The Triangle

Let $\mathbf{T}$ be the triangle $\triangle ABC$. Following the usual convention, $\angle A$ or just $A$ itself will also denote the interior angle at vertex $A$. The edge opposite vertex $A$ is lowercase $a$, etc. Also, $a$ denotes the length of edge $a$.

Tools. Recall the Law of Cosines, which asserts that

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(C).$$

Also note that

1: \hspace{1cm} \text{Area} (\triangle ABC) = \frac{1}{2}ab \cdot \sin(C),

since $b \cdot \sin(C)$ is $\text{Len}(A$-altitude); i.e., down to edge $a$.

2: Heron’s formula. Fix $\mathbf{T} := \triangle ABC$. Then

$$[4 \cdot \text{Area}(\mathbf{T})]^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

$$\text{note} \hspace{1cm} 2 \left[ a^2b^2 + b^2c^2 + a^2c^2 \right] - \left[ a^4 + b^4 + c^4 \right].$$

Equivalently (and classically),

2ii: \hspace{1cm} \text{Area}(\mathbf{T}) = \sqrt{\sigma \cdot [\sigma - a][\sigma - b][\sigma - c]},

where $\sigma := \frac{a+b+c}{2}$ is the semi-perimeter of $\mathbf{T}$. 

\begin{proof}

The Pythag thm, in form $\sin^2 \theta = 1^2 - \cos^2 \theta$, gives

$$[4 \cdot \text{Area}]^2 \stackrel{\text{Pythag}}{=} [2ab \cdot \sin(C)]^2 \stackrel{\text{LoCos}}{=} [2ab]^2 - [2ab \cdot \cos(C)]^2 \stackrel{\text{Pythag}}{=} [2ab]^2 - [a^2 + b^2 - c^2]^2.$$

This last doesn’t look symmetric in $a,b,c$, but squaring, then adding, produces $\text{RHS}(2i)$, as desired.

\end{proof}

Inscribed radius and circum-radius. Let $r$ and $O$ denote the radius and center of the in-circle of $\mathbf{T} := \triangle ABC$. Evidently $\text{Area}(\triangle AOB) = \frac{1}{2}rc$. Adding this to the areas of $\triangle BOC$ and $\triangle COA$ yields that $\text{Area}(\mathbf{T}) = \frac{1}{2}r \cdot [a + b + c]$. Equivalently

3.1: \hspace{1cm} \text{InRadius}(\mathbf{T}) = \frac{2 \cdot \text{Area}(\mathbf{T})}{\text{Perimeter}(\mathbf{T})}.

Now let $R$ denote the radius of $\text{CircumCircle(} \mathbf{T} \text{)}$. Let $2\gamma$ denote the central angle $\angle AOB$ of the circle-arc not owning $C$. Dropping a perpendicular from $O$ to chord $AB$ we see that

$$\sin(\gamma) = \frac{1}{2} \frac{c}{R} = \frac{c}{2R}.$$ 

Thus

$$R = \frac{c}{2 \sin(\gamma)} = \frac{a \cdot b \cdot c}{2ab \cdot \sin(\gamma)}.$$ 

The Central-angle thm asserts that the inscribed $\angle C$ equals $\gamma$. So (1) hands us

3.2: \hspace{1cm} \text{CircumRadius}(\mathbf{T}) = \frac{a \cdot b \cdot c}{4 \cdot \text{Area}(\mathbf{T})}.

We get this curious corollary for the radii-ratio:

3.3: \hspace{1cm} \frac{\text{CircumRadius}}{\text{InRadius}} = \frac{[a \cdot b \cdot c][a + b + c]}{8 \cdot [\text{Area}^2]}.

Reciprocating, then using Heron’s formula (2ii), gives

$$\frac{2 \cdot \text{InRadius}}{\text{CircumRad}} = \frac{[a+b+c][a-b+c][a+b-c]}{[a \cdot b \cdot c]}$$

3.4: \hspace{1cm} = \left[ \frac{b+c}{a} - 1 \right] \left[ \frac{a+c}{b} - 1 \right] \left[ \frac{a+b}{c} - 1 \right].
Boomerangs cannot tile a convex polygon

(Problem from David Gale.) A boomerang is a non-convex quadrilateral; call its \( > \frac{\pi}{2} \) interior-angle “fat”. Conversely, a quadrilateral with each angle \( \leq \frac{\pi}{2} \) is a kite. A dissection of a polygon \( P \) into \( \text{finitely many} \) quadrilaterals is a “quadritiling of \( P \)”. The tiles in a quadritiling need not be congruent to each other.

4.1: Boom-Kite Thm. Each quadritiling of a convex polygon \( P \) must use a kite.

4.2: Fails with “Quad” replaced by “Penta”. Let \( P \) be the square with vertices \( (\pm2,\pm2) \). Cut \( P \) with a polygonal path going from/to \((2,2) \to (-1,1) \to (1,-1) \to (-2,-2)\).

This cuts \( P \) (which is convex) into two non-convex pentagons (which are congruent to each other).

Exercise: Each polygon \( Q \), convex or not, admits a (finite) tiling by non-convex pentagons.

\[ \square \]

Finding an equation of a circle, given a non-colinear triple of points.

\[ \text{[jk: The 4×4 Det is from edgar@mps.ohio-state.edu]} \]

We have three points \( (A, \alpha), (B, \beta) \) and \( (C, \gamma) \) in the plane and we let

\[
E := \begin{bmatrix} A & \alpha & 1 \\ B & \beta & 1 \\ C & \gamma & 1 \end{bmatrix}.
\]

[More generally, the coordinates can come from an arbitrary commutative ring.]

6.1: Colinearity lemma. The triple of points is colinear IFF \( \text{Det}(E) = 0 \).

\[ \diamond \]

\( Pf \) of \( (\Rightarrow) \). The triple lies in a line, so the three rows of \( E \), viewed as points in \( \mathbb{R}^3 \), lie in a lift of that line to the \( z=1 \) plane, hence lie in a line in \( \mathbb{R}^3 \). Consequently, \( \text{Spn(Lifted points)} \) is at-most 2-dimensional.

\[ \diamond \]

\( Pf \) of \( (\Leftarrow) \). By hyp., the triple of \( E \)-rows [viewed as points in \( \mathbb{R}^2 \)] lie in plane, \( P \), through the origin. But they also lie in the \( z=1 \) plane; it misses the origin, so does not equal \( P \). Hence the intersection of these two planes lies in a line in the \( z=1 \) plane; and this line projects to a line in \( \mathbb{R}^2 \).

\[ \diamond \]

6.2: Circle-eqn lemma. An equation \( Fnc(x,y) = 0 \) of the circle through the non-colinear triple of points can be given as a 4×4 determinant-eqn:

\[
\text{Det} \begin{bmatrix} x^2 + y^2 & x & y & 1 \\ A^2 + \alpha^2 & A & \alpha & 1 \\ B^2 + \beta^2 & B & \beta & 1 \\ C^2 + \gamma^2 & C & \gamma & 1 \end{bmatrix} = 0.
\]

\[ \diamond \]

\( Pf \). Expanding \( \text{LhS}(6.3) \) along the first row shows it to be a polynomial with a common coeff for \( x^2 \) and for \( y^2 \) of \( \text{Det}(E) \); this latter is non-zero, courtesy (6.1). Hence (6.3) is the equation of some circle [which possibly is degenerate or empty].

Certainly \( (x,y) := (A, \alpha) \) satisfies (6.3), since a matrix with two rows equal has \( \text{Det}=0 \). Ditto \( (B, \beta) \) and \( (C, \gamma) \) lie on the circle. Hence the circle is not degenerate, since a non-colinear triple lies on it.

\[ \diamond \]