Inclusion-Exclusion principle

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Prolegomenon. Consider a finite set, \( \Omega \), of tokens.
and a finite indexing set \( N \); let \( N := \#N \). For each \( j \in N \) we have a patch, a subset \( A_j \subset \Omega \), and its complement \( V_j := \Omega \setminus A_j \). Our goal is an expression for the cardinality of the union
\[
U := \bigcup_{j \in N} A_j.
\]

[N.B: This note uses \( \# \) to indicate the cardinality of an index-set, and \( |\cdot| \) for the cardinality of a token set.]

Notation. For each index-set \( I \subset N \), define the patch-intersection

\[
A_I := \bigcap_{j \in I} A_j,
\]

and \( V_I := \bigcap_{j \in I} V_j \) and note\(^1\) \( A_\emptyset \) is all of \( \Omega \).

Finally, let \( \mathcal{E}_k \) comprise those index-sets \( I \subset N \) with \( \#I = k \); this, for \( k = 0, 1, \ldots, N \). Easily
\[
|\mathcal{E}_k| = \binom{N}{k}.
\]

2: Inclusion-Exclusion Lemma. With notation from above:

2a: \[ \left| \Omega \setminus U \right| = \sum_{k=0}^{N} \left[ \left( -1 \right)^{k} \cdot \sum_{I : |I| \in \mathcal{E}_k} |A_I| \right]. \]

Alternatively,
\[
\left| U \right| = \sum_{k=1}^{N} \left[ \left( -1 \right)^{k-1} \cdot \sum_{I : |I| \in \mathcal{E}_k} |A_I| \right].
\]

\( Pf (2a) \Rightarrow (2b) \). Adding the RhSes causes cancellation, with only the \( k=0 \) term remaining. So
\[
\left| \Omega \setminus U \right| + \text{RhS}(2b) = \left( -1 \right)^{0} \cdot |A_\emptyset| = |\Omega|.
\]

So RhS(2b) equals \( |U| \).

To establish (2a), let’s prove something a little stronger. Each subset \( S \subset \Omega \) yields a function \( 1_S: \Omega \to \{0, 1\} \), the “indicator function of \( S \),
\[
1_S(x) := \begin{cases} 1 & \text{when } x \in S \\ 0 & \text{when } x \in \Omega \setminus S \end{cases}.
\]

Indicator fnCs allow us to restate intersection ITOf multiplication: For a set \( I \) of indices,
\[
1_I := \prod_{j \in I} 1_{A_j}.
\]

Let’s strengthen (2a) to equality of functions,
\[
3.1: 1_{\Omega \setminus U} = \sum_{k=0}^{N} \left[ \left( -1 \right)^{k} \cdot \sum_{I : |I| \in \mathcal{E}_k} 1_A \right].
\]

Proof of (3.1). The RhS(3.1) equals
\[
\prod_{j \in N} \left[ 1 - 1_{A_j} \right] \overset{\text{note}}{=} \prod_{j \in N} 1_{V_j}.
\]

After all, the LhS(*) product expands to the sum, over all subsets \( I \subset N \), of \( \prod_{j \in I} \left( -1 \right)^{|I|} \cdot 1_{A_j} \). This latter product, letting \( k := \#I \), equals \( \left( -1 \right)^{k} \cdot 1_{A_I} \), courtesy (1.1').

By De Morgan’s law, \( \Omega \setminus U = \bigcap_{j \in N} V_j \). So LhS(3.1) is \( 1_{V_N} \). Hence (3.1) boils down to the triviality that
\[
1_{V_N} = \prod_{j \in N} 1_{V_j}.
\]

4: Rem. We can rewrite (2a) in simpler [but often less convenient] form, ITOf summing over all subsets of \( N \):

4a: \[ \left| \Omega \setminus U \right| = \sum_{I : |I| \in \mathcal{N}} \left[ \left( -1 \right)^{|I|} \cdot |A_I| \right]. \]

\(^1\) For index-sets \( I, J \subset N \), by definition \( A_I \cap A_J = A_{I \cap J} \). In particular, \( A_I \cap A_\emptyset = A_I \). I.e, \( A_\emptyset \) is the identity element for intersection, on the powerset of \( \Omega \). So \( A_\emptyset \) is \( \Omega \).
**Incl-Excl Examples**

**Counting candy.** The store sells jelly-Beans and Chocolate squares and Dates. Mom allows you a total of 20 candies. 

Alas!, the store only has 8B and 5C and 13D. Stars-and-Bars counts how to pick out of multiset \( \{\infty B, \infty C, \infty D\} \). The relevant multiset is \( \{8B, 5C, 13D\} \); so how do we count? 

**Candy soln.** Let \( \Omega \) be the set of natnum triples \((B,C,D)\) with \( B+C+D = 20 \). We’ll count the “good” \([B \leq 8 \text{ & } C \leq 5 \text{ & } D \leq 13]\) triples, using Incl-Excl. 

Let \( A_B \) be the set of triples that are “bad” because \( B > 8 \). Hence, 

\[
|A_B| = \binom{3}{20-8+1} = \binom{21}{2} = 78.
\]

So \( |A| = \binom{3}{20-5+1} = \binom{21}{2} = 120 \), and \( |A_D| = 28 \). 

For pairwise intersections

\[
|A_B \cap A_C| = \binom{3}{20-8+5+2} = \binom{25}{2} = 21.
\]

Also, \( |A_B \cap A_D| = \binom{3}{20-8+13+2} = \binom{3}{10} \) and \( |A_C \cap A_D| = \binom{3}{20-5+13+2} = \binom{3}{12} = 1 \). 

For the sole three-fold intersection \( |A_B \cap A_C \cap A_D| = \binom{3}{20-8+5+13+3} = \binom{3}{23} = 0 \). 

Since \( \binom{3}{20} = 231 \), the number of good triples is 

\[
|\Omega| - (|A_B| + |A_C| + |A_D|) + (|A_B \cap A_C| + |A_B \cap A_D| + |A_C \cap A_D|) - |A_B \cap A_C \cap A_D|
\]

\[
= 231 - [78+120+28] + [21+0+1] - 0.
\]

This equals 27.

**Doubting Thomas.** Here are the 27 good triples:

\[
(2 \ 5 \ 13) \ (3 \ 4 \ 13) \ (3 \ 5 \ 12) \ (4 \ 3 \ 13) \ (4 \ 4 \ 12) \ (4 \ 5 \ 11) \\
(5 \ 2 \ 13) \ (6 \ 3 \ 12) \ (5 \ 4 \ 11) \ (6 \ 5 \ 10) \ (6 \ 1 \ 13) \ (6 \ 2 \ 12) \\
(6 \ 6 \ 11) \ (6 \ 4 \ 10) \ (6 \ 5 \ 9) \ (7 \ 0 \ 13) \ (7 \ 1 \ 12) \ (7 \ 2 \ 11) \\
(7 \ 3 \ 10) \ (7 \ 4 \ 9) \ (7 \ 5 \ 8) \ (8 \ 0 \ 12) \ (8 \ 1 \ 11) \ (8 \ 2 \ 10) \\
(8 \ 3 \ 9) \ (8 \ 4 \ 8) \ (8 \ 5 \ 7)
\]

5: **Cardinality independence.** In some combinatorial applications, cardinality \( |A_I| \) depends only on the number of patches being intersected. In that instance, let \( F(k) \) be \( |A_I| \), for each and every index-set \( I \) satisfying \( \#I = k \). So rewrite (2a) as

\[
|\Omega \setminus U| = \sum_{k=0}^{N} \left[ (-1)^k \cdot \binom{N}{k} \cdot F(k) \right].
\]

6: **Probability of getting your own hat.** The \( N \) guests leaving your party grab their hats at random from your dark closet. What does \( \mathbb{P}(N) \), the probability that no one gets his own hat, tend to as \( N \to \infty \)?

**Hat soln.** For each subset \( I \) of your guests, let \( H_I \) denote the probability that each person in \( I \) took his own hat. It equals \( H_k := [N-k]! / N! \) where \( k \) is the cardinality of \( I \), i.e \( k := \# I \). By the principle of inclusion/exclusion, our \( \mathbb{P}(N) \) equals 

\[
H_\emptyset - \sum_{I: \#I=1} H_I - \sum_{I: \#I=2} H_I - \sum_{I: \#I=3} H_I + \ldots + \sum_{I: \#I=N} H_I
\]

\[
= 1 - \binom{N}{1} H_1 + \binom{N}{2} H_2 - \binom{N}{3} H_3 + \ldots + \sum_{I: \#I=N} \frac{N! \cdot \left[ (-1)^{N-I} H_N \right]}{N!}
\]

This last is the first \( k \)-1 terms of the Taylor series for \( e^{-1} \). Thus \( \lim_{n \to \infty} \mathbb{P}(n) \) equals 1/e.

**Derangements.** A **derangement** of an \( N \)-set is a fixed-point free permutation of that set; let \( \mathbb{D}_N \) be the set of derangements. Thus

6a: 

\[
|\mathbb{D}_N| = N! \sum_{k=0}^{N} \frac{(-1)^k}{k!}
\]

is a restatement of the above probability.
Two Stirling numbers. The “Stirling # of the 2nd kind”, written $\mathcal{S}(L, N)$, counts the number of partitions of an $L$-set into $N$ nvt-atoms.

7: Two Stirling Theorem. For all natnums $L, N$:

$$\star: \quad \mathcal{S}(L, N) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \cdot \frac{[N - k]^L}{[N - k]!}. \quad \diamondsuit$$

Proof. Let $\mathcal{L}$ be some $L$-set, and $\mathcal{N}$ an $N$-set; the 

**token** set. Previous work showed that

$$N! \cdot \mathcal{S}(L, N)$$

equals be the number of surjections $\mathcal{L} \rightarrow \mathcal{N}$. This equals the number of all fnscs, minus those that miss 
a specified token, plus those that miss a specified pair of tokens, minus... For a subset $I \subset \mathcal{N}$, define

$$A_I := \{ h: \mathcal{L} \rightarrow \mathcal{N} \mid \text{Range}(h) \cap I \}.$$

Notice that $0 \leq k \leq N$, where $k := \#I$. And that cardinality $|A_I| = [N - k]^L$ depends on $k$, independent of the contents of $I$. Hence, let $F(k) := [N - k]^L$. Our (5a) reads as

$$N! \cdot \mathcal{S}(L, N) = \sum_{k=0}^{N} \left[ (-1)^k \cdot \binom{N}{k} \cdot [N - k]^L \right].$$

Dividing by $N!$ produces ($\star$).

8: Comps avoiding a part. What is $f(N)$, the number of 3-compositions of $N$ with no PartSize=2? \diamondsuit

Remark. Bona’s text asks for $f(12)$. \square

Proof. Not yet typed. \diamondsuit