

GENERATING FUNCTION examples

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Tools. Use GF for “generating function”, and OGF/EGF for “Ordinary/Exponential GF”. We derived in class, the following:

$$1.1: \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

$$1.2: \quad \frac{1}{[1-x]^2} = \sum_{k=0}^{\infty} [k+1] \cdot x^k.$$

More generally, for L a posint,

$$1.3: \quad \frac{1}{[1-x]^L} = \sum_{k=0}^{\infty} \binom{k+L-1}{L-1} \cdot x^k.$$

$$1.4: \quad \log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \cdot x^k.$$

Consider $A(x) \xrightarrow{\text{OGF}} \vec{a}$ and $B(x) \xrightarrow{\text{OGF}} \vec{b}$. Recall that product $A(x)B(x)$ is the OGF of $\vec{c} := \vec{a} \otimes \vec{b}$, the **convolution** of \vec{a} with \vec{b} , where

$$1.5: \quad c_n := \sum_{j+k=n} [a_j \cdot b_k],$$

where (j, k) ranges over all ordered-pairs of natnums with $j+k = n$. As a special case, note $\vec{a} \otimes (1, 1, \dots)$ is the partial-sum seq \vec{c} , where $c_n = \sum_{j=0}^n a_j$.

Counting Involutions in \mathbb{S}_N

Let t_N be the # of involutions in the N^{th} symmetric group; permutations of token set $\Omega_N := \{\widehat{1}, \widehat{2}, \dots, \widehat{N}\}$. Involutions are the perms composed only of 1-cycles and 2-cycles. Easily, $t_0 = 1$, $t_1 = 1$ and $t_2 = 2$.

Seq. \vec{t} grows factorial-ishly because, just counting perms with the maximum number, $h := \lfloor \frac{N}{2} \rfloor$, of 2-cycles, shows that

$$2a: \quad t_N \geq [N-1][N-3][N-5] \cdots [N-h].$$

This, since \widehat{N} can be paired $[N-1]$ other tokens. Now the highest unpaired token has $[N-3]$ candidate tokens to be paired with; etc. Note that RhS(2a) dominates $[N-2][N-4][N-6] \cdots [2 \text{ or } 1]$. Thus

$$2a': \quad t_N \geq \sqrt{[N-1]!}, \quad \text{for all } N \geq 1.$$

2b: Lemma. *Involution-sequence \vec{t} satisfies*

$$t_{n+2} = t_{n+1} + [n+1]t_n$$

for all natnums n . ◇

Proof. There are t_{n+1} involutions in \mathbb{S}_{n+2} which fix token $\widehat{n+2}$, since the remaining tokens are permuted via an involution.

The other case is that $\widehat{n+2}$ is in 2-cycle. He can be paired with $n+1$ many other tokens, leaving n tokens to be involved. ◇

% (involut 10)

Use Low_k for Ceil(Sqrt(k!))

n:	t_n	Low_nMO	Low_n	t_n/n!
0:	1	*	1	1
1:	1	1	1	1
2:	2	1	2	1
3:	4	2	3	2/3
4:	10	3	5	5/12
5:	26	5	11	13/60
6:	76	11	27	19/180
7:	232	27	71	29/630
8:	764	71	201	191/10080
9:	2620	201	603	131/18144
10:	9496	603	1905	1187/453600

OGF or EGF?. The rapid growth of \vec{t} , (2a), suggests using an EGF rather than an OGF. Define EGF

$$\mathbf{G} = \mathbf{G}(x) := \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n. \quad \text{Note that}$$

$$\forall: \quad x \cdot \mathbf{G}' = \sum_{n=0}^{\infty} n \cdot \frac{t_n}{n!} x^n.$$

2c: Thm. *The EGF of \vec{t} , the count-involutions sequence, is*

$$\mathbf{G}(x) = \exp\left(\frac{x^2}{2} + x\right) = \exp\left(x \cdot \left[\frac{x}{2} + 1\right]\right). \quad \diamond$$

Proof. Multiplying (2b) by x^{n+2} gives

$$t_{n+2}x^{n+2} = x \cdot t_{n+1}x^{n+1} + x^2 \cdot [n+1]t_nx^n.$$

Dividing by $[n+1]!$ produces

$$*: [n+2] \cdot \frac{t_{n+2}}{[n+2]!}x^{n+2} = x \cdot \frac{t_{n+1}}{[n+1]!}x^{n+1} + x^2 \cdot \frac{t_n}{n!}x^n.$$

Courtesy (\forall), applying $\sum_{n=0}^\infty$ to LhS(*) gives

$$[x\mathbf{G}'] - 0 \cdot \frac{t_0}{0!} - 1 \cdot \frac{t_1}{1!}x \stackrel{\text{note}}{=} [x\mathbf{G}'] - x,$$

since $t_1 = 1$. And summing RhS(*) hands us

$$x \cdot [\mathbf{G} - \frac{t_0}{0!}] + x^2 \cdot \mathbf{G} \stackrel{\text{note}}{=} x\mathbf{G} - x + x^2 \cdot \mathbf{G}.$$

Equating these, then dividing by x , results in

$$2d: \quad \mathbf{G}' - [x+1]\mathbf{G} = 0.$$

This is a FOLDE (First-Order Linear DE), solved by antidifferentiating coefficient-fnc $-[x+1]$, then negating, producing $\frac{x^2}{2} + x$. Exponentiating this gives $W(x) := \exp(\frac{x^2}{2} + x)$. All solns to the DE have form $\alpha \cdot W(x)$, for $\alpha \in \mathbb{C}$. We need *the* α such that

$$1 \stackrel{\text{note}}{=} \frac{t_0}{0!} = \alpha \cdot W(0) \stackrel{\text{note}}{=} \alpha \cdot 1.$$

So $\alpha = 1$. ♦

Non-closed formula for t_n . From (2c),

$$\mathbf{G}(x) = \sum_{k=0}^\infty \frac{1}{k!} x^k \left[\frac{x}{2} + 1 \right]^k.$$

Courtesy the Binomial thm,

$$\left[\frac{x}{2} + 1 \right]^k = \sum_{j=0}^k \binom{k}{j} \left[\frac{x}{2} \right]^j.$$

Hence $\mathbf{G}(x)$ equals

$$\sum_{k=0}^\infty \sum_{j=0}^k \frac{1}{k!} \frac{1}{2^j} \binom{k}{j} x^{j+k} = \sum_{k=0}^\infty \sum_{j=0}^k \frac{1}{[k-j]! j! 2^j} x^{j+k}.$$

Fix $n := j+k$. Then $k = n-j$ so $k-j = n-2j$. Also, the largest value of j is $\lfloor n/2 \rfloor$, since $j \leq k$. Thus, in the above sum, the coeff of x^n is

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{[n-2j]! j! 2^j}.$$

We have established the following.

2e: Theorem. For all natnums n ,

$$t_n = n! \cdot \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{[n-2j]! j! 2^j} \stackrel{\text{note}}{=} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{n-2j, j, j} \frac{j!}{2^j},$$

where $\binom{n}{n-2j, j, j}$ denotes a multinomial coefficient. ♦

Remark. The above summand

$$\binom{n}{n-2j, j, j} \cdot \frac{j!}{2^j}$$

has the combinatorial interpretation of counting the number of involutions with precisely j many 2-cycles.

Pocket-1 holds the $n-2j$ fixed-pts. The j many tokens in Pocket-2 will be paired with the j tokens in Pocket-3, and there are $j!$ many ways to do the pairing.

Finally, we've over-counted by a factor of 2^j since, for each pair, we can reverse which Pocket each is in. □

Matrix description of t_n . With matrix

$$\mathbf{M}_n := \begin{bmatrix} 0 & 1 \\ n & 1 \end{bmatrix} \quad \text{and column-vector} \quad \mathbf{v}_n := \begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix},$$

we can restate recurrence (2b) as

$$2f: \quad \mathbf{v}_{n+1} = \mathbf{M}_{n+1} \cdot \mathbf{v}_n.$$

Hence $\mathbf{v}_n = \mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_2 \mathbf{M}_1 \mathbf{v}_0$. And $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$$\begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ n & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ n-1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Observations. Since $\text{Det}(\mathbf{M}_n) = -n$, we have

$$\text{Det}(\mathbf{M}_n \mathbf{M}_{n-1} \cdots \mathbf{M}_2 \mathbf{M}_1) = [-1]^n \cdot n!.$$

The char-poly of \mathbf{M}_n is

$$\wp(z) = z^2 - z - n = [z - \lambda_n^+] \cdot [z - \lambda_n^-],$$

where the eigenvalues of \mathbf{M}_n are

$$\lambda_n^\pm := \frac{1}{2} [1 \pm \sqrt{1+4n}], \quad \text{with} \\ \lambda_n^+ + \lambda_n^- = 1 \quad \text{and} \quad \lambda_n^+ \cdot \lambda_n^- = -n.$$

Corresponding eigenvectors are

$$\mathbf{e}_n^+ := \begin{bmatrix} \lambda_n^-/n \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_n^- := \begin{bmatrix} \lambda_n^+/n \\ -1 \end{bmatrix}.$$

Unfinished: as of 7Jun2018

Remark.

Do eigenvalues bound expand/shrink rate

Consider 2×2 matrix M . Let $\|M\|_{\text{op}}$ denote the *maximum*, taken over all unit-vectors \mathbf{v} (i.e $\|\mathbf{v}\| = 1$), of the ratio

$$*: \quad \frac{\|M\mathbf{v}\|}{\|\mathbf{v}\|}.$$

This $\|M\|_{\text{op}}$ is called the “*operator norm* of M ”.

For number $\mathcal{S} > 0$ to be determined later, define

$$E := \begin{bmatrix} \mathcal{S}^2 + \mathcal{S} & \mathcal{S}^3 - \mathcal{S}^2 \\ \mathcal{S} - 1 & \mathcal{S}^2 + \mathcal{S} \end{bmatrix}.$$

Verify that

$$3: \quad \mathbf{a} := \begin{bmatrix} \mathcal{S} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} -\mathcal{S} \\ 1 \end{bmatrix}$$

are E -eigenvectors, with respective eigenvalues

$$\alpha := 2\mathcal{S}^2 \quad \text{and} \quad \beta := 2\mathcal{S}.$$

But for unit-vector $\mathbf{u} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, product

$$E\mathbf{u} = \begin{bmatrix} \mathcal{S}^3 - \mathcal{S}^2 \\ \mathcal{S}^2 + \mathcal{S} \end{bmatrix}.$$

So ratio $\frac{\|E\mathbf{u}\|}{\|\mathbf{u}\|} \approx \mathcal{S}^3$. For large \mathcal{S} , then, this ratio is much larger than α , the largest eigenvalue.

Similarly, for $\mathbf{w} := \begin{bmatrix} 1 - \mathcal{S} \\ [\mathcal{S} + 1]/\mathcal{S} \end{bmatrix}$, note

$$E\mathbf{w} = \begin{bmatrix} 0 \\ 4\mathcal{S} \end{bmatrix}.$$

For large \mathcal{S} , note $\|\mathbf{w}\| \approx \mathcal{S}$, so $\frac{\|E\mathbf{w}\|}{\|\mathbf{w}\|} \approx 4$. □