Generating Function examples

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Tools. Use GF for "generating function", and OGF/EGF for "Ordinary/Exponential GF". We derived in class, the following:

1.1: $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

1.2: $$\frac{1}{[1-x]^2} = \sum_{k=0}^{\infty} [k+1] \cdot x^k.$$

More generally, for L a posint,

1.3: $$\frac{1}{[1-x]^L} = \sum_{k=0}^{\infty} \binom{k+L-1}{L-1} \cdot x^k.$$

1.4: $$\log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \cdot x^k.$$

Consider $A(x) \overset{\text{OGF}}{\longleftrightarrow} \vec{\mathbf{a}}$ and $B(x) \overset{\text{OGF}}{\longleftrightarrow} \vec{\mathbf{b}}$. Recall that product $A(x)B(x)$ is the OGF of $\vec{\mathbf{c}} := \vec{\mathbf{a}} \circledast \vec{\mathbf{b}}$, the ***convolution*** of $\vec{\mathbf{a}}$ with $\vec{\mathbf{b}}$, where

1.5: $$c_n := \sum_{j+k=n} [a_j \cdot b_k],$$

where (j,k) ranges over all ordered-pairs of natnums with $j+k=n$. As a special case, note $\vec{\mathbf{a}} \circledast (1,1,\dots)$ is the partial-sum seq $\vec{\mathbf{c}}$, where $c_n = \sum_{j=0}^{n} a_j$.

Counting Involutions in $\mathbb{S}_N$

Let t_N be the # of involuTions in the N^{th} symmetric group; permutations of token set $\Omega_N := \{\widehat{1}, \widehat{2}, \dots, \widehat{N}\}$. Involutions are the perms composed only of 1-cycles and 2-cycles. Easily, $t_0 = 1$, $t_1 = 1$ and $t_2 = 2$.

Seq. $\vec{\mathbf{t}}$ grows factorial-ishly because, just counting perms with the maximun number, $h := \lfloor \frac{N}{2} \rfloor$, of 2-cycles, shows that

2a: $$t_N \geqslant [N-1][N-3][N-5]\cdots[N-h].$$

This, since $\widehat{N}$ can be paired $[N-1]$ other tokens. Now the highest unpaired token has $[N-3]$ candidate tokens to be paired with; etc. Note that RhS(2*a*) dominates $[N-2][N-4][N-6]\cdots[2 \text{ or } 1]$. Thus

2a′: $$t_N \geqslant \sqrt{[N-1]!}\,, \quad \textit{for all } N \geqslant 1.$$

2b: Lemma. *Involution-sequence* $\vec{\mathbf{t}}$ *satisfies*

$$t_{n+2} = t_{n+1} + [n+1]t_n$$

for all natnums n. ◊

Proof. There are t_{n+1} involutions in $\mathbb{S}_{n+2}$ which fix token $\widehat{n+2}$, since the remaining tokens are permuted via an involution.

The other case is that $\widehat{n+2}$ is in 2-cycle. He can be paired with $n+1$ many other tokens, leaving n tokens to be involved. ♦

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% (involut 10)

 Use Low_k for Ceil(Sqrt(k!))
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n:	t_n	Low_nM0	Low_n	t_n/n!
0:	1	*	1	1
1:	1	1	1	1
2:	2	1	2	1
3:	4	2	3	2/3
4:	10	3	5	5/12
5:	26	5	11	13/60
6:	76	11	27	19/180
7:	232	27	71	29/630
8:	764	71	201	191/10080
9:	2620	201	603	131/18144
10:	9496	603	1905	1187/453600

OGF or EGF?. The rapid growth of $\vec{\mathbf{t}}$, (2a), suggests using an EGF rather than an OGF. Define EGF

$$\mathbf{G} = \mathbf{G}(x) := \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n. \quad \textit{Note that}$$

¥: $$x \cdot \mathbf{G}' = \sum_{n=0}^{\infty} n \cdot \frac{t_n}{n!} x^n.$$

2c: Thm. *The EGF of* $\vec{\mathbf{t}}$*, the count-involutions sequence, is*

$$\mathbf{G}(x) = \exp\left(\frac{x^2}{2} + x\right) = \exp\left(x \cdot \left[\frac{x}{2} + 1\right]\right). \quad ◊$$

Proof. Multiplying (2b) by x^{n+2} gives

$$t_{n+2}\,x^{n+2} \;=\; x\cdot t_{n+1}\,x^{n+1} \;+\; x^2\cdot[n{+}1]t_n x^n\,.$$

Dividing by $[n{+}1]!$ produces

$$*:\quad [n{+}2]\cdot\frac{t_{n+2}}{[n{+}2]!}x^{n+2} \;=\; x\cdot\frac{t_{n+1}}{[n{+}1]!}x^{n+1} \;+\; x^2\cdot\frac{t_n}{n!}x^n\,.$$

Courtesy (¥), applying $\sum_{n=0}^{\infty}$ to LhS($*$) gives

$$[x\mathbf{G}'] - 0\cdot\tfrac{t_0}{0!} - 1\cdot\tfrac{t_1}{1!}x \;\stackrel{\text{note}}{=\!=}\; [x\mathbf{G}'] - x\,,$$

since $t_1 = 1$. And summing RhS($*$) hands us

$$x\cdot[\mathbf{G} - \tfrac{t_0}{0!}] \;+\; x^2\cdot\mathbf{G} \;\stackrel{\text{note}}{=\!=}\; x\mathbf{G} - x + x^2\cdot\mathbf{G}\,.$$

Equating these, then dividing by x, results in

2d: $$\mathbf{G}' \;-\; [x+1]\mathbf{G} \;=\; 0\,.$$

This is a FOLDE (First-Order Linear DE), solved by antidifferentiating coefficient-fnc -$[x{+}1]$, then negating, producing $\frac{x^2}{2}+x$. Exponentiaing this gives $W(x) := \exp(\frac{x^2}{2}+x)$. All solns to the DE have form $\alpha\cdot W(x)$, for $\alpha\in\mathbb{C}$. We need *the* α such that

$$1 \;\stackrel{\text{note}}{=\!=}\; \tfrac{t_0}{0!} \;=\; \alpha\cdot W(0) \;\stackrel{\text{note}}{=\!=}\; \alpha\cdot 1\,.$$

So $\alpha = 1$. ♦

Non-closed formula for t_n.

From (2c),

$$\mathbf{G}(x) \;=\; \sum_{k=0}^{\infty}\frac{1}{k!}\,x^k\Big[\frac{x}{2}+1\Big]^k.$$

Courtesy the Binomial thm,

$$\Big[\frac{x}{2}+1\Big]^k \;=\; \sum_{j=0}^{k}\binom{k}{j}\Big[\frac{x}{2}\Big]^j.$$

Hence $\mathbf{G}(x)$ equals

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\frac{1}{2^j}\binom{k}{j}x^{j+k} \;=\; \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{[k-j]!\;j!\;2^j}x^{j+k}.$$

Fix $n := j+k$. Then $k = n-j$ so $k-j = n-2j$. Also, the largest value of j is $\lfloor n/2\rfloor$, since $j \leqslant k$. Thus, in the above sum, the coeff of x^n is

$$\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{[n-2j]!\;j!\;2^j}\,.$$

We have established the following.

2e: Theorem. *For all natnums n,*

$$t_n = n!\cdot\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{[n{-}2j]!\;j!\;2^j} \;\stackrel{\text{note}}{=\!=}\; \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{n{-}2j,\;j,\;j}\frac{j!}{2^j},$$

where $\binom{n}{n-2j,\,j,\,j}$ denotes a multinomial coefficient. ♢

Remark. The above summand

$$\binom{n}{n-2j,\;j,\;j}\cdot\frac{j!}{2^j}$$

has the combinatorial interpretation of counting the number of involutions with precisely j many 2-cycles.

Pocket-1 holds the $n{-}2j$ fixed-pts. The j many tokens in Pocket-2 will be paired with the j tokens in Pocket-3, and there are $j!$ many ways to do the pairing.

Finally, we've over-counted by a factor of 2^j since, for each pair, we can reverse which Pocket each is in. □

Matrix description of t_n.

With matrix

$$\mathsf{M}_n \;:=\; \begin{bmatrix}0 & 1\\ n & 1\end{bmatrix} \quad\textit{and column-vector}\quad \mathbf{v}_n \;:=\; \begin{bmatrix}t_n\\ t_{n+1}\end{bmatrix},$$

we can restate recurrence (2b) as

2f: $$\mathbf{v}_{n+1} \;=\; \mathsf{M}_{n+1}\cdot\mathbf{v}_n\,.$$

Hence $\mathbf{v}_n = \mathsf{M}_n\mathsf{M}_{n-1}\cdots\mathsf{M}_2\mathsf{M}_1\mathbf{v}_0$. And $\mathbf{v}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, so

$$\begin{bmatrix}t_n\\ t_{n+1}\end{bmatrix} = \begin{bmatrix}0 & 1\\ n & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ n{-}1 & 1\end{bmatrix}\cdots\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}.$$

Observations. Since $\mathrm{Det}(\mathsf{M}_n) = $ -n, we have

$$\mathrm{Det}(\mathsf{M}_n\mathsf{M}_{n-1}\cdots\mathsf{M}_2\mathsf{M}_1) \;=\; [\text{-}1]^n\cdot n!\,.$$

The char-poly of M_n is

$$\wp(z) \;=\; z^2 - z - n \;=\; [z-\lambda_n^+]\cdot[z-\lambda_n^-]\,,$$

where the eigenvalues of M_n are

$$\lambda_n^{\pm} \;:=\; \tfrac{1}{2}\big[1\pm\sqrt{1+4n}\,\big]\,,\quad\textit{with}$$
$$\lambda_n^+ + \lambda_n^- \;=\; 1 \qquad\textit{and}\qquad \lambda_n^+\cdot\lambda_n^- \;=\; \text{-}n\,.$$

Corresponding eigenvectors are

$$\mathbf{e}_n^+ := \begin{bmatrix} \lambda_n^- / n \\ \text{-}1 \end{bmatrix} \quad \textit{and} \quad \mathbf{e}_n^- := \begin{bmatrix} \lambda_n^+ / n \\ \text{-}1 \end{bmatrix}.$$

Unfinished: as of 19Feb2018

Remark.

Do eigenvalues bound expand/shrink rate

Consider 2×2 matrix M. Let $\|\mathsf{M}\|_{\text{op}}$ denote the *maximum*, taken over all unit-vectors $\mathbf{v}$ (i.e $\|\mathbf{v}\| = 1$), of the ratio

*: $$\frac{\|\mathsf{M}\mathbf{v}\|}{\|\mathbf{v}\|}.$$

This $\|\mathsf{M}\|_{\text{op}}$ is called the "***operator norm*** of M".

For number $\mathcal{S}>0$ to be determined later, define

$$\mathsf{E} := \begin{bmatrix} \mathcal{S}^2 + \mathcal{S} & \mathcal{S}^3 - \mathcal{S}^2 \\ \mathcal{S} - 1 & \mathcal{S}^2 + \mathcal{S} \end{bmatrix}.$$

Verify that

3: $$\mathbf{a} := \begin{bmatrix} \mathcal{S} \\ 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} := \begin{bmatrix} \text{-}\mathcal{S} \\ 1 \end{bmatrix}$$

are E-eigenvectors, with respective eigenvalues

$$\alpha := 2\mathcal{S}^2 \quad \textit{and} \quad \beta := 2\mathcal{S}.$$

But for unit-vector $\mathbf{u} := \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, product

$$\mathsf{E}\mathbf{u} = \begin{bmatrix} \mathcal{S}^3 - \mathcal{S}^2 \\ \mathcal{S}^2 + \mathcal{S} \end{bmatrix}.$$

So ratio $\frac{\|\mathsf{E}\mathbf{u}\|}{\|\mathbf{u}\|} \approx \mathcal{S}^3$. For large $\mathcal{S}$, then, this ratio is much larger than α, the largest eigenvalue.

Similarly, for $\mathbf{w} := \begin{bmatrix} 1 - \mathcal{S} \\ [\mathcal{S}+1]/\mathcal{S} \end{bmatrix}$, note

$$\mathsf{E}\mathbf{w} = \begin{bmatrix} 0 \\ 4\mathcal{S} \end{bmatrix}.$$

For large $\mathcal{S}$, note $\|\mathbf{w}\| \approx \mathcal{S}$, so $\frac{\|\mathsf{E}\mathbf{w}\|}{\|\mathbf{w}\|} \approx 4$. □