JK Putnam Competition Solns

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1.0: B3.2009. *(Let $\equiv$ mean $\equiv_2$, congruence mod 2.) A subset* $\mathbf{S} \subset [1..N]$ *is* ***mediocre***[♡1] *if:*

1.1: $\forall x,y \in \mathbf{S}$: *If* $x \equiv y$ *then* $\mathbf{S} \ni \frac{x+y}{2}$.

For $n \in \mathbb{N}$, *let* V_n *be the number of mediocre subsets (including* $\varnothing$*) of* $[1..n]$. *Characterize those* ***"good"*** n *st.*

1.2: $$V_{n+2} - 2V_{n+1} + V_n = 1\,. \quad \lozenge$$

Prelims. Note that $V_0 = 1$, $V_1 = 2$, $V_2 = 4$ and $V_3 = 7$. So $n{=}0$ and $n{=}1$ are each good.

The defn of *mediocre* makes sense for arbitrary subsets of $\mathbf{S} \subset \mathbb{Z}$. The collection, $\mathcal{M}$, of mediocres is sealed under:

Intersection:	$\mathbf{S}_1 \cap \mathbf{S}_2$ is mediocre.
Translation:	$\mathbf{S} + k$ is mediocre.
Odd scaling:	$D\mathbf{S}$ is mediocre, for each odd integer D.

Stronger than *mediocrity* is *convexity*; $\mathbf{S}$ is ***convex*** if:

> $\forall x,y \in \mathbf{S}, \forall \lambda \in [0,1]$: *If* $\lambda x + [1-\lambda]y$ *is an integer, then* $\mathbf{S}$ *owns it.*

Evidently, an $\mathbf{S} \subset \mathbb{Z}$ is convex IFF $\mathbf{S}$ is an ***IntOfInt***, an interval-of-integers (possibly infinite, or empty).

For a set T of integers, generalize to let V_T be the number of mediocre subsets of T. Inclusion/exclusion gives that

1.3: $$V_{[0..n+1]} = V_{[0..n]} + V_{[1..n+1]} - V_{[1..n]} + E_{n+1}\,.$$

Here, E_{n+1} is the number of ***extreme*** mediocre subsets $\mathbf{S} \subset [0..n{+}1]$, in that $\mathbf{S} \supset \{0, n{+}1\}$. So (1.2) is asking for those natnums n having $E_{n+1} = 1$. □

[♡1] A pun, I suppose, on "average". The term **"mean"** could also have been used.

Soln B3.2009 (JK). I will show that

1.4: *GOOD = Reduced-POTs, i.e,* n *in* $\{0,1,3,7,15,\dots\}$. *So* $E_{n+1} = 1$, *i.e (1.2), IFF* $n{+}1$ *is a power-of-two.*

Take consecutive members $x<y$ of a mediocre $\mathbf{S}$; so $\{x,y\} \stackrel{\text{note}}{=\!=} \mathbf{S} \cap [x..y]$ is mediocre. Average $\frac{x+y}{2}$ is not in $\mathbf{S}$, necessitating $x \not\equiv y$. Thus:

> *In a mediocre set, each* ***gap*** *between consecutive elements is* odd.

Now consider *three* consecutive elements and translate so that the middle is zero; call them $-G < 0 < H$. Since each gap, G and H, is odd, necessarily $\mathbf{S}$ owns $\frac{-G+H}{2}$. So this average *must* be the middle term. Consequently $H = G$; consective gaps are equal. Thus:

1.5: *A subset* $\mathbf{S} \subset \mathbb{Z}$ *is mediocre IFF* $\mathbf{S}$ *is an arithmetic progression (finite, 1-sided infinite or 2-sided infinite) with odd gap-length.*

(Here, we generously allow $\varnothing$ and singletons to be called "arithmetic progressions".)

For an $L \in \mathbb{Z}_+$, what are the "extreme" subsets of interval $[0..L]$? There is one such for each positive odd divisor of L, since these are the possible gap-lengths. Write $L = D{\cdot}2^m$ with D odd and m a natnum. Since 1 and D are each divisors of D, interval $[0..L]$ has just one extreme subset IFF $D = 1$, i.e, L is a power-of-two. Hence (1.5). ♦

Post mortem. For a posint $D = p_1^{e_1} \cdots p_K^{e_K}$ (distinct primes, with each $e_j \in \mathbb{Z}_+$), recall $\boldsymbol{\tau}(D)$ is the number of positive divisors D has. This equals the product

$*$: $$[1+e_1]\cdot[1+e_2]\cdots[1+e_K]\,.$$

So the number of "extreme" subsets of $[0..L]$, E_L, is $(*)$, having written $L = D{\cdot}2^m$ with D as above. For all n, then,

1.2′: $$V_{n+2} - 2V_{n+1} + V_n = \boldsymbol{\tau}\Big(\mathrm{Odd}(n{+}1)\Big)\,. \quad \square$$

Typeset 1 Putnam problems...
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