Probability II
MAP6473 4810

Homework-1b

Prof. JLF King
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Reading. Please get your copy of his text as soon as possible.

Please read Sect.7, pages 92–108 by Monday, 19Jan.

Please read part of Sect.8, pages 111–121 by Friday, 23Jan.

1b1: Fix a posint $\mathfrak{D}$. Let $\mathbb{P} = \mathbb{P}^{\mathfrak{D}-1}$ be the simplex of probability vectors $\mathbf{v} \in \mathbb{R}^{\mathfrak{D}}$. Fix a $\mathfrak{D}\times\mathfrak{D}$ ***(column)-stochastic*** matrix M; each column is a prob.vec.

Given a vector $\mathbf{v} \in \mathbb{P}$, define the ***Cesàro average***

$$\mathbf{v}_N \;:=\; \mathbb{A}_N(\mathbf{v}) \;:=\; \tfrac{1}{N}\sum\nolimits_{j\in[0\,..\,N)} \mathsf{M}^j\mathbf{v}\,.$$

Prove that

$$\lim_{N\to\infty} \mathbb{A}_N(\mathbf{v})$$

exists, and is in $\mathbb{P}$. [*Hint:* See EE1 on Markov pamphlet.]

1b2: A $\mathfrak{D}\times\mathfrak{D}$ Markov matrix M determines transition probabilities. Use $\tau(A\,B)$ to denote the transition prob from state A to B (it is the A,B-entry in M). A distribution $\sigma()$ on the states (i.e, an M-left-invariant col-vector) determines a one-sided Markov process $\vec{\mathsf{Y}} = \mathsf{Y}_0\mathsf{Y}_1\mathsf{Y}_2\ldots$ where the prob of $\mathsf{Y}_0\mathsf{Y}_1\ldots\mathsf{Y}_N$ equaling word $w_0w_1\ldots w_N$ is the product

†: $$\sigma(w_0)\tau(w_0\,w_1)\tau(w_1\,w_2)\cdots\tau(w_{N-1}\,w_N)\,.$$

Invariance. Now suppose that $\sigma()$ is an invariant distribution. This gives us a *doubly-infinite* Markov chain

$$\overleftrightarrow{\mathsf{Y}} \;=\; \ldots\mathsf{Y}_{\text{-}2}\mathsf{Y}_{\text{-}1}\mathsf{Y}_0\mathsf{Y}_1\mathsf{Y}_2\ldots$$

by using (†) to define finite joint-distributions, starting from anywhere in time.

Define the time-reversal r.var $\mathsf{Z}_n := \mathsf{Y}_{\text{-}n}$ and call the resulting bi-infinite process $\overleftrightarrow{\mathsf{Z}}_\sigma$ (in principle, it depends on σ). Proof that $\overleftrightarrow{\mathsf{Z}}_\sigma$ is itself a Markov process.

Consider this 3-state Markov matrix from the last HW:

- State A goes to B and C each with prob=$\frac{1}{2}$.
- State B goes to states A,B,C with probabilities $\frac{1}{4},\frac{1}{4},\frac{1}{2}$.
- State C goes to A,B with probabilities $\frac{1}{3},\frac{2}{3}$. This has a unique invariant vector (distribution) σ. *Compute the invariant vector* for the time-reversal process $\overleftrightarrow{\mathsf{Z}}_\sigma$.