Probability II
MAP6473 4810

Homework-6

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Reading. Please read the ergodic theory chapter of Billingsley.

Notation in force. For sets $E, A \subset \mathbb{N}$, say that "E ***eventually includes*** A" if

$$E \;\supset\; A \cap [N \,..\, \infty)$$

for *some* sufficiently large N.

6a: Suppose that $A_1, A_2, \ldots$ are zero-density subsets of $\mathbb{N}$. Then there exists a zero-density set E which eventually-includes each A_j. [*Hint:* Think Cantor diagonalization.]

Consider a sequence $\vec{\mathsf{X}}$ of non-negative reals. For each $\varepsilon > 0$, let

$$\mathcal{N}_\varepsilon \;:=\; \{n \mid x_n \geqslant \varepsilon\}\,.$$

For each ε, suppose that each $\mathcal{N}_\varepsilon$ is a zero-density set. **Prove** that there exists a zero-density set $\mathcal{Z} \subset \mathbb{N}$ so that $x_n \to 0$ off of $\mathcal{Z}$; that is, as $n \to \infty$ with $n \in \mathbb{N} \smallsetminus \mathcal{Z}$.

6b: Do Billingsley's problem, $24.7P.326$. Here $(T\colon X, \mathcal{X}, \mathsf{P})$ is a *mixing* transformation. A fnc $\delta\colon X \to [0, \infty)$, with $\int_X \delta()\, \mathrm{d}\mathsf{P} = 1$, gives rise to a *new* probability measure μ (the text calls it P_0) by

$$\mu(A) \;:=\; \int_A \delta()\, \mathrm{d}\mathsf{P}\,.$$

Prove, for each measurable B, that

$$\mu(T^{-n}(B)) \;\to\; \mathsf{P}(B)$$

as $n \to \infty$. [*Hint:* You might first want to consider the case where δ is a (scaled) indicator function $[1/\mathsf{P}(A)]\mathbf{1}_A$.]