y,R\rangle$. I.e, it adds-up +1 or -1 for each edge, as follows: Going clockwise around the polygon, traversing an edge adds +1 if the vertex-colors increase; adds -1 if the colors decrease.

Proof. No such modification-sequence exists.

In a coloring, let $\overrightarrow{BY}$ denote the number of BY adjacent-pairs, when traveling CW. In (†), then, $\overrightarrow{BY}$ is 1, and $\overrightarrow{YB} = 0$. Still in (†), $\overrightarrow{RY} = 0$ and $\overrightarrow{YR} = 1$ (because CW from the "end" Y is the "starting" R).

Define the ***discrepancy*** $\langle B,Y\rangle$ as the difference $\overrightarrow{BY} - \overrightarrow{YB}$. In the (†)-coloring, $\langle B,Y\rangle = 1-0 = 1$. And $\langle R,B\rangle$ equals $49-48=1$.

Invariance of discrepancy. Claim: Modifying a coloring does not change $\langle B,Y\rangle$.

For consecutive sides α,β,γ, in order to change color β to β', the α color must *differ* from β and β'. Ditto γ; hence $\alpha \overset{\text{must}}{=\!=} \gamma$.

When this end-color is R: Changing RBR↦RYR alters neither $\overrightarrow{BY}$ nor $\overrightarrow{YB}$.

Blue end-color: Changing BYB↦BRB decrements $\overrightarrow{BY}$ *and* $\overrightarrow{YB}$. So discrepancy $\langle B,Y\rangle$ is unaffected.

Yellow end-color: Changing YRY↦YBY *increments* both $\overrightarrow{BY}$ and $\overrightarrow{YB}$. Hence $\langle B,Y\rangle$ is unchanged.

Impossibility of modifying (†)⇝(‡). Well...

The (†)-coloring: $\langle B,Y\rangle = 1-0 = 1\,.$
The (‡)-coloring: $\langle B,Y\rangle = 0-1 = -1\,.$

Since $-1 \neq 1$, there is no such modification-sequence.♦

22.2: *Observation.* Fix a cyclic order of the colors, say [$R\to B\to Y\to$]. For (†), let's compute the discrepancies:

$$\begin{aligned}\langle R,B\rangle &= 49-48 &= 1\,;\\ \langle B,Y\rangle &= 1-0 &= 1\,;\\ \langle Y,R\rangle &= 1-0 &= 1\,.\end{aligned}$$

They're equal. *Magic? Coincidence?*

EQUAL-DISCREPANCIES CONUNDRUM 22.2: *For each cyclic-order of the colors, and for each coloring, must the three discrepancies be equal?*

If the answer is "yes", does this depend on 99 being a multiple of 3? □

22.3: ?? [99]Modif-Equiv Question. *For $N \in [2..\infty)$, let Ω_N be the set of (legal) colorings of the N-gon. Say two colorings are* ***modif-equivalent*** *if there exists a modification-seq carrying one to the other.*

Which N are ***rotational****?, i.e, each N-coloring is modif-equivalent to itself rotated by one click.*

Which N are ***transitive****?, i.e, each N-coloring is modif-equivalent to each other.* ♢

22.4: ?? [99]Discrepancy-complete Question. *Is the discrepancy-triple a* ***complete invariant****? That is, whenever two colorings have the same discrepancy-triples, they necessarily are modif-equivalent.* ♢

(*Commentary temporarily deleted.*)

23.1: ?? Pentagon (USAMO2011.2). *An integer is assigned to each vertex of a regular pentagon so that they sum to 2011. A* ***move*** *of a solitaire game consists of subtracting an integer β from each of the integers at two neighboring vertices and adding 2β to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount β and the vertices chosen can vary from move to move.) The game is* ***won*** *at a certain vertex if, after some number of moves, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for each choice of the initial integers, there is exactly one vertex at which the game can be won.* ♢

24: Three aces expectation (USAMO 1975.5). *A deck of N playing cards, with three aces, is shuffled "at random" (the $N!$ many orderings are equally-likely). The cards are then turned up one-by-one from the top until the second ace appears. Prove that T, the expected-number of cards to be turned up, is $[N+1]/2$.* ♢

SOLVED BY: *Lizzie [Donna] N-C., 2017g.*

Proof. Consider a particular card in a deck: Let t be its index counting from the top, and b be its index counting from the bottom. Then $t+b$ equals $N+1$.

Apply this to the middle-ace, using T for its expected-index counting from the Top, and B the expectation counting from the Bottom. Then

$$\mathsf{T} + \mathsf{B} = N+1 .$$

The probability distribution on decks has this property: *Turning the deck over, is probability preserving.* Thus $\mathsf{T} = \mathsf{B}$. Hence $\mathsf{T} = [N+1]/2$. ♦

Combinatorial Graphs

[Some, but not all, of these problems use induction.]

For these problems, you should draw pictures of your combinatorial graphs.

25: ?? Desegregation problem. *A* ***coloring*** *of a graph assigns to each vertex either "black" or "white". It is* ***desegregated****, if each vertex has at least one neighbor of the opposite color from his. [Two vertices are* ***neighbors*** *IFF they are connected by an edge.] Prove that each finite connected graph G with $N \geqslant 2$ vertices, admits a desegregated coloring.* ♢

Remark. Find two proofs of this. Can you generalize this problem? □

26.1: N-towns Theorem. *Consider a network of $N \geqslant 1$ towns, each connected to every other town by a one-way*[♡3] *road. Then . . .*

A: There exists a ***universal*** *town; from it, you can bike to each other town (possibly passing through intermediate towns).*

SOLVED BY: *John P., 2011t. Zach N., 2012t. Michael E., 2013t. Lizzie [Donna] N-C., 2017g.*

B: There exists a ***2-universal*** *town; it can access each town using at most two roads [i.e, at most one intermediate town].*

SOLVED BY: *Michael V., Terry T., Alex H., Stephen H., 2011t. Ken D., 2017g.*

C: In a network of $N \geqslant 3$ towns, it is always possible to reverse at most one road so that, now, every town is universal.

SOLVED BY: *Ken D., 2017g.* ♢

[♡3] We have a ***directed graph***; a "***digraph***". This one is a "***complete digraph*** on N vertices"; it has $\binom{N}{2}$ directed-edges, that is, $\frac{1}{2}N[N-1]$ many ***oriented edges***. A complete digraph is called a ***tournament***. (Why?). Exer: *As a fnc of N, how many N-tournaments are there?*

Pf of (A). Proposition $A(1)$ holds; we are already *in* the unique town.

Fixing posint N, consider $N+1$ many towns

26.2: $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_N, \mathbf{B}$. [The **"full network"**.]

We'll induct from the "Green" towns $\mathbf{G}_1, \ldots, \mathbf{G}_N$ [call the roads running between them "green"] to the full network which includes the Brand new "Blue" town $\mathbf{B}$ and its roads. Prop. $A(N)$ gives us a town, WLOG $\mathbf{G}_1$, which can access $\mathbf{G}_1, \ldots, \mathbf{G}_N$ using *only* green roads.

CASE: Road $\mathbf{G}_1$—$\mathbf{B}$ is $\mathbf{G}_1 \to \mathbf{B}$ Suppose the full network has a road *from* $\mathbf{G}_1$ *to* $\mathbf{B}$. Then, in the full network, town $\mathbf{G}_1$ can access all $N+1$ many towns.

CASE: Road $\mathbf{G}_1$—$\mathbf{B}$ is $\mathbf{G}_1 \leftarrow \mathbf{B}$ Town $\mathbf{B}$ accesses $\mathbf{G}_1$ using a new road, whence it can access all the other towns via green roads. ♦

Pf of (B). FTSOC, suppose (26.2) has **no** 2-universal town; necessarily, then, $N \geqslant 1$.

By minimality, WLOG $\mathbf{G}_1$ is 2-universal for the green network [using notation from the proof of (A).] Road $\mathbf{G}_1$—$\mathbf{B}$ must be $\mathbf{B} \to \mathbf{G}_1$; otherwise $\mathbf{G}_1$ would be 2-universal for the full network.

Partition $\mathbf{G}_2, \ldots, \mathbf{G}_N$ as $\mathcal{O} \sqcup \mathcal{W}$, where

$$\mathcal{O} := \{\text{Towns green-reachable in } One \text{ step from } \mathbf{G}_1\};$$

$$\mathcal{W} := \left\{\begin{matrix}\text{Towns green-reachable in } tWo \text{ steps} \\ \text{from } \mathbf{G}_1\text{, but not in one step}\end{matrix}\right\}.$$

CASE: $\exists \mathbf{T} \in \mathcal{O}$ with $\mathbf{T} \to \mathbf{B}$ In consequence, $\mathbf{G}_1 \to \mathbf{T} \to \mathbf{B}$ is a 2-step path. Thus $\mathbf{G}_1$ is 2-universal for the full network.

CASE: $\forall \mathbf{T} \in \mathcal{O}$, the road goes $\mathbf{T} \leftarrow \mathbf{B}$ I claim that $\mathbf{B}$ is 2-universal for the full network.

Recall that $\mathbf{B}$ reaches $\mathbf{G}_1$, and all of $\mathcal{O}$, in one step. Consider a town $\widetilde{\mathbf{G}} \in \mathcal{W}$. It is one (green-)step away from **some** town $\mathbf{T}$ in $\mathcal{O}$, since $\widetilde{\mathbf{G}}$ is *two* (green-)steps from $\mathbf{G}_1$. Thus $\mathbf{B} \to \mathbf{T} \to \widetilde{\mathbf{G}}$ is a 2-step path. ♦

Finding dials to turn. What degrees of freedom do we have that we haven't exploited yet?

What corollaries do we get from (A)? from (B)?

What operations preserve "digraph-ness"? □

Finding dials to turn. Given a town digraph Γ, let Γ' mean the digraph obtained by reversing all the roads. (Thus Γ'' is the original graph). This reversed graph, Γ', is itself a complete graph, so the the above theorems apply to it.

What we called a "universal town", we really should call ***start-universal***. Analogously, an ***end-universal*** town can be reached *from* each town. Define also a ***start-2-universal*** town (from which we can get to every other with a length⩽2 path), and an ***end-2-universal*** town. □

27.1: ?? Polygamy Problem. *A polygamous community comprises 100 women and 101 men. Every man has at least one wife. Prove that there is a married couple such that the wife has more husbands than the husband has wives.* ♢

History. Dear Jonathan, ...Jack Feldman just came through here and we were given a problem which should be right up my alley but I haven't been able to crack it. [The above problem was stated.]

(jk: I started my reply "David, how about this..." but I didn't record *which* David! [27Feb2017: My guess is that it was David Gale.])

Proof. Consider a bipartite graph on vertex-set pair (Boys, Girls) such that *–for the sake of contradiction–*

$*$: *each married couple* $\mathbf{b}{:}\mathbf{g}$ *has* $W_\mathbf{b} \geqslant H_\mathbf{g}$,

where $W_\mathbf{b}$ is the number of wives boy $\mathbf{b}$ has, and $H_\mathbf{g}$ is the number of husbands girl $\mathbf{g}$ has.

Banish! all unmarried woman; this changes none of the spouse-counts.

Now put weights (non-negative rationals) on the edges as follows. On each edge emanating from girl $\mathbf{g}$, put weight $1/H_\mathbf{g}$. Thus, the sum of the weights at each girl-vertex is 1.

Boy $\mathbf{b}$ has $W_\mathbf{b}$ many wives. *Suppose* you were to put weight $1/W_\mathbf{b}$ on each of his edges. Then the total weight emanating from him *would be* 1.

However, for each of his wives $\mathbf{g}$, the value of $H_\mathbf{g}$ is less-equal $W_\mathbf{b}$. So $1/H_\mathbf{g}$, the *actual* weight on the edge, is greater-equal $1/W_\mathbf{b}$. Hence the sum of the edge-weights emanating from $\mathbf{b}$ is *at least* 1

Let T be the total weight of all the edges. Then

$$\begin{aligned} {}^{\#}\text{GIRLS} \cdot 1 &= T \\ &= \sum_{\mathbf{b}} \text{WeightEmanating}(\mathbf{b}) \geqslant {}^{\#}\text{BOYS} \cdot 1. \end{aligned}$$

Thus ${}^{\#}$GIRLS $\geqslant$ ${}^{\#}$BOYS; contradiction. ♦

Using Extema

Here is an example argument.

28: Obs. *Consider two vertices, $\mathbf{v}, \mathbf{w}$, in a connected (possibly infinite) graph. Prove there exists a path between them, with no repeated-vertex.* ◊

Proof. Take a minimum-length path. Etc. ♦

Note: The above DESEGREGATION PROBLEM can be done via an extremal argument.

.

Bashful Boyfriends Story. For a natnum N we have two sets of points, with $|\mathbf{B}| = N = |\mathbf{G}|$, and $\mathbf{B} \cup \mathbf{G}$ comprises $2N$ distinct points.

Set $\mathbf{B}$ comprises the boys' homes, $\mathbf{G}$ the girls' homes. Each boy wants to build a straight sidewalk from his home to his girlfriend's. Boys are bashful, hence don't want to meet other boys when girlfriend-visiting. So the boys want their sidewalks to be disjoint. Indeed, the boys are so bashful that they are willing to change girlfriends in order to not meet another boy.□

29: ⁇ Bashful Boyfriends. *In the plane, consider sets $|\mathbf{B}| = N = |\mathbf{G}|$, with $|\mathbf{B} \cup \mathbf{G}| = 2N$ and no-three-points-colinear. Then there exists a bijection $\mathcal{D}{:}\mathbf{B} \hookrightarrow\!\!\!\!\to \mathbf{G}$ such that the collection of line-segments $\left\{ \text{Seg}(b, \mathcal{D}(b)) \;\middle|\; b \in \mathbf{B} \right\}$ is pairwise-disjoint.* ◊

Remark. Can you come up with an extremal proof? An induction proof? Does the result hold if $|\mathbf{B}| = \infty = |\mathbf{G}|$ [the smallest infinity]? Can *no-three-points-colinear* be weakened? Generalize to 3 dimensions?□

Number theory

The first few problems can be approached via factoring, or by modular arithmetic.

30.1: $a^2 - b^4$ Problem (HMMT2009.1.algebra). *Suppose posints a, b have $a^2 - b^4 = 2009$. Compute $a + b$.* ◊

SOLVED BY: *Yifei L.*, 2017g.

Ans. Sum $a + b$ equals 47, since $a = 45$ and $b = 2$. □

Proof. Factoring gives

$$[a - b^2] \cdot [a + b^2] = 7 \cdot 7 \cdot 41\,.$$

As $a + b^2 > 0$, so $a - b^2 > 0$. Since $a + b^2 > a - b^2$, the three possibilites are:

$$\begin{aligned} a + b^2 &= 2009 \text{ or } 41{\cdot}7 \text{ or } 49, && \textit{as} \\ a - b^2 &= 1 \text{ or } 7 \text{ or } 41. && \textit{Their average is} \\ a &= 505 \text{ or } 21{\cdot}7 \text{ or } 45. && \textit{Subtraction gives} \\ b^2 &= 504 \text{ or } 20{\cdot}7 \text{ or } 4. \end{aligned}$$

But $504 = 4 \cdot 126$ is not a square, since 126 isn't. Nor is $20 \cdot 7$. Consequently $b = 2$, hence $a = 45$. ♦

31.1: Power-sum Problem. *For each odd $n \geqslant 3$, the integer $f(n) := \frac{1}{2} \cdot [15^n + 19^n]$ is composite.* ◊

SOLVED BY: *Class of*, 2017g.

1st proof. Induction on posints n easily gives this identity:

$$\begin{aligned} x^n - y^n &= [x - y] \cdot H_n(x, y)\,, \quad \text{where} \\ H_n(x, y) &:= x^{n-1}y^0 + x^{n-2}y^1 + x^{n-3}y^2 + \ldots \\ &\qquad \ldots + x^1 y^{n-2} + x^0 y^{n-1}\,. \end{aligned}$$

Replacing y by $-z$ gives:

When n is odd: $x^n + z^n = [x + z] \cdot H_n(x, -z)$.

Value $H_n(15, -19)$ is an *integer*, since it is a sum of integers. Hence $f(\text{odd } n)$ factors as

$$\tfrac{1}{2}[15 + 19] \cdot H_n(15, -19) \overset{\text{note}}{=\!=} 17 \cdot H_n(15, -19)\,.$$

So: For each posodd n, sum $f(n)$ is a multiple of 17.

Finally

31.2: *To see that each $f(\text{odd } n \geqslant 3)$ is a non-trivial multiple of 17, note that $f(3)$ exceeds 17, and f is an increasing function.* ♦

2nd proof. Here, let $\equiv$ mean "congruent mod-17".

For each natnum n, note

$$15^n + 19^n \equiv [-2]^n + 2^n = 2^n \cdot [[-1]^n + 1] .$$

Henceforth, n is odd. Thus $[[-1]^n + 1]$ is zero, so $[15^n + 19^n]$ is divisible by 17. This sum is also divisible by 2, since 15 and 19 have the same parity. Since $2 \perp 17$ [they are coprime], it follows that $[15^n + 19^n]$ is divisible by $[2 \cdot 17]$. Hence $f(n)$ is divisible by 17.

Finally, finish the proof as in (31.2), above. ♦

32.1: Power-4Term Problem. *For natnum n, define*

$$S_n := 3^n + 7^n + 11^n - 6^n .$$

Prove, for odd posints n, that S_n is composite. ◊

SOLVED BY: *Ken D., 2017g.*

1st proof. Working mod-3, observe that

$$S_n \equiv_3 0 + 1^n + [-1]^n - 0 .$$

For odd n, then, S_n is a multiple of 3; indeed, a *non-trivial* multiple, since $S_n \geqslant 11^n \geqslant 11 > 3$, once $n \geqslant 1$. ♦

2nd proof. Working mod-5, note

$$S_n \equiv_5 [-2]^n + 2^n + 1^n - 1^n$$
$$= 2^n \cdot \Big[[-1]^n + 1\Big] .$$

So for odd n, our S_n is a multiple of 5; indeed, a *non-trivial* multiple, since $S_n \geqslant 11^n \geqslant 11 > 5$. ♦

33.1: PoT-plus-Square Question. *(Dis)Prove: There are at least seven primes p such that sum*

$$f(p) := 2^p + p^2$$

is prime. ◊

Non-examples. Note 5 is prime, but $f(5) = 57 = 19 \cdot 3$ is composite. In the other direction, the composite 15 yields $f(15) = 32993$, which is prime. Finally, $f(1) = 3$ is prime but the unit 1, alas, is not. □

SOLVED BY: *Keven H., 2013t. Rabon M., 2017g.*

Soln, false. Well, $f(2) = 8$, which is not prime. Note $f(3) = 17$, which *is* prime. We will show that no primes other than 3 produce a prime. Letting $\equiv$ mean $\equiv_3$, we will prove:

For each odd posint $n \perp 3$, nec. $f(n) \equiv 0$.

For since n is odd, our $2^n \equiv [-1]^n = -1$. And $n \perp 3$, so n is $\equiv$ to one of ± 1. Hence $n^2 \equiv 1$. Consequently, $f(n) \equiv -1 + 1 = 0$. ♦

34: *bcq-bc* Problem (USAMO 1976.3). *Determine all integral solutions of*

$$\dagger: \quad b^2 + c^2 + q^2 = b^2 \cdot c^2 .$$ ◊

SOLVED BY: *Lizzie [Donna] N-C., 2017g.*

Answer. The only solution is $b = c = q = 0$. □

Proof. Since $[b = 0 = c]$ implies $[q = 0]$, WLOG $\boxed{b \neq 0}$. Hence there is a largest natnum K st. 2^K divides both b and c. Then $[2^K]^2$ must divide q^2, so $2^K \bullet| q$. Rename b to $2^K b$, ditto for c and q. Dividing (†) by $[2^K]^2$ produces

$$\dagger\dagger: \quad b^2 + c^2 + q^2 = 4^K \cdot b^2 c^2 .$$

The improvement is: *At least one of b, c is odd.*

CASE: $K > 0$ Looking mod-4 gives

$$*: \quad b^2 + c^2 + q^2 \equiv 0 .$$

But mod-4 squares are either 0 or 1. So the only $(*)$-soln is $0+0+0 \equiv 0$, forcing both b and c even, and contradicting that K was maximal. Thus $\boxed{K = 0}$.

CASE: $K = 0$ We can now rewrite (††) as

$$\ddagger: \quad q^2 + 1 = [b^2 - 1] \cdot [c^2 - 1] .$$

Since K was maximal, at least one of b,c is odd, say b, and thus $[b^2 - 1] \equiv_4 0$. Hence (‡) becomes $q^2 + 1 \equiv_4 0$. And that has no solution. ♦

35a: ??? 2-to-2 Theorem (USAMO1991). *Sequence*

$$\vec{\mathbf{b}} := \left(1, 2, 2^2, 2^{[2^2]}, 2^{\left[2^{[2^2]}\right]}, \ldots\right)$$

can be recursively defined as

$$b_0 := 1, \quad \text{and} \quad b_{t+1} := 2^{b_t} ,$$

for $t = 0, 1, 2, \ldots$. Then for each modulus M, sequence $\vec{\mathbf{b}}$ is eventually mod-M constant. ◊

Rings. Consider a commutative ring $(\Gamma, +, 0, \cdot, 1)$. An elt $\alpha \in \Gamma$ is a ***zero-divisor*** (abbrev **ZD**) if there exists a *non-zero* $\beta \in \Gamma$ st. $\alpha\beta = 0$. In contrast, an element $u \in \Gamma$ is a ***unit*** if $\exists w \in \Gamma$ st. $u \cdot w = 1$. (This w is the "multiplicative inverse" of u, is unique, and is often written u^{-1}.) Exer: In an arbitrary ring Γ, the set $\mathrm{ZD}(\Gamma)$ is *disjoint* from $\mathrm{Units}(\Gamma)$.

An element α is:

i: Γ-***irreducible*** if α is a non-unit, non-ZD, such that for each Γ-factorization $\alpha = x \cdot y$, either x or y is a Γ-unit.

ii: Γ-***prime*** if α is a non-unit, non-ZD, such that for each pair $c,d \in \Gamma$: If $[c \cdot d] \bullet\!\!| \, \alpha$ then *either* $c \bullet\!\!| \, \alpha$ or $d \bullet\!\!| \, \alpha$.

Two ring-elements α and β are ***associates***, written $\alpha \overset{\text{as}}{\sim} \beta$, if $\alpha \, |\!\!\bullet \, \beta$ *and* $\alpha \bullet\!\!| \, \beta$ [i.e, $\alpha \in \beta\Gamma$ and $\beta \in \alpha\Gamma$]. They are ***strong associates*** if there exists a unit u st. $\beta = u\alpha$. Exer: Prove *Strong-Assoc* $\Rightarrow$ *Assoc*.

Exer: Ring $\mathbb{Z}_N$ has no irreducible elements, since $\mathbb{Z}_N$ is the disjoint-union $\mathrm{ZD} \sqcup \mathrm{Units}$.

Exer: PRIME⇒IRRED. However there are rings[♡4] with irreducible elements p which are nonetheless <u>not</u> prime.

Examples. Every ring has the "trivial zero-divisor" —zero itself. The ring of integers doesn't have others. In contrast, the non-trivial zero-divisors of $\mathbb{Z}_{12}$ comprise $\{\pm 2, \pm 3, \pm 4, 6\}$.

In $\mathbb{Z}$ the units are ± 1. But in $\mathbb{Z}_{12}$, the ring of integers mod-12, the set of units, $\Phi(12)$, is $\{\pm 1, \pm 5\}$. In the ring $\mathbb{Q}$ of rationals, *each* non-zero element is a unit. In the ring $\mathbb{G} := \mathbb{Z} + i\mathbb{Z}$ of ***Gaussian integers***, the units group is $\{\pm 1, \pm i\}$. [Aside: $\mathrm{Units}(\mathbb{G})$ is cyclic, generated by i. And $\mathrm{Units}(\mathbb{Z}_{12})$ is not cyclic. For which N is $\Phi(N)$ cyclic?] □

36.1: *Example.* The set of ***Leah-numbers*** is

$$\mathcal{L} := \{1, 4, 7, 10, \dots\} = \{n \in \mathbb{Z}_+ \mid n \equiv_3 1\}.$$

Ok, so $\mathcal{L}$ is not a ring. But $\mathcal{L}$ is sealed under multiplication, has no ZDs, and the only $\mathcal{L}$-unit is 1; we can make sense of "$\mathcal{L}$-irreducible" and "$\mathcal{L}$-prime".

Factoring 100, these two Leah-factorizations

$$4 \cdot 25 = 100 = 10 \cdot 10,$$

show that none of $4, 10, 25$ is Leah-prime. Yet each *is* Leah-irreducible. [This, since their only non-trivial $\mathbb{N}$-factorizations use non-Leah numbers]. □

36.2: ⁇ Leah problem. *Given a "target" $\mathsf{T} \in [2..\infty)$ write its usual $\mathbb{N}$-prime factorization,*

36.3: $$\mathsf{T} = p_1^{E_1} \cdot p_2^{E_2} \cdot \ldots \cdot p_L^{E_L},$$

with $p_1, \dots, p_L$ distinct, and each E_ℓ a posint.

In terms of (36.3), give an IFF-characterization of:

i: *When T is Leahian.*

ii: *When T is Leah-irreducible.*

iii: *When T is Leah-prime.*

In particular, are there ∞ly many Leah-primes? –or any at all? [*Hint:* Look up Dirichlet's theorem on arithmetic progressions.] ♢

SOLVED BY: *Keven H., 2013t.*

37: ⁇ The $x + \frac{1}{x}$ Theorem. *A real [or complex] number x is **good** if $x + \frac{1}{x}$ is an integer. Prove, for each posint N, that x^N is good, i.e, $x^N + \frac{1}{x^N}$ is integral.* ♢

E.g: Let $F := \sqrt{5}$ and $y := \frac{3+F}{2}$. Then $\frac{1}{y}$ equals

$$\frac{2}{3+F} = \frac{2 \cdot [3-F]}{9-5} = \frac{3-F}{2}.$$

Hence $y + \frac{1}{y}$ equals 3, so y is good. The Recip. Thm implies that $y^2 \overset{\text{note}}{=\!=} \frac{7+3F}{2}$ is good; *is it?* □

Pf of (37), start. For $N \in \mathbb{N}$, let $S_N := [x^N + \frac{1}{x^N}]$. Now . . . [*Hint:* Think binomial coefficents[♡5].] ♦

SOLVED BY: *John P., 2011t.*

[♡4] Consider the ring, Γ, of polys with coefficients in $\mathbb{Z}_{12}$. There, $x^2 - 1$ factors as $[x-5][x+5]$ and as $[x-1][x+1]$ Thus none of the four linear terms is prime. Yet each is Γ-irreducible. (Why?) This ring Γ has zero-divisors (yuck!), but there are natural subrings of $\mathbb{C}$ where Irred⇏Prime.

[♡5] For a natnum n, use "$n!$" to mean "n ***factorial***"; the product of all positive-integers less-equal n. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1$ and $1! = 1$.

The ***binomial coefficient*** $\binom{7}{3}$, read "7 choose 3", means

Similar to. `~/Problems/Putnam/1959A1Putnam-.latex` □

Proof of (37). The Elem. Binomial Thm says

$$\Big[x+\frac{1}{x}\Big]^N = \sum_{\substack{j,k\in\mathbb{N}\\ j+k=N}} \binom{N}{j,k}\cdot x^j\cdot\Big[\frac{1}{x}\Big]^k = \sum_{\text{same}} \tbinom{N}{j,k}\cdot x^{j-k}\,.$$

We now use the $\binom{N}{j,k} = \binom{N}{k,j}$ identity.

Case: $N = 1+2H$ is odd. Restate the above, using difference $d := k-j = [N-j]-j = N-2j$:

$$[S_1]^{1+2H} = \sum_{j=0}^{H} \tbinom{N}{j}\,[x^d + x^{-d}] = \sum_{j=0}^{H} \tbinom{N}{j} S_{N-2j}\,.$$

The $j{=}0$ term yields S_N. Solving,

$$\dagger:\qquad S_N = [S_1]^N - \sum_{j=1}^{H} \tbinom{N}{j}\cdot S_{N-2j}\,.$$

Case: $N = 2H$ is even. Again, with $d := N-2j$, the Binomial theorem gives

$$[S_1]^{2H} = \tbinom{N}{H}x^{H-H} + \sum_{j=0}^{H-1} \tbinom{N}{j}\,[x^d + x^{-d}]\,.$$

Isolating S_N tells us that

$$\ddagger:\quad S_N = [S_1]^N - \tbinom{N}{H} - \sum_{j=1}^{H-1} \tbinom{N}{j} S_{N-2j}\,.$$

By hyp., S_1 is an integer –so to finish by "strong" induction, fix an $N \in [2\,..\,\infty)$ and assume that values $S_1, S_2, \ldots, S_{N-1}$ are integers. Since binom-coeffs are also integers, both RhS(†) and RhS(‡) are integral. ♦

38.1: ⁇ Squarish problem. *Call $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_L)$ an **L-bit-tuple** if each ε_k is +1 or −1. Integer T is **squarish** if there exists a natnum L and an L-bit-tuple $\vec{\varepsilon}$ st.*

$$T = \sum_{k=1}^{L} [\varepsilon_k \cdot k^2]\,.$$

Prove that every integer is squarish. ◊

Solved by: *John P., 2011t.*

---

the number of ways of choosing 3 objects from 7 distinguishable objects. If we think of putting these objects in our left pocket, and putting the remaining 4 things in our right pocket, then we write the coefficient as $\binom{7}{3,4}$. [Read as "7 choose 3-comma-4."] Note that $\binom{7}{0} = \binom{7}{0,7} = 1$. Also note this identity:

$$[x+y]^N = \sum_{j+k=N} \tbinom{N}{j,k}\cdot x^j y^k\,,$$

where (j,k) ranges over all *ordered* pairs of natural numbers with sum N.

In general, for natnums $N = K_1 + \ldots + K_L$, the ***multinomial coefficient*** $\binom{N}{K_1,K_2,\ldots,K_L}$ means the number of ways of partitioning N different things, by putting K_1 of them in pocket 1 and K_2 of them in pocket 2, and so on. Easily

$$\binom{N}{K_1,K_2,\ldots,K_L} = \frac{N!}{K_1!\cdot K_2!\cdots K_L!}\,.$$

And $[x_1 + \cdots + x_L]^N$ indeed equals the sum of

$$\tbinom{N}{K_1,\ldots,K_L}\cdot {x_1}^{K_1}\cdot {x_2}^{K_2}\cdots {x_L}^{K_L}\,,$$

taken over all natnum-tuples $\vec{K}{=}(K_1,\ldots,K_L)$ that sum to N.

39: Recip-sum-is-one (USAMO1978.3). *An integer G is **good** if there exist posints $\sigma_1, \ldots, \sigma_N$ (not necessarily distinct) with*

$$*: \quad \Big[\sum_{j=1}^{N} \sigma_j\Big] = G \quad \text{and} \quad \Big[\sum_{j=1}^{N} \frac{1}{\sigma_j}\Big] = 1\,.$$

Given that $\Gamma \supset [33\,..\,73]$, prove that $\Gamma \supset [33\,..\,\infty)$, where $\Gamma \subset \mathbb{Z}_+$ denotes the set of good numbers. ♢

Solved by: *Rabon M., 2017g.*

Call $(*)$ a "**(good) decomposition** of G".

39a: Lemma. *(All squares, 1, 4, 9, 25, …, are good, but our proof doesn't need this.) These assertions[♡6] hold:*

i: Consider a decomposition $(*)$, and good numbers $H_1, \ldots, H_N$. Then the sum $\sum_{j=1}^{N} \sigma_j H_j$ is good.

In particular, $[G, H \text{ good}] \Rightarrow [G \cdot H \text{ is good}]$; simply apply the above with each H_j equaling H.

ii: Consider posints A, B such that $8 + 2A \leqslant B$. If interval $[A\,..\,B)$ is all good, then interval $[A\,..\,2+2B)$ is all good. ♢

Looking ahead. Note (39a.i) is a closure property of Γ. It will turn out that

$$\text{39b:} \quad \begin{aligned} \Gamma &= \mathbf{S} \sqcup [24\,..\,\infty)\,, \quad \text{where the sporadic set is} \\ \mathbf{S} &:= \{1, 4, 9, 10, 11, 16, 17, 18, 20, 22\}. \end{aligned} \quad \square$$

A little Lisp program computed all decompositions of good numbers, with these results:

G	Decomps(G).
1	$\{1\}$.
4	$\{2,2\}$.
9	$\{3,3,3\}$.
10	$\{2,4,4\}$.
11	$\{2,3,6\}$.
16	$\{4,4,4,4\}$.
17	$\{3,4,4,6\}$.
18	$\{3,3,6,6\}$.
20	$\{2,6,6,6\}$.
22	$\{3,3,4,12\}$ $\{2,4,8,8\}$ $\{2,5,5,10\}$.
24	$\{2,4,6,12\}$.
25	$\{5,5,5,5,5\}$.
26	$\{4,4,6,6,6\}$.
27	$\{3,6,6,6,6\}$.
28	$\{4,4,5,5,10\}$ $\{4,4,4,8,8\}$.
29	$\{2,3,12,12\}$ $\{3,4,6,8,8\}$ $\{3,5,5,6,10\}$.
30	$\{2,3,10,15\}$ $\{4,4,4,6,12\}$.
31	$\{2,4,5,20\}$ $\{3,4,6,6,12\}$.
32	$\{2,3,9,18\}$.
33	$\{3,5,5,5,15\}$ $\{3,3,9,9,9\}$.
34	$\{3,3,8,8,12\}$ $\{2,8,8,8,8\}$.
35	$\{2,6,9,9,9\}$ $\{3,4,4,12,12\}$.
36	$\{6,6,6,6,6,6\}$ $\{2,6,8,8,12\}$ $\{3,3,6,12,12\}$ $\{3,4,4,10,15\}$.
⋮	⋮

Proof of (39a.i). Goodly-decompose H_j as

$$H_j = \sum_{k=1}^{L_j} \eta_j(k)\,, \quad \text{with each } \eta_j(k) \in \mathbb{Z}_+.$$

Summing $[L_1 + \ldots + L_N]$ many posints gives

$$\sum_{j=1}^{N} \sum_{k=1}^{L_j} [\sigma_j \cdot \eta_j(k)] = \sum_{j=1}^{N} \sigma_j H_j\,.$$

And summing their reciprocals,

$$\sum_{j=1}^{N} \sum_{k=1}^{L_j} \frac{1}{\sigma_j \cdot \eta_j(k)} = \sum_{j=1}^{N} \frac{1}{\sigma_j} \sum_{k=1}^{L_j} \frac{1}{\eta_j(k)} = \sum_{j=1}^{N} \Big[\frac{1}{\sigma_j} \cdot 1\Big] \stackrel{\text{note}}{=\!=} 1\,.$$ ♦

Proof of (39a.ii). As H ranges over $[A\,..\,B)$, apply (i) with decomp $2 + 2 = 4$, to produce good number $2 \cdot 1 + 2 \cdot H$. We get a set $\mathcal{E}$, comprising the even numbers in

$$[2 + 2A\,..\,2 + 2B)\,.$$

Decomp $3 + 6 + 2 = 11$ produces $3 \cdot 1 + 6 \cdot 1 + 2 \cdot H$, i.e, $9 + 2H$. We obtain $\mathcal{D}$, the odd numbers in

$$[9 + 2A\,..\,9 + 2B)\,.$$

[♡6] As examples, here are some good decompositions:

$$4 = 2 + 2, \quad 9 = 3 + 3 + 3,$$
$$10 = 2 + 4 + 4, \quad 11 = 2 + 3 + 6\,.$$

Courtesy (i), integers $2 \cdot 4 + 2 \cdot 4 = 16$ and $3 \cdot 1 + 3 \cdot 1 + 3 \cdot 4 = 18$ are good. Also:

$$\begin{aligned} 2 \cdot 9 + 3 \cdot 11 + 6 \cdot 4 &= 75\,, \\ 2 \cdot 10 + 3 \cdot 11 + 6 \cdot 4 &= 77\,, \\ 2 \cdot 11 + 3 \cdot 11 + 6 \cdot 4 &= 79\,, \\ \text{and} \quad 9 \cdot 9 &= 81\,. \end{aligned}$$

.

To see that union $\mathcal{E} \cup \mathcal{D}$ includes all of

$$\mathcal{I} \quad := \quad [8 + 2A \,..\, 2 + 2B)\,,$$

we need but verify that $9 + 2A \leqslant 2 + 2B$, i.e, that $7 + 2A \leqslant 2B$. This is certainly implied by the given $\boxed{8 + 2A \leqslant B}$.

Lastly, to see that $[A \,..\, B) \cup \mathcal{I}$ is one large interval, again inequality $\boxed{8 + 2A \leqslant B}$ suffices. ♦

Proof of (39). Let $A := 33$, $B_0 := 74$ and, for each natnum n, let

$$B_{n+1} \quad := \quad 2 + [2B_n]\,.$$

Note $8 + 2A = 8 + 66 = B_0 \leqslant B_n$. Hence each A, B_n pair satisfies the inequality of (ii). Thus

$$\forall n \in \mathbb{N}: \quad [A \,..\, B_n) \text{ good} \implies [A \,..\, B_{n+1}) \text{ good}\,.$$

We were given that $[A \,..\, B_0)$ is good. By induction on n, then, each $[A \,..\, B_n)$ is good. Consequently, their union

$$\bigcup_{n=1}^{\infty} [33 \,..\, B_n) \quad \stackrel{\text{note}}{=\!=} \quad [33 \,..\, \infty)$$

is all good. ♦

39c: ⁇ #Decomps to ∞? *For a good $\boldsymbol{G}$, let $\mathcal{N}(\boldsymbol{G})$ be the number of decompositions of $\boldsymbol{G}$. Do there exist $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots$ so that $\mathcal{N}(\boldsymbol{G}_k) \nearrow \infty$ as $k \to \infty$?* ♢

39d: ⁇ Length-difference to ∞? *Let $L(\boldsymbol{H})$ be the longest decomp-length of $\boldsymbol{H}$, and $S(\boldsymbol{H})$ the shortest. $\exists\ \boldsymbol{H}_1, \boldsymbol{H}_2, \ldots$ with $[L(\boldsymbol{H}_k) - S(\boldsymbol{H}_k)] \nearrow \infty$?* ♢

39e: *Egyptian fractions?* Is there a connection to https://en.wikipedia.org/wiki/Egyptian_fraction □

40: ⁇ Odd-divisor Fibonacci. *Arbitrary posints f_0 and f_1 determine an **oddish** sequence $\vec{\mathbf{f}}$, defined thereafter by letting f_n be the largest odd divisor of $f_{n-2} + f_{n-1}$.*

Prove that $\vec{\mathbf{f}}$ is eventually-constant, and determine what this constant $C = C(f_0, f_1)$ is. ♢

Remark. Given a posint $F = 2^e \cdot D$, where $e \in \mathbb{N}$ and D is odd, define $[\![F]\!]$ to be this D. Thus

$$[\![f_{n-2} + f_{n-1}]\!] \quad =: \quad f_n$$

is the update rule. □

41a: ⁇ Integer-product seq. Thm (USAMO 2009.6). *Suppose $\vec{\mathbf{s}} = (s_0, s_1, s_2, \ldots)$ is an infinite, nonconstant sequence [i.e, not $s_0 = s_1 = s_2 \cdots$] of rational numbers. Suppose $\vec{\mathbf{t}}$ is also an infinite, nonconstant, rational sequence with the property that*

†: *For all j and k:* Product $[s_j - s_k] \cdot [t_j - t_k]$ is an integer.

Prove that there exists a rational number $r \neq 0$ st.

‡: *For all j and k:* Values $[s_j - s_k]/r$ and $[t_j - t_k] \cdot r$ are integers. ♢

Misc. Problems

These problems are "routine" *given* the right tools, e.g, calculus, binomial coeffs, algebra identities, complex numbers.

42.1: Sum-of-Tan (HMMT2009.4.gen). *Angles x, y satisfy that*

$$\tan(x) + \tan(y) = 4\,, \quad \textit{and} \quad \cot(x) + \cot(y) = 5\,.$$

Compute $\tan(x + y)$. ♢

Solved by: *Ken D., 2017g.*

Proof. Let t, s, c denote the tan, sin, cos of angle x, and use T, S, C for angle y. So

$$4 = t + T = \frac{s}{c} + \frac{S}{C} = \frac{sC + Sc}{cC} \stackrel{\text{note}}{=\!=} \frac{\sin(x+y)}{cC}$$

by the formula for the sine of a sum. Similarly,

$$5 = \frac{1}{t} + \frac{1}{T} = \frac{c}{s} + \frac{C}{S} = \frac{cS + Cs}{sS} = \frac{\sin(x+y)}{sS}\,.$$

Cross-multiplying, then subtracting, gives

$$\sin(x+y) \cdot \left[\tfrac{1}{4} - \tfrac{1}{5}\right] = cC - sS = \cos(x+y)\,.$$

Thus

$$\tan(x+y) = \frac{1}{\frac{1}{4} - \frac{1}{5}} = \frac{4 \cdot 5}{5 - 4} = 20\,.$$ ♦

43: Factorial-cosine limit (*Domain specific*). *With n taking on values $1,2,3,\ldots$, prove that limit*

$$L \quad := \quad \lim_{n\to\infty} \cos(n!\cdot 2\boldsymbol{\pi}\mathsf{e})$$

exists, and compute it. ♢

SOLVED BY: *Daniel B. & Rabon M., 2017g.*

Pf. We'll show that $\boxed{L=1}$. Recall, for $|r|<1$, that

$$*: \qquad \sum_{d=1}^{\infty} r^d \quad = \quad r\cdot\frac{1}{1-r} \quad = \quad \frac{r}{1-r}\,.$$

Define product P_N to be

$$N!\cdot\mathsf{e} \stackrel{\text{note}}{=\!=} N!\cdot\sum_{j=0}^{\infty}\frac{1}{j!} \;=\; \text{Integer}+S_N$$

$$\text{where} \qquad S_N \;:=\; \sum_{j=N+1}^{\infty}\frac{N!}{j!}\,.$$

Happily, cos() has period $2\boldsymbol{\pi}$, thus

$$\cos(2\boldsymbol{\pi}\cdot[N!\cdot\mathsf{e}]) \quad = \quad \cos(2\boldsymbol{\pi}\cdot S_N)\,.$$

So to prove that $L=1$, ISTShow that $S_N\to 0$, since cos() is continuous at zero, and cos(0) equals 1.

With d the difference $d:=j-N$, we can upper-bound as

$$S_N \;\leqslant\; \sum_{d=1}^{\infty}\left[\tfrac{1}{N+1}\right]^d \stackrel{\text{by }(*)}{=\!=} \frac{1/[N+1]}{1-\frac{1}{N+1}} \;=\; \frac{1}{N}\,.$$

Hence $0\leqslant S_N\to 0$. ♦

44.1: Coefficient-Sum (HMMT2009.2.algebra). *Let S be the sum of all the real coefficients of the expansion of $[1+\boldsymbol{i}x]^{2009}$. What is $\log_2(S)$?* ♢

SOLVED BY: *Ken D., 2017g.*

Soln. Let $N:=2009$ and $\boldsymbol{\tau}:=\sqrt{2}$. Plugging $x{=}1$ in the expansion gives the sum of *all* the coefficients. Hence[♡7]

$$2S \quad = \quad [1+\boldsymbol{i}]^N + [1-\boldsymbol{i}]^N\,.$$

Unit vectors $u:=[1+\boldsymbol{i}]/\boldsymbol{\tau}$ and $w:=[1-\boldsymbol{i}]/\boldsymbol{\tau}$ allow

$$*: \; 2S \quad = \quad [u\boldsymbol{\tau}]^N + [w\boldsymbol{\tau}]^N \;=\; [u^N + w^N]\cdot\boldsymbol{\tau}^N\,.$$

Note $u^8=1=w^8$. And $N\equiv_8 1$. So u^N+w^N equals $u+w$ equals $\frac{2}{\boldsymbol{\tau}}=\boldsymbol{\tau}$. Hence $(*)$ says $\boldsymbol{\tau}^2 S \;=\; \boldsymbol{\tau}\cdot\boldsymbol{\tau}^N$, i.e $S=\boldsymbol{\tau}^{N-1}=2^{1004}$. So $\log_2(S)=1004$. ♦

[♡7] Too cryptic. What sentence should have come before the "Hence"?

45.1: Reciprocal Sum (HMMT2009.5.algebra). *With A,B,C denoting the roots of cubic $f(x):=x^3-x+1$, compute the sum*

$$\frac{1}{A+1}+\frac{1}{B+1}+\frac{1}{C+1}\,.$$ ♢

SOLVED BY: *Yifei L., 2017g.*

Soln. We seek the RRS, ***reciprocal-root sum***, of translated polynomial

$$\dagger: \qquad g(x) \;:=\; f(x{-}1) \stackrel{\text{note}}{=\!=} x^3-3x^2+2x+1\,.$$

Consider a general factored cubic

$$\begin{aligned} h(x) \;&:=\; x^3 - Sx^2 + Dx - T \\ &=\; [x-a]\cdot[x-b]\cdot[x-c]\,. \end{aligned}$$

The coeffs are symmetric-polynomials in the roots:

$$\begin{aligned} S \;&=\; a+b+c\,; \\ D \;&=\; bc+ac+ab\,; \\ T \;&=\; abc\,. \end{aligned}$$

[Mnemonic: Single, Double, Triple.] Thus

$$\ddagger: \quad \mathrm{RRS}(h) \stackrel{\text{def}}{=\!=} \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{bc+ac+ab}{abc} = \frac{D}{T}\,.$$ ♦

The g in $(\dagger)$ has $D=2$ and $T=\text{-}1$. Hence,

$$\frac{1}{A+1}+\frac{1}{B+1}+\frac{1}{C+1} \;=\; \mathrm{RRS}(g) \;=\; \frac{2}{\text{-}1} \;=\; \text{-}2\,.$$

46.1: ⁇ Counting idempotent fncs. *Consider a set Γ of cardinality $N:=|\Gamma|\in\mathbb{Z}_+$. A map $f{:}\Gamma{\to}\Gamma$ is* ***idempotent*** *if $f\circ f=f$. Give a formula for I_N, the number of such involutions. Compute I_5.* ♢

Removing Foliage

47a: Polynomial-deriv-divisible (Putnam 2016.A1). *Find the smallest natnum J such that for every intpoly $p()$ and for every $\mathbf{k} \in \mathbb{Z}$, the integer*

$$*: \quad p^{(J)}(\mathbf{k}) \quad \begin{bmatrix}\text{The } J\text{-th derivative} \\ \text{of } p()\text{, evaluated at } \mathbf{k}.\end{bmatrix}$$

is divisible by 2016. ♢

Solved by: *Rabon M., 2017g.*

Soln A1.2016. Our $J = 8$.

Note that $q(x) := p(x - \mathbf{k})$ is an intpoly, with $q^{(J)}(0) = p^{(J)}(\mathbf{k})$. So we only need evaluate $(*)$ at zero. Writing $p(x) = \sum_{j=0}^{N} C_j \cdot x^j$, note that $p^{(j)}(0)$ equals $C_j \cdot [j!]$. Hence $C_j = 1$ is the most difficult case. Consequently,

Our J is the smallest natnum st. $[j!] \bullet 2016$.

Lastly, $2016 = \mathbf{2}^5 \cdot \mathbf{3}^2 \cdot \mathbf{7}$. Thus $J \geqslant 7$. Numbers 3 and 6 give us the needed two copies of $\mathbf{3}$. But $2, 4, 6$ only give us $1+2+1 = 4$ copies of $\mathbf{2}$. Hence we need $J{=}8$ for the required 5^{th}-copy of $\mathbf{2}$. ♦

Challenging misc. Problems

48: ⁇ Chessboard-config Problem. *In some order, put the numbers $1, 2, \ldots, 64$ on the cells (squares) of a chessboard; call this a **configuration**. For a cell α, let $\alpha_{\mathbf{G}}$ denote the number placed there by $\mathbf{G}$. Two cells α, β are **adjacent** if they touch vertically, horizontally or diagonally. Define the worst-case difference,*

$$48a: \quad \widehat{\mathbf{G}} := \text{Max}\Big\{ |\alpha_{\mathbf{G}} - \beta_{\mathbf{G}}| \;\Big|\; \text{Cells } \alpha \text{ and } \beta \text{ are adjacent.} \Big\}$$

What is the minimum (taken over all configurations $\mathbf{G}$) of $\widehat{\mathbf{G}}$? ♢

49.1: ⁇ Rational 6×6 grid (USAMO.2004.4). *Alice and Bob play a game on a 6×6 grid. On his turn, a player chooses a rational number not yet in the grid and writes it in an empty cell (i.e, square) of the grid. Alice starts, then players alternate. After all cells have numbers: In each row, color black the cell with the greatest number in that row. Alice wins if she can draw a (polygonal) line from the top of the grid to the bottom of the grid that stays in black cells; Bob wins if she can't. [Defn: Two cells in adjacent rows are **connected** IFF they share a vertex.] Find, with proof, a winning strategy for one of the players.* ♢

50: ⁇ 7-5-Prob. *For $k = 0, 1, \ldots$, let $\mathbf{S}_k := 7^k + 5^k$.*

Produce a a simple formula so that, for coprime natnums $L \geqslant N$,

$$\text{Gcd}(\mathbf{S}_L, \mathbf{S}_N) = \textit{SimpleFormula}(L, N)\,.$$

[Guessing a formula may be easy; our goal is a proof!] ♢

Valiant polynomial. A polynomial f is ***valiant***[♡8] if $[w \in \mathbb{Z}] \Rightarrow [f(w) \in \mathbb{Z}]$. Define the k^{th} ***binomial polynomial***

$$\mathbf{b}_K(x) := \frac{x[x-1][x-2]\cdots[x-[K-1]]}{K!}\,,$$

which we can think of as $\binom{x}{K}$.

We proved in class that $\mathbf{b}_3$ is valiant.

[♡8] I.e, its ***VAL***ues are ***INT***egers.

51.1: **??** Binomial-polys are Valiant. *For each $K \in \mathbb{N}$, polynomial $\mathbf{b}_K$ is valiant.* ♢

51.2: **??** Valiants are lin-combs. *Each valiant poly f can be written as a finite linear-combination, with integer coefficients, of the binomial polys.* [*I.e,* $\{\mathbf{b}_k\}_{k=0}^{\infty}$ *is a $\mathbb{Z}$-basis for* VALIANT.] ♢

52.1: **??** Half-intersection Problem. *Consider a set Λ with $|\Lambda| = 4028$, along with subsets $C_1, C_2, \ldots, C_{2014} \subset \Lambda$, where each $|C_j| = 2014$. Prove that there exist distinct indices i,j with*

$$|C_i \cap C_j| \geqslant 1007\,.$$ ♢

53.1: **??** Polynomial fit (USAMO.1975.3). *Fix $N \in \mathbb{Z}_+$ and $J := [0\,..\,N)$. Let $P()$ denote the unique polynomial st.* $\text{Deg}(P) \leqslant N{-}1$ *and*

†: $$\forall k \in J\colon \quad P(k) = \frac{k}{k+1}\,.$$

Determine the value of $P(N)$. ♢

Defn. The ***geometric-mean*** of a set of m non-negative numbers is the m^{th}-root of their product. □

54.1: **??** Integral geometric-mean Problem (USAMO1984.2). *A subset $\mathcal{G} \subset \mathbb{N}$ is **good** if: The geometric-mean of each* (non-void) *<u>finite</u> subset of $\mathcal{G}$ is an integer.*

i: Which posints N admit a good-set of cardinality N? (*Such an N is also called **good**.*)

ii: Is there an <u>infinite</u> good set? ♢

55.1: **??** Heart-isomorphism. *The $f(x) := 2^x$ map, from $\mathbb{R}\to\mathbb{R}_+$, is a group-isomorphism from $(\mathbb{R}, +, 0)$ onto $(\mathbb{R}_+, \cdot, 1)$. More than a group, the reals form a ring, $(\mathbb{R}, +, 0, \cdot, 1)$, So f carries this ring to a ring $(\mathbb{R}_+, \cdot, 1, \heartsuit, @)$, where $\heartsuit$ is a binary operation on $\mathbb{R}_+$, and @ is an element of $\mathbb{R}_+$.*

What is @? And what is the $\heartsuit$ binop? What does $5 \heartsuit 8$ equal?

Theorems with Example Proofs

Prolegomenon. The following exposition is more loquacious that I would ask you to be, because I wish to illustrate various ideas and mis-steps in proofs.

An easy inequality

Suppose I ask you to demonstrate the following assertion.

56.1: Statement A. *For each posint n:*

∗: $$5 \cdot 2^n < 3^n\,.$$ ♢

You would detect the error and write:

Dear Prof. King:

Something is amiss; assertion (∗) fails for $n = 1$, since $5 \cdot 2 \not< 3$. [*Inequality (∗) also fails for $n{=}2$ and $n{=}3$.*] *I, Bubba, correct the statement below, and prove my correction.*

56.2: Theorem A′. *For each $n \in [4\,..\,\infty)$:*

$B(n)$: $$5 \cdot 2^n < 3^n\,.$$ ♢

Proof. Let $L(k) := 5{\cdot}2^k$ and $R(k) := 3^k$.

Base case: Note that

$$L(4) = 5 \cdot 16 = 80 < 81\,,$$

which equals $R(4)$. Hence $B(4)$.

Induction: Fix an index $n \in [4\,..\,\infty)$. [Henceforth, "n" plays the role of a constant.]

Assuming $B(n)$, my goal is to establish $B(n{+}1)$. So I want to examine how $L(n{+}1)$ relates to $L(n)$, and ditto for $R()$.

Easily

$$\begin{aligned} L(n{+}1) &= 2 \cdot L(n) \\ &< 2 \cdot R(n)\,, \end{aligned}$$

courtesy $B(n)$ and that 2 is positive. [Multiplication by a positive number is order-preserving.] Thus

$$\begin{aligned} L(n{+}1) &< 2 \cdot R(n)\,, \\ &< 3 \cdot R(n)\,, \quad \text{since } R(n) \text{ is positive}, \\ &\stackrel{\text{def}}{=} R(n{+}1)\,, \end{aligned}$$

as desired. ♦

Autopsy. Of course, your proof used this elementary tool.

56.3: Lemma. *For all reals $\alpha<\beta$, and "multiplier" $M \in \mathbb{R}$: If M is positive, then $\alpha M < \beta M$.* ♢

Exer.: *You used this lemma twice in your proof of Thm A′; where are the two occurrences?*

57: ∞ly-many-Primes Thm (Euclid). *There are ∞ly many primes.* ♢

Pf. Given primes $p_1, \ldots, p_N$ (not-nec. distinct), we construct a new prime. Let $Q := [p_1 \cdot p_2 \cdot \ldots \cdot p_N]$; this Q is at least 1. [Even for N=0; the void-product is 1.]

Now add 1; let $R := Q+1$. Necessarily, $R \perp Q$. Thus R is coprime to *each* p_j. Moreover, $R \geqslant 2$, so R has *at least one* prime factor (which might be R itself). And each of these prime factors is new. ♦

57a: Joke (Hendrik Lenstra). *There are ∞ly many composite numbers.* ♢

Proof. To obtain a new composite number, multiply together the first N composite numbers, then don't add 1. ♦

A two-term recurrence

We discuss a sequence like the Fibonacci sequence.

58.1: Thm B. *Define a sequence $\vec{\mathbf{b}} = (b_0, b_1, b_2, b_3, \ldots)$ by $b_0 := -1$ and $b_1 := 2$; moreover, for each integer $n \geqslant 2$, define*

$$\dagger: \qquad b_n := 5b_{n-1} - 6b_{n-2}.$$

Prove, for each natnum k, that[♡9]

$$\ddagger: \qquad b_k = [4 \cdot 3^k] - [5 \cdot 2^k].$$ ♢

Preliminaries. Define $f{:}\mathbb{N}\to\mathbb{Z}$ by

$$\ddagger\ddagger: \qquad f(k) := [4{\cdot}3^k] - [5{\cdot}2^k].$$

Before starting work, do I even *believe* the outlandish assertion of the thm? From (†) I can compute

$$b_2 \overset{\text{def}}{=} 5{\cdot}2 - 6{\cdot}[-1] = 10+6 = 16.$$

And $f(2)$ equals $[4{\cdot}9] - [5{\cdot}4] = 36 - 20$, which indeed equals 16. Also,

$$b_3 \overset{\text{def}}{=} 5{\cdot}16 - 6{\cdot}[2] = 80 - 12 = 68.$$

And $f(3)$ equals $[4{\cdot}27] - [5{\cdot}8] = 108 - 40$, which –wow!– also equals 68. So now I [Bubba Student] think the stmt is plausible, and I am willing to work on it. □

Observation. When k is large, the value 3^k swamps 2^k. So a corollary of Thm B is that $\vec{\mathbf{b}}$ grows like $k \mapsto 3^k$, in the sense that ratio $b_k/[4{\cdot}3^k] \to 1$, as $k \nearrow \infty$.

And that is not obvious from the recursive *definition* of $\vec{\mathbf{b}}$, in (†). □

Proof of Thm B. Since (†) needs the *two* previous values in $\vec{\mathbf{b}}$ in order to determine the next, we'll need to check two base cases.

Base cases: Firstly [or should I say "Zerothly"?],

$$f(0) = [4{\cdot}1] - [5{\cdot}1] = -1 \overset{\text{Hooray!}}{=\!=} b_0.$$

And secondly ["firstly"?],

$$f(1) = [4{\cdot}3] - [5{\cdot}2] = 12 - 10 = 2 \overset{\text{note}}{=\!=} b_1,$$

as was needed.

[♡9] I use natnum for "natural number". BTWay, do you understand why (†) uses ":=", but (‡) uses the "=" relation?

Induction: We just need to show that fnc $f()$ behaves like (†). So say that a fnc $\varphi{:}\mathbb{N}{\to}\mathbb{Z}$ is ***good*** if

$$*:\quad \forall k \in \mathbb{N}: \qquad \varphi(k{+}2) \;=\; 5\varphi(k{+}1) \;-\; 6\varphi(k)\,.$$

Restated, our goal is to show that f is good.

We can, of course, show goodness directly, but let's "look ahead", and see if we can shorten our work.

We glance at (‡‡) and note that f is built from two simpler fncs, namely

$$H(k) \;:=\; 3^k \quad \text{and} \quad W(k) \;:=\; 2^k\,.$$

["H" is for tHree, and "W" is for tWo.] Our beloved f is simply the linear combination

$$f() \quad = \quad 4{\cdot}H() \;-\; 5{\cdot}W()\,.$$

Evidently, if a fnc $\varphi()$ is good, then for α an arbitrary real, the product $\alpha\varphi()$ is also good; this follows from ($*$) since mult is associative and commutative.

Moreover, the *sum* of two good fncs is good; this, since mult distributes-over addition. So we've established:

$**$: *A linear combination of good functions is always good.*

Hence our task has simplified to the following.

Goal: *Fnc $H()$ is good, and so is $W()$.* Letting $Y := 3$, in order to show $H()$ good, we covet

$$\forall k \in \mathbb{N}: \quad Y^{k+2} = 5Y^{k+1} - 6Y^k\,.$$

But this is implied by establishing

$$Y^2 \quad \stackrel{?}{=} \quad 5Y \;-\; 6\,,$$

simply by multiplying by Y^k. And this nice quadratic equality[♡10] is the same as saying that $Y{=}3$ is a root of polynomial

$$P(x) \quad := \quad x^2 - 5x + 6\,.$$

Similarly, showing $W()$ good is equivalent to showing that $P(2) = 0$. So we could simply check that both $P(3)$ and $P(2)$ are each zero. Or note that

$$P(x) \quad = \quad [x-3]\cdot[x-2]\,;$$

i.e, we simply factor the P polynomial. *Elegant!* ♦

[♡10] We *could* just compute that 9 equals $[5{\cdot}3]-6$, but let's take an approach that illustrates how the problem was created.

Autopsy. Indeed, to *create* the problem, Prof. K simply started with the factored poly $[x-3]\cdot[x-2]$, then multiplied to get $x^2 - 5x + 6$. This gave him the coeffs for (†).

The Upshot?: We learn a lot about a subject/technique by *creating* problems with that technique. So I encourage you to create and post induction problems, and to post solns to others' posted problems.

We adults tend to learn by synthesis, more than by analysis. [Or at least, we retain more.] □

58.2: Exercise. *For distinct reals α, β, define a sequence $\vec{\mathbf{b}}$ by (58.1†) together with $b_0 := \alpha$ and $b_1 := \beta$. Derive formulas for numbers $\mathsf{H}_{\alpha,\beta}$ and $\mathsf{W}_{\alpha,\beta}$ so that:*

$$58.3:\quad \forall k \in \mathbb{N}: \qquad b_k \;=\; \left[\mathsf{H}_{\alpha,\beta}\cdot 3^k\right] \;-\; \left[\mathsf{W}_{\alpha,\beta}\cdot 2^k\right].$$ ◊

Generalizing the Triangle Inequality

Some wag thinks he can stump us with this baby conundrum.

59.1: General Triangle-Inequality. *For each natnum N, and sequence $s_1, \ldots, s_N$ of complex numbers, this inequality holds:*

$$P_N\colon \qquad \left|\sum_{j=1}^N s_j\right| \;\leqslant\; \sum_{j=1}^N |s_j|\,. \qquad \Diamond$$

Remark. Looking ahead, our tool will be P_2. □

59.2: Weak Tri-Ineq. *For all complex numbers α,β:*

$$*\colon \qquad |\alpha+\beta| \;\leqslant\; |\alpha|+|\beta|\,. \qquad \Diamond$$

Rem. For α,β real, this follows by a case-by-case argument [Both negative? Mixed sign?] For complexes, this takes a bit of development of the complex plane. □

Proof of Gen. Tri-Ineq. We use the vacuous base-case.

Base case: Evidently (P_0), since $0 \leqslant 0$. [And (P_1), since $|s_1| \leqslant |s_1|$. However, we don't need this argument, since the induction gets the same result.]

Induction: Fix a natnum N, and sequence $s_1, \ldots, s_N, s_{N+1}$. Assuming (P_N), our goal is to establish (P_{N+1}).

Applying (59.2∗) with $\alpha := \sum_{j=1}^N s_j$ and $\beta := s_{N+1}$, gives

$$\left|\sum_{j=1}^{N+1} s_j\right| \;\leqslant\; |\alpha|+|\beta|\,.$$

And (P_N) yields $|\alpha| \leqslant \sum_{j=1}^N |s_j|$. Adding these gives

$$\left|\sum_{j=1}^{N+1} s_j\right| \;\leqslant\; \left[\sum_{j=1}^N |s_j|\right] + |\beta|\,,$$

which equals $\mathrm{RhS}(P_{N+1})$, as was sought. ♦

Prelim lemmas, sqrt-harmonic sum

By looking ahead in our induction proof, we may find a result that we wish to prove as a separate lemma.

60.1: Recip-Squareroot theorem. *For each $N \in \mathbb{Z}_+$,*

$$\dagger\colon \qquad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{N}} \;<\; 2\sqrt{N}\,. \qquad \Diamond$$

Hmmm [Bubba thinks to him/her/it-self]*: After playing with (†) for a bit, I realize I need a little inequality involving square-roots. Let me state and prove that separately, to be nice to those reading my proof.*

60.2: Lemma. *For each real $x \geqslant 1$, we have that*

$$*\colon \qquad \frac{1}{\sqrt{x}} \;<\; 2\left[\sqrt{x} - \sqrt{x-1}\right].$$

(We needed $x \geqslant 1$ for $\sqrt{x-1}$ to make sense in $\mathbb{R}$.) ◊

Proof of (60.2). Since $\sqrt{x} > 0$, our $(*)$ is implied by

$$1 \;<\; 2\left[x - \sqrt{x^2 - x}\,\right],$$

hence by $2\sqrt{x^2-x} \;<\; 2x-1$. Both sides are non-negative, so this follows from the squared-version,

$$4[x^2 - x] \;<\; 4x^2 - 4x + 1\,.$$

And this last is trivially true. ♦

Proof of Recip-Squareroot thm. Let L_N and R_N denote the left/right-hand sides of (60.1†).

Base case. Since $L_1 = 1 < 2 = R_1$, we can start[♡11] our induction at $N{=}2$.

Induction: ISTEstablish, for each $N \in [2\,..\,\infty)$, that $L_N - L_{N-1} < R_N - R_{N-1}$, i.e, that

$$\ddagger\colon \qquad \frac{1}{\sqrt{N}} \;\overset{?}{<}\; 2\left[\sqrt{N} - \sqrt{N-1}\right].$$

Happily, this is implied by Lemma 60.2. ♦

[♡11] Actually, *in a sense* we could use $N{=}0$ as our base case. True, $L_0 = 0 = R_0$, so we do not have the strict inequality of (†). But as (‡) is strict, we would obtain (†) for $N = 1, 2, \ldots$.

Après-proof. In developing our induction argument, at (‡) we realized we needed another result. Not only is it clearer to split the result out to a separate lemma, but we got a *slightly stronger* result, since (60.2) holds for reals, not just integers. □

60.3**:** *Alternative.* We can sharpen (60.1), using calculus. For an arbitrary decreasing fnc $f:\mathbb{R}_+\to\mathbb{R}_+$ and integer $N \in [2 \,..\, \infty)$, a picture easily shows that

$$¥: \qquad \sum_{j=2}^{N} f(j) \quad < \quad \int_1^N f(x) \cdot \mathrm{d}x\,.$$

Applying this[12] with $f(x) := 1/\sqrt{x}$ yields that

$$L_N - 1 \quad < \quad 2x^{1/2}\Big|_{x=1}^{x=N} \;=\; 2[\sqrt{N} - \sqrt{1}]\,.$$

Adding 1 to each side yields $\boxed{L_N \;<\; [2\sqrt{N}] - 1}$, for $N = 2, 3, 4, \ldots$. □

[12]Well... The inequality in (¥) is strict *unless* f is *the* step-function which mimics the summation.

§Index, with symbols and abbrevs at the End

Filename: `Problems/Misc/induction-SELO-jk.latex`
As of: *Wednesday 25Jan2017.* Typeset: *12Apr2017* at *11:30.*