Inclusion-Exclusion principle

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Prolegomenon. Consider a finite set, Ω, of objects. and a finite indexing set $\mathbb{J}$; let $N := |\mathbb{J}|$. For each $j \in \mathbb{J}$, we have a subset $A_j \subset \Omega$, and its complement $V_j := \Omega \smallsetminus A_j$. Our goal is an expression for the cardinality of the union

$$\mathbf{U} := \bigcup_{j\in\mathbb{J}} A_j .$$

Notation. For each $I \subset \mathbb{J}$, define

1.1: $$\mathbf{A}_I := \bigcap_{j\in I} A_j ,$$

[and $\mathbf{V}_I := \bigcap_{j\in I} V_j$] and note[♡1] $\mathbf{A}_\varnothing$ is all of Ω.

Finally, let $\mathcal{C}_k$ comprise those subsets $I \subset \mathbb{J}$ with $|I| = k$; this, for $k = 0, 1, \ldots, N$. Easily

1.2: $$|\mathcal{C}_k| = \binom{N}{k} .$$

2: Inclusion-Exclusion Lemma. *With notation from above:*

2a: $$\big|\Omega \smallsetminus \mathbf{U}\big| = \sum_{k=0}^{N} [-1]^k \cdot \sum_{I\,:\,I\in\mathcal{C}_k} |\mathbf{A}_I| .$$

Alternatively,

2b: $$\big|\mathbf{U}\big| = \sum_{k=1}^{N} [-1]^{k-1} \cdot \sum_{I\,:\,I\in\mathcal{C}_k} |\mathbf{A}_I| . \qquad \diamondsuit$$

Pf (2a) $\Rightarrow$ (2b). Adding the RhSes causes cancellation, with only the k=0 term remaining. So

$$\big|\Omega \smallsetminus \mathbf{U}\big| + \text{LhS}(2b) = [-1]^0 \cdot |\mathbf{A}_\varnothing| \overset{\text{note}}{=\!=} |\Omega| .$$

So LhS(2b) equals $\big|\mathbf{U}\big|$. ♦

To establish (2a), let's prove something a little stronger. Each subset $S \subset \Omega$ yields a function $\mathbf{1}_S{:}\Omega\to\{0,1\}$, the "***indicator function*** of S",

$$\mathbf{1}_S(x) := \left\{\begin{matrix} 1 & \text{when } x \in S \\ 0 & \text{when } x \in \Omega\smallsetminus S \end{matrix}\right\} .$$

Indicator fncs allow us to restate intersection ITOf multiplication: *For a set I of indicies,*

1.1′: $$\mathbf{1}_{\mathbf{A}_I} = \textstyle\prod_{j\in I} \mathbf{1}_{A_j} .$$

Let's strengthen (2a) to equality of *functions*,

3.1: $$\mathbf{1}_{\Omega\smallsetminus\mathbf{U}}() = \sum_{k=0}^{N} [-1]^k \cdot \sum_{I\,:\,I\in\mathcal{C}_k} \mathbf{1}_{\mathbf{A}_I}() .$$

Proof of (3.1). The RhS(3.1) equals

$*$: $$\prod_{j\in\mathbb{J}} [1 - \mathbf{1}_{A_j}] \overset{\text{note}}{=\!=} \prod_{j\in\mathbb{J}} \mathbf{1}_{V_j} .$$

After all, this lefthand product expands to the sum, over all subsets $I \subset \mathbb{J}$, of $\prod_{j\in I} [-\mathbf{1}_{A_j}]$. And this latter product, courtesy (1.1′), equals $[-1]^k \cdot \mathbf{1}_{\mathbf{A}_I}$.

By De Morgan's law, $\Omega \smallsetminus \mathbf{U} = \bigcap_{j\in\mathbb{J}} V_j$. So LhS(3.1) is $\mathbf{1}_{\mathbf{V}_\mathbb{J}}$. Hence (3.1) boils down to the triviality that

$$\mathbf{1}_{\mathbf{V}_\mathbb{J}} = \textstyle\prod_{j\in\mathbb{J}} \mathbf{1}_{V_j} . \qquad ♦$$

4: *Rem.* We can rewrite (2a) in simpler [but often less convenient] form, ITOf summing over all subsets of $\mathbb{J}$:

4a: $$\big|\Omega \smallsetminus \mathbf{U}\big| = \sum_{I\,:\,I\subset\mathbb{J}} (-1)^{|I|} \cdot |\mathbf{A}_I| . \qquad \square$$

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[♡1] For subsets $I,J \subset \mathbb{J}$, by definition $\mathbf{A}_I \cap \mathbf{A}_J = \mathbf{A}_{I\cup J}$. In particular, $\mathbf{A}_I \cap \mathbf{A}_\varnothing$ equals $\mathbf{A}_\varnothing$. I.e, $\mathbf{A}_\varnothing$ is the identity element for intersection, on the powerset of Ω. So $\mathbf{A}_\varnothing$ is Ω.