Inclusion-Exclusion principle

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14 October, 2018 (at *21:58*)

Prolegomenon. Consider a finite set, Ω, of ***tokens***. and a finite indexing set $\mathcal{N}$; let $N := {}^{\#}\mathcal{N}$. For each $j \in \mathcal{N}$ we have a ***patch***, a subset $A_j \subset \Omega$, and its complement $V_j := \Omega \smallsetminus A_j$. Our goal is an expression for the cardinality of the union

$$\mathbf{U} := \bigcup_{j\in\mathcal{N}} A_j .$$

[N.B: This note uses $^{\#}$ to indicate the cardinality of an index-set, and $|\cdot|$ for the cardinality of a token set.]

Notation. For each ***index-set*** $I \subset \mathcal{N}$, define the ***patch-intersection***

1.1: $$\mathbf{A}_I := \bigcap_{j\in I} A_j ,$$

[and $\mathbf{V}_I := \bigcap_{j\in I} V_j$] and note[♡1] $\mathbf{A}_\varnothing$ is all of Ω.

Finally, let $\mathcal{C}_k$ comprise those index-sets $I \subset \mathcal{N}$ with ${}^{\#}I = k$; this, for $k = 0, 1, \ldots, N$. Easily

1.2: $$|\mathcal{C}_k| = \binom{N}{k} .$$

2: Inclusion-Exclusion Lemma. *With notation from above:*

2a: $$\big|\Omega \smallsetminus \mathbf{U}\big| = \sum_{k=0}^{N} \Big[[-1]^k \cdot \sum_{I\,:\,I\in\mathcal{C}_k} |\mathbf{A}_I|\Big] .$$

Alternatively,

2b: $$\big|\mathbf{U}\big| = \sum_{k=1}^{N} \Big[[-1]^{k-1} \cdot \sum_{I\,:\,I\in\mathcal{C}_k} |\mathbf{A}_I|\Big] . \quad \diamond$$

Pf (2a) $\Rightarrow$ (2b). Adding the RhSes causes cancellation, with only the k=0 term remaining. So

$$|\Omega \smallsetminus \mathbf{U}| + \mathrm{RhS}(2b) = [-1]^0 \cdot |\mathbf{A}_\varnothing| \stackrel{\text{note}}{=\!=} |\Omega| .$$

So RhS(2b) equals $|\mathbf{U}|$. ♦

To establish (2a), let's prove something a little stronger. Each subset $S \subset \Omega$ yields a function $\mathbf{1}_S{:}\Omega{\to}\{0,1\}$, the "***indicator function*** of S",

$$\mathbf{1}_S(x) := \begin{Bmatrix} 1 & \text{when } x \in S \\ 0 & \text{when } x \in \Omega\smallsetminus S \end{Bmatrix} .$$

Indicator fncs allow us to restate intersection ITOf multiplication: *For a set I of indicies,*

1.1′: $$\mathbf{1}_{\mathbf{A}_I} = \prod_{j\in I} \mathbf{1}_{A_j} .$$

Let's strengthen (2a) to equality of *functions*,

3.1: $$\mathbf{1}_{\Omega\smallsetminus\mathbf{U}} = \sum_{k=0}^{N} \Big[[-1]^k \cdot \sum_{I\,:\,I\in\mathcal{C}_k} \mathbf{1}_{\mathbf{A}_I}\Big] .$$

Proof of (3.1). The RhS(3.1) equals

$*$: $$\prod_{j\in\mathcal{N}} [1 - \mathbf{1}_{A_j}] \stackrel{\text{note}}{=\!=} \prod_{j\in\mathcal{N}} \mathbf{1}_{V_j} .$$

After all, the LhS($*$) product expands to the sum, over all subsets $I \subset \mathcal{N}$, of $\prod_{j\in I} [-\mathbf{1}_{A_j}]$. This latter product, letting $k := {}^{\#}I$, equals $[-1]^k \cdot \mathbf{1}_{\mathbf{A}_I}$, courtesy (1.1′).

By De Morgan's law, $\Omega \smallsetminus \mathbf{U} = \bigcap_{j\in\mathcal{N}} V_j$. So LhS(3.1) is $\mathbf{1}_{\mathbf{V}_\mathcal{N}}$. Hence (3.1) boils down to the triviality that

$$\mathbf{1}_{\mathbf{V}_\mathcal{N}} = \prod_{j\in\mathcal{N}} \mathbf{1}_{V_j} . \quad ♦$$

4: *Rem.* We can rewrite (2a) in simpler [but often less convenient] form, ITOf summing over all subsets of $\mathcal{N}$:

4a: $$\big|\Omega \smallsetminus \mathbf{U}\big| = \sum_{I\,:\,I\subset\mathcal{N}} \big[[-1]^{\#I} \cdot |\mathbf{A}_I|\big] . \quad \square$$

[♡1] For index-sets $I, J \subset \mathcal{N}$, by definition $\mathbf{A}_I \cap \mathbf{A}_J = \mathbf{A}_{I\cup J}$. In particular, $\mathbf{A}_I \cap \mathbf{A}_\varnothing$ equals $\mathbf{A}_I$. I.e, $\mathbf{A}_\varnothing$ is the identity element for intersection, on the powerset of Ω. So $\mathbf{A}_\varnothing$ is Ω.

Incl-Excl Examples

Counting candy. The store sells *jelly-$\mathcal{B}$eans* and $\mathcal{C}$hocolate squares and $\mathcal{D}$ates. Mom allows you a total of 20 candies.

Alas!, the store only has $8\mathcal{B}$ and $5\mathcal{C}$ and $13\mathcal{D}$. Stars-and-Bars counts how to pick out of multiset $\{\infty\mathcal{B}, \infty\mathcal{C}, \infty\mathcal{D}\}$. The relevant multiset is $\{8\mathcal{B}, 5\mathcal{C}, 13\mathcal{D}\}$; *so how do we count?*

Candy soln. Let Ω be the set of natnum triples $(\mathcal{B}, \mathcal{C}, \mathcal{D})$ with $\mathcal{B}+\mathcal{C}+\mathcal{D} = 20$. We'll count the "***good***" $[\mathcal{B}\leqslant 8 \ \& \ \mathcal{C}\leqslant 5 \ \& \ \mathcal{D}\leqslant 13]$ triples, using Incl-Excl.

Let $A_\mathcal{B}$ be the set of triples that are "***bad***" because $\mathcal{B} > 8$. Hence,

$$|A_\mathcal{B}| \overset{\textit{Why?}}{=\!=} \left[\!\!\left[\begin{matrix}3\\ 20-[8+1]\end{matrix}\right]\!\!\right] = \binom{2+11}{2} = 78.$$

So $|A_\mathcal{C}| = \left[\!\left[\begin{smallmatrix}3\\ 20-[5+1]\end{smallmatrix}\right]\!\right] = \binom{2+14}{2} = 120$, and $|A_\mathcal{D}| = 28$.

For *pairwise* intersections

$$|A_\mathcal{B} \cap A_\mathcal{C}| \overset{\textit{Why?}}{=\!=} \left[\!\!\left[\begin{matrix}3\\ 20-[8+5+2]\end{matrix}\right]\!\!\right] = \binom{2+5}{2} = 21.$$

Also, $|A_\mathcal{B} \cap A_\mathcal{D}| = \left[\!\left[\begin{smallmatrix}3\\ 20-[8+13+2]\end{smallmatrix}\right]\!\right] = \left[\!\left[\begin{smallmatrix}3\\ \text{negative}\end{smallmatrix}\right]\!\right] \overset{\textit{Why?}}{=\!=} 0$,

and $|A_\mathcal{C} \cap A_\mathcal{D}| = \left[\!\left[\begin{smallmatrix}3\\ 20-[5+13+2]\end{smallmatrix}\right]\!\right] = \left[\!\left[\begin{smallmatrix}3\\ 0\end{smallmatrix}\right]\!\right] = 1$.

For the sole *three-fold* intersection

$$|A_\mathcal{B} \cap A_\mathcal{C} \cap A_\mathcal{D}| = \left[\!\!\left[\begin{matrix}3\\ 20-[8+5+13+3]\end{matrix}\right]\!\!\right] = \left[\!\!\left[\begin{matrix}3\\ \text{neg}\end{matrix}\right]\!\!\right] = 0.$$

Since $\left[\!\left[\begin{smallmatrix}3\\ 20\end{smallmatrix}\right]\!\right] = 231$, the number of good triples is

$$\begin{aligned} &|\Omega| - \Big(|A_\mathcal{B}| + |A_\mathcal{C}| + |A_\mathcal{D}|\Big) \\ &\quad + \Big(|A_\mathcal{B}\cap A_\mathcal{C}| + |A_\mathcal{B}\cap A_\mathcal{D}| + |A_\mathcal{C}\cap A_\mathcal{D}|\Big) \\ &\quad - |A_\mathcal{B}\cap A_\mathcal{C}\cap A_\mathcal{D}| \\ &= 231 - [78+120+28] + [21+0+1] - 0. \end{aligned}$$

This equals 27. ♦

Doubting Thomas. Here are the 27 good triples:

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(2 5 13) (3 4 13) (3 5 12) (4 3 13) (4 4 12) (4 5 11)
(5 2 13) (5 3 12) (5 4 11) (5 5 10) (6 1 13) (6 2 12)
(6 3 11) (6 4 10) (6 5  9) (7 0 13) (7 1 12) (7 2 11)
(7 3 10) (7 4  9) (7 5  8) (8 0 12) (8 1 11) (8 2 10)
(8 3  9) (8 4  8) (8 5  7)
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□

5: *Cardinality independence.* In some combinatorial applications, cardinality $|\mathbf{A}_I|$ depends only on the number of patches being intersected. In that instance, let $F(k)$ be $^{\#}I$, for each and every index-set I satisfying $^{\#}I = k$. So rewrite (2a) as

5a:
$$\begin{aligned} |\Omega \smallsetminus \mathbf{U}| &= \sum_{k=0}^{N} \Big[[-1]^k \cdot \tbinom{N}{k} \cdot F(k)\Big] \\ &= \sum_{k=0}^{\infty} \Big[[-1]^k \cdot \tbinom{N}{k} \cdot F(k)\Big], \end{aligned}$$

where, for $k{>}N$, our $\binom{N}{k} := \frac{[\![N\downarrow k]\!]}{k!}$ is zero. □

6: Probability of getting your own hat. *The N guests leaving your party grab their hats at random from your dark closet. What does $\mathcal{P}(N)$, the probability that no one gets his own hat, tend to as $N \nearrow \infty$?* ♢

Hat soln. For each subset I of your guests, let H_I denote the probability that each person in I took his own hat. It equals $H_k := [N-k]!/N!$ where k is the cardinality of I, i.e $k := {}^{\#}I$. By the principle of inclusion/exclusion, our $\mathcal{P}(N)$ equals

$$H_\varnothing - \sum_{I:\,^{\#}I=1} H_I + \sum_{I:\,^{\#}I=2} H_I - \sum_{I:\,^{\#}I=3} H_I + \ldots + [-1]^N \sum_{I:\,^{\#}I=N} H_I$$

$$= 1 - \tbinom{N}{1}H_1 + \tbinom{N}{2}H_2 - \tbinom{N}{3}H_3 + \ldots + [-1]^N \tbinom{N}{N} H_N$$

$$= \sum_{k=0}^{N} [-1]^k \cdot \binom{N}{k} \cdot \frac{[N-k]!}{N!} = \sum_{k=0}^{N} \frac{[-1]^k}{k!}.$$

This last is the first $k{+}1$ terms of the Taylor series for e^{-1}. Thus $\lim_{n\to\infty} \mathcal{P}(n)$ equals $1/\mathsf{e}$. ♦

Derangements. A ***derangement*** of an N-set is a *fixed-point free* permutation of that set; let $\mathbb{D}_N$ be the set of derangements. Thus

6a:
$$|\mathbb{D}_N| = N! \cdot \sum_{k=0}^{N} \frac{[-1]^k}{k!}$$

is a restatement of the above probability. □

TwoStirling numbers. The "***Stirling # of the 2nd kind***", written $\boldsymbol{\mathcal{S}}(L, N)$, counts the number of partitions of an L-set into N nv-atoms.

7: TwoStirling Theorem. *For all natnums L,N:*

$$*: \qquad \boldsymbol{\mathcal{S}}(L, N) = \sum_{k=0}^{N} [\text{-}1]^k \cdot \frac{[N-k]^L}{k! \cdot [N-k]!} . \qquad \diamondsuit$$

Proof. Let $\mathcal{L}$ be some L-set, and $\mathcal{N}$ an N-set; the ***token*** set. Previous work showed that

$$N! \cdot \boldsymbol{\mathcal{S}}(L, N)$$

equals be the number of surjections $\mathcal{L} \to \mathcal{N}$. This equals the number of all fncs, minus those that miss *a* specified token, plus those that miss a specified *pair* of tokens, minus.... For a subset $I \subset \mathcal{N}$, define

$$\mathbf{A}_I := \left\{ h{:}\mathcal{L} \to \mathcal{N} \;\middle|\; \text{Range}(h) \sqcap I \right\}.$$

Notice that $0 \leqslant k \leqslant N$, where $k := {}^{\#}I$. And that cardinality $|\mathbf{A}_I| = [N-k]^L$ depends on k, *independent* of the contents of I. Hence, let $F(k) := [N-k]^L$. Our (5a) reads as

$$N! \cdot \boldsymbol{\mathcal{S}}(L, N) = \sum_{k=0}^{N} \Big[[\text{-}1]^k \cdot \binom{N}{k} \cdot [N-k]^L \Big].$$

Dividing by $N!$ produces ($*$). ♦

8: Comps avoiding a part. *What is $f(N)$, the number of 3-compositions of N with no PartSize=2?* ♢

(*Next part is commented-out.*)