Ergodic HW Zwei

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(*Due Monday, 21Sep2009, hopefully. Please* ***staple this sheet*** *as the* ***first page*** *of your write-up.*)

Notation. On a normed vectorspace, the ***operator-norm*** of a linear operator $\mathsf{U}:\mathbf{H}\to\mathbf{H}$, written $\|\mathsf{U}\|_{\text{op}}$, is the supremum of $\|\mathsf{U}\mathbf{v}\|$ over all unit-vectors $\mathbf{v}\in\mathbf{H}$. If $\|\mathsf{U}\|_{\text{op}}\leqslant 1$ then we call U a ***weak contraction***. Finally, use

$$\text{Fix}_{\mathsf{U}} := \{\mathbf{v}\in\mathbf{H} \mid \mathsf{U}\mathbf{v}=\mathbf{v}\}$$

for the set of fixed-points of U.

On an *inner-product space* $\mathbf{H}$, the "***adjoint*** of U" is the linear operator U^* that satisfies

$$\forall\mathbf{v},\mathbf{w}\in\mathbf{H}:\quad \langle\mathsf{U}^*\mathbf{v},\mathbf{w}\rangle \quad=\quad \langle\mathbf{v},\mathsf{U}\mathbf{w}\rangle\,.$$

H4: Henceforth $\mathsf{U}:\mathbf{H}\to\mathbf{H}$ is a weak contraction on a Hilbert space. (Below, are two facts we used in our proof of the full $\mathbb{L}^2$ Ergodic Thm.)

a Please prove that U^* is a weak contraction. (You are free to establish a stronger statement.)

b Prove that $\text{Fix}_{\mathsf{U}^*}=\text{Fix}_{\mathsf{U}}$.

H5: Fix a rotation number $\boldsymbol{\alpha}\in\mathbb{R}$. Let X and Y denote copies of $[0,1)$, viewed (and topologized) as the circle-group, with $\oplus$ and $\ominus$ denoting addition and subtraction mod 1. Use $m()$ for arclength measure, and let $R{=}R_{\boldsymbol{\alpha}}$ be the rotation $x\mapsto x\oplus\boldsymbol{\alpha}$.

Prove that $R\times R$ is measure-theoretically isomorphic to $\mathit{Id}\times R$, by producing an explicit (very simple) bi-mp map $f:X{\times}Y\to X{\times}Y$ such that this diagram

$$\begin{array}{ccc} X\times Y & \xrightarrow{R\times R} & X\times Y \\ f\big\downarrow & & f\big\downarrow \\ X\times Y & \xrightarrow{\mathit{Id}\times R} & X\times Y \end{array}$$

commutes. (Here, $\mathit{Id}=\mathit{Id}_X$ is the identity map on X.) The f that I'm imagining is a homeomorphism, and is "algebraic". Make sure to prove that *your* f is measure-preserving. Please draw a *picture* showing how your f works. (By the way, how does your f vary as a function of $\boldsymbol{\alpha}$?)

Finally, use your isomorphism to show that $R\times R$ is *never* ergodic (no matter what $\boldsymbol{\alpha}$ is). Produce an explicit non-trivial $R{\times}R$-invariant set.