Homework-5

Below, $(T\colon X, \mathcal{X}, \mu)$ and $(S\colon Y, \mathcal{Y}, \nu)$ are bi-mpts on probability spaces. Say that T and S are "***disjoint*** in the sense of Furstenberg" if the space of joinings $\mathcal{J}(T,S)$ has but one point, $\mu\times\nu$. We write $T \perp S$ to indicate that T and S are disjoint.

It is not difficult to see that $T \perp S$ implies that T and S are co-prime. For if they had isomorphic factors, then the relative independent joining over this factor would be a non-product-measure joining of T with S.

Symmetric powers. Let $T^{\times n}$ mean the cartesian n^{th}-power of T, that is, $T \times \overset{n}{\cdots} \times T$.

Let $T^{\odot n}$ mean the ***symmetric cartesian n^{th}-power*** of T. It is $T^{\times n}/\equiv$ where two points $\vec{x},\vec{y} \in X^{\times n}$ are equivalent, $\vec{x} \equiv \vec{y}$, iff there is a permutation π of $[1..n]$ so that each $y_j = x_{\pi(j)}$. Thus $T^{\odot n}$ is a factor of $T^{\times n}$, and fibers have $n!$ many points.

General Notation. If $f\colon X\to\mathbb{R}$ and $g\colon Y\to\mathbb{R}$, let $f \times g$ denote the fnc $(x,y) \mapsto f(x)g(y)$.

Let $\mathbf{1}_X$ denote the constant 1 function and let Id_X denote the identity map, each on space X. If the space is understood, we may just write $\mathbf{1}$ and Id.

For conditional expectation (abbrev: ***c.expectation***), we may write the subfield as a subscript, e.g, $\mathsf{E}_{\mathcal{X}}(f)$ rather than $\mathsf{E}(f \mid \mathcal{X})$.

5.1: Fix a joining $\eta \in \mathcal{J}(\mu,\nu)$. For each fnc $g \in \mathbb{L}^1(\nu)$, let $\bar{g}$ denote the c.expectation function

$$\bar{g} \;:=\; \mathsf{E}^{\eta}_{\mathcal{X}}(\mathbf{1}_X \times g)\,.$$

Here, $\mathcal{X}$ is interpreted as a subfield of $\mathcal{X}\times\mathcal{Y}$, so $\bar{g}$ is a fnc on $X\times Y$. Furthermore, the c.expectation is taken w.r.t measure η.

a Suppose, for each fnc $g \in \mathbb{L}^1(\nu)$, that $\bar{g}$ is η-a.e constant. Prove that η is product measure. (Argue that η agrees with $\mu\times\nu$ on measurable rectangles.)

b Let $Id := Id_Y$ and suppose that $\eta \in \mathcal{J}(T, Id)$ has the property, for each fnc $g \in \mathbb{L}^1(\nu)$, that $\bar{g}$ is η-a.e invariant under $T\times Id$. If T is ergodic then argue that η must be product measure. Where do you use that η itself is invariant under $T\times Id$?

Conclude that every ergodic trn is disjoint from every identity trn.

5.2: Fix a trn T and let $S := T^{\odot 2}$.

a Show that T and S are *not* disjoint.

b If T has MSJ of order 2, show that T and S *are* co-prime. You may use without proof that T is necessarily prime.

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