Tools. Use GF for “generating function”, and OGF/EGF for “Ordinary/Exponential GF”. We derived in class, the following:

1.1: \( \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n. \)
1.2: \( \frac{1}{[1-x]^2} = \sum_{k=0}^{\infty} (k+1) \cdot x^k. \)

More generally, for \( L \) a posint,

1.3: \( \frac{1}{[1-x]^L} = \sum_{k=0}^{\infty} \binom{k+L-1}{L-1} \cdot x^k. \)
1.4: \( \log\left(\frac{1}{1 - x}\right) = \sum_{n=k=1}^{\infty} \frac{1}{n} \cdot x^k. \)

Consider \( A(x) \xrightarrow{\text{OGF}} \vec{a} \) and \( B(x) \xrightarrow{\text{OGF}} \vec{b}. \) Recall that product \( A(x)B(x) \) is the OGF of \( \vec{c} := \vec{a} \otimes \vec{b}, \) the convolution of \( \vec{a} \) with \( \vec{b}, \) where

1.5: \( c_n := \sum_{j+k=n} [a_j \cdot b_k], \)

where \( (j, k) \) ranges over all ordered-pairs of natnums with \( j+k = n. \) As a special case, note \( \vec{a} \otimes (1, 1, \ldots) \) is the partial-sum seq \( \vec{c}, \) where \( c_n = \sum_{j=0}^{n} a_j. \)

Counting Involutions in \( S_N \)

Let \( t_N \) be the \# of involutions in the \( N^{th} \) symmetric group; permutations of token set \( \Omega_N := \{1, 2, \ldots, \tilde{N}\}. \) Involutions are the perms composed only of 1-cycles and 2-cycles. Easily, \( t_0 = 1, t_1 = 1 \) and \( t_2 = 2. \)

Seq. \( \vec{t} \) grows factorial-ishy because, just counting perms with the maximum number, \( h := \left\lfloor \frac{N}{2} \right\rfloor, \) of 2-cycles, shows that

2a: \( t_N \geq [N-1][N-3][N-5] \cdots [N-h]. \)

This, since \( \tilde{N} \) can be paired \([N-1]\) other tokens. Now the highest unpaired token has \([N-3]\) candidate tokens to be paired with; etc. Note that RhS(2a) dominates \([N-2][N-4][N-6] \cdots [2 \text{ or } 1]. \) Thus

2a': \( t_N \geq \sqrt{[N-1]!}, \text{ for all } N \geq 1. \)

2b: Lemma. Involution-sequence \( \vec{t} \) satisfies

\[ t_{n+2} = t_{n+1} + [n+1]t_n \]

for all natnums \( n. \)

Proof. There are \( t_{n+1} \) involutions in \( S_{n+2} \) which fix token \( n+2, \) since the remaining tokens are permuted via an involution.

The other case is that \( n+2 \) is in 2-cycle. He can be paired with \( n+1 \) many other tokens, leaving \( n \) tokens to be involved.

\% (involut 10)

Use Low_k for Ceil(Sqrt(k!))

\[
\begin{align*}
\text{n:} & \quad \text{t_n} \quad \text{Low_nMO} \quad \text{Low_n} \quad \text{t_n/n!} \\
0: & \quad 1 \quad * \quad 1 \quad 1 \\
1: & \quad 1 \quad 1 \quad 1 \quad 1 \\
2: & \quad 2 \quad 1 \quad 2 \quad 1 \\
3: & \quad 4 \quad 2 \quad 3 \quad 2/3 \\
4: & \quad 10 \quad 3 \quad 5 \quad 5/12 \\
5: & \quad 26 \quad 5 \quad 11 \quad 13/60 \\
6: & \quad 76 \quad 11 \quad 27 \quad 19/180 \\
7: & \quad 232 \quad 27 \quad 71 \quad 29/630 \\
8: & \quad 764 \quad 71 \quad 201 \quad 191/10080 \\
9: & \quad 2620 \quad 201 \quad 603 \quad 131/18144 \\
10: & \quad 9496 \quad 603 \quad 1905 \quad 1187/453600
\end{align*}
\]

OGF or EGF?. The rapid growth of \( \vec{t}, \) (2a), suggests using an EGF rather than an OGF. Define EGF

\[
G = G(x) := \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n. \quad \text{Note that}
\]

\[
Y := x \cdot G' = \sum_{n=0}^{\infty} n \cdot \frac{t_n}{n!} x^n.
\]

2c: Thm. The EGF of \( \vec{t}, \) the count-involutions sequence, is

\[
G(x) = \exp\left(\frac{x^2}{2} + x\right) = \exp\left(x \left[\frac{x}{2} + 1\right]\right). \quad \Diamond
\]
Proof. Multiplying (2b) by \( x^{n+2} \) gives
\[
 t_{n+2} x^{n+2} = x \cdot t_{n+1} x^{n+1} + x^2 \cdot [n+1] t_n x^n .
\]
Dividing by \([n+1]!\) produces
\[
*:\quad [n+2] \cdot \frac{t_{n+2}}{[n+2]!} x^{n+2} = x \cdot \frac{t_{n+1}}{[n+1]!} x^{n+1} + x^2 \cdot \frac{t_n}{n!} x^n .
\]
Courtesy (Y), applying \( \sum_{n=0}^{\infty} 0 \) to \text{LhS}(*) gives
\[
[x G'] - 0 \cdot \frac{t_0}{0!} - 1 \cdot \frac{t_1}{1!} x \quad \text{note} \quad [x G'] - x ,
\]
since \( t_1 = 1 \). And summing \text{Rhs}(*) hands us
\[
x \cdot [G - \frac{t_0}{0!}] + x^2 \cdot G \quad \text{note} \quad x G - x + x^2 \cdot G .
\]
Equating these, then dividing by \( x \), results in
2d:
\[
G' - [x+1]G = 0 .
\]
This is a FOLDE (First-Order Linear DE), solved by antidifferentiating coefficient-fnc \(-[x+1]\), then negating, producing \( \frac{x}{2} + x \). Exponentiating this gives
\( W(x) := \exp(x^2 + x) \). All solns to the DE have form
\( \alpha W(x) \), for \( \alpha \in \mathbb{C} \). We need \( \alpha \) such that
\[
1 \quad \text{note} \quad \frac{t_0}{0!} = \alpha \cdot W(0) \quad \text{note} \quad \alpha \cdot 1 .
\]
So \( \alpha = 1 \).

Non-closed formula for \( t_n \). From (2c),
\[
G(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \left[ \frac{x}{2} + 1 \right]^k .
\]
Courtesy the Binomial thm,
\[
\left[ \frac{x}{2} + 1 \right]^k = \sum_{j=0}^{k} \binom{k}{j} \left[ \frac{x}{2} \right]^j .
\]
Hence \( G(x) \) equals
\[
\sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{2j} \binom{k}{j} x^{j+k} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(k-j)!} j! \cdot 2^j x^{j+k} .
\]
Fix \( n := j + k \). Then \( k = n - j \) so \( k - j = n - 2j \).
Also, the largest value of \( j \) is \( \lfloor n/2 \rfloor \), since \( j \leq k \). Thus, in the above sum, the coeff of \( x^n \) is
\[
\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{[n-2j]!} j! 2^j .
\]
We have established the following.

2e: Theorem. For all natnums \( n \),
\[
t_n = n! \cdot \sum_{j=0}^{[n/2]} \frac{1}{[n-2j]!} \frac{1}{j!} \cdot \text{note} \ \sum_{j=0}^{[n/2]} \binom{n}{n-2j} \frac{n}{2j} ! j! \cdot \frac{1}{2j} ,
\]
where \( \binom{n}{n-2j} \) denotes a multinomial coefficient. ♦

Remark. The above summand
\[
\left( \frac{n}{n-2j}, j, j \right) \cdot \frac{j!}{2j}!
\]
has the combinatorial interpretation of counting the number of involutions with precisely \( j \) many 2-cycles.
Pocket-1 holds the \( n-2j \) fixed-pts. The \( j \) many tokens in Pocket-2 will be paired with the \( j \) tokens in Pocket-3, and there are \( j! \) many ways to do the pairing.
Finally, we’ve over-counted by a factor of \( 2^j \) since, for each pair, we can reverse which Pocket each is in. □

Matrix description of \( t_n \). With matrix
\[
M_n := \begin{bmatrix} 0 & 1 & 0 \\ n & 1 & 1 \end{bmatrix} \quad \text{and column-vector} \quad v_n := \begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix} ,
\]
we can restate recurrence (2b) as
\[
2f: \quad v_{n+1} = M_{n+1} \cdot v_n .
\]
Hence \( v_n = M_n M_{n-1} \cdots M_2 M_1 v_0 \). And \( v_0 = \begin{bmatrix} 1 \end{bmatrix} \), so
\[
\begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ n & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ n-1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .
\]

Observations. Since \( \text{Det}(M_n) = -n \), we have
\[
\text{Det}(M_n M_{n-1} \cdots M_2 M_1) = \begin{bmatrix} -1 \end{bmatrix}^n \cdot n ! .
\]
The char-poly of \( M_n \) is
\[
\varphi(z) = z^2 - z - n = [z - \lambda_+^n] \cdot [z - \lambda_-^n] ,
\]
where the eigenvalues of \( M_n \) are
\[
\lambda_+^n := \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4n} \right] , \quad \text{with} \quad \lambda_+^n + \lambda_-^n = 1 \quad \text{and} \quad \lambda_+^n \cdot \lambda_-^n = -n .
\]
Corresponding eigenvectors are
\[ e^+_n := \begin{bmatrix} \lambda_n / n \\ 1 \end{bmatrix} \quad \text{and} \quad e^-_n := \begin{bmatrix} \lambda_n^* / n \\ 1 \end{bmatrix}. \]

Unfinished: as of 29Jan2019

**Remark.**

---

**Do eigenvalues bound expand/shrink rate**

Consider $2 \times 2$ matrix $M$. Let $\|M\|_{\text{op}}$ denote the maximum, taken over all unit-vectors $v$ (i.e $\|v\| = 1$), of the ratio
\[
\star: \quad \frac{\|Mv\|}{\|v\|}.
\]

This $\|M\|_{\text{op}}$ is called the “operator norm of $M$”.

For number $S > 0$ to be determined later, define
\[
E := \begin{bmatrix} S^2 + S & S^3 - S^2 \\ S - 1 & S^2 + S \end{bmatrix}.
\]

Verify that
\[
3: \quad a := \begin{bmatrix} S \\ 1 \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} -S \\ 1 \end{bmatrix}
\]
are $E$-eigenvectors, with respective eigenvalues
\[
\alpha := 2S^2 \quad \text{and} \quad \beta := 2S.
\]

But for unit-vector $u := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, product
\[
Eu = \begin{bmatrix} S^3 - S^2 \\ S^2 + S \end{bmatrix}.
\]

So ratio $\frac{\|Eu\|}{\|u\|} \approx S^3$. For large $S$, then, this ratio is much larger than $\alpha$, the largest eigenvalue.

Similarly, for $w := \begin{bmatrix} 1 - S \\ [S+1]/S \end{bmatrix}$, note
\[
Ew = \begin{bmatrix} 0 \\ 4S \end{bmatrix}.
\]

For large $S$, note $\|w\| \approx S$, so $\frac{\|Ew\|}{\|w\|} \approx 4.$

---

**Warm-up.** Consider sequence $\tilde{a} := (a_0, a_1, a_2, \ldots)$ where
\[ a_0 := 0 \quad \text{and} \quad a_{n+1} := 1 + 3a_n. \]

We’ll get a formula for $a_n$, by manipulating the entire sequence $\tilde{a}$. The method is to define the corresponding (ordinary) generating function [abbrev. OGF]
\[
A(x) := \sum_{n=0}^{\infty} a_n \cdot x^n.
\]

The RHS, here, converges for all $|x| < \frac{1}{3}$. However, it turns out that, for most problems, we can view the RHS as a formal power series, and never worry about convergence.

Let’s manipulate OGF $A(x)$ so as to get a recurrence formula for it. Note first that
\[
\star: \quad \sum_{n=0}^{\infty} a_{n+1} \cdot x^n = \left[ \sum_{n=0}^{\infty} a_n \cdot x^n \right] + 3 \left[ \sum_{n=0}^{\infty} a_n \cdot x^n \right]
= \frac{1}{1-x} + 3A(x).
\]

OTOHand, product $x \cdot \text{Lhs}(\star)$ equals
\[
x \cdot \sum_{n=0}^{\infty} a_{n+1} \cdot x^n = \sum_{k=1}^{\infty} a_k \cdot x^k = A(x),
\]
since $a_0 = 0$. Thus, multiplying ($\star$) by $x$, gives
\[
A = x \cdot \left[ \frac{1}{1-x} + 3A \right] = \frac{x}{1-x} + 3xA.
\]

Hence $A \cdot [1 - 3x] = \frac{x}{1-x}$, so $A = \frac{x}{[1-3x][1-x]}$. The partial-fraction decomposition now yields
\[
A = \frac{1/2}{1-3x} - \frac{1/2}{1-x} = \frac{1}{2} \left[ \sum_{n=0}^{\infty} [3^n] x^n \right] - \left[ \sum_{n=0}^{\infty} x^n \right].
\]

Combining terms gives the power series expansion,
\[
A(x) = \sum_{n=0}^{\infty} \left[ \frac{1}{2} [3^n - 1] \right] \cdot x^n.
\]

And you can easily check that the formula
\[
a_n := \frac{1}{2} [3^n - 1], \quad \text{for } n = 0, 1, 2, \ldots,
\]
indeed satisfies the original (†) recurrence.

Unless mentioned otherwise, the following problems are from Bona’s text.

5.1: #48P.207. Let $g_n$ be the number of (combinatorial) simple graphs on $[1..n]$ in which each vertex has degree 2. With $G(x) \xrightarrow{\text{Egf}} \tilde{g}$, prove that
\[
G(x) = \frac{1}{\sqrt{1-x}} \cdot e^{x^2 - \frac{x^3}{2}}.
\]

**Soln.** On a $k$-set, let $a_k$ be the number of cyclic simple graphs using all $k$ of the vertices. So $a_0 = a_1 = a_2 = 0$ (The $a_0 = 0$ needs comment). For $k \geq 3$, there are $[k-1]!$ circular permutations.
### 6.1: #43P176 (People)

Let \( q_n \) be the number of ways of partitioning \( n \) people into groups labeled “E”, “D” and “A”, and asking each group to form a line, where group E has an even number of folk, group D oddly many, and group A has an arbitrary number of people. Get a closed-formula for \( q_n \).

---

#### Ans.

We’ll give an answer first, then show three derivations. The \( N \)th **triangular number** is

\[
\tau_N := \sum_{k=1}^{N} k = \frac{1}{2} \cdot N[N+1].
\]

Define a sequence \( \vec{s} = (s_0, s_1, \ldots) \) by

\[
6.2: \quad \vec{s} = \left( \tau_0, \tau_1, \tau_1, \tau_2, \tau_2, \tau_3, \tau_3, \tau_4, \tau_4, \ldots \right) = (0, 1, 1, 3, 3, 6, 6, 10, 10, \ldots).
\]

Then, for each natnum \( n \),

\[
6.3: \quad q_n = n! \cdot s_n. \quad \square
\]

---

#### 1st Soln.

For the three groups, the corresponding EGFs are

\[
A(x) := \sum_{n=0}^{\infty} \frac{n!}{n!} \cdot x^n \stackrel{\text{note}}{=} \frac{1}{1-x};
\]

\[
E(x) := \sum_{j=0}^{\infty} \frac{1 \cdot x^{2j}}{1-x} \stackrel{\text{note}}{=} \frac{1}{1-[x^2]};
\]

\[
D(x) := \sum_{k=0}^{\infty} \frac{1 \cdot x^{2k+1}}{1-x} \stackrel{\text{note}}{=} \frac{x}{1-[x^2]^2}.
\]

Hence the EGF for \( \vec{s} \) is \( Q := A \cdot E \cdot D \). I.e

\[
6.4: \quad Q(x) = \frac{x}{[1-x^2]^2} \cdot \frac{1}{1-x}.
\]

This Rhs(6.4) is the OGF of some sequence \( \vec{s} \).

Tool (1.2) says \( \frac{1}{[1-y]^2} \stackrel{\text{OGF}}{\leftrightarrow} (1, 2, 3, \ldots) \). Hence

\[
(0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots) \stackrel{\text{OGF}}{\leftrightarrow} \frac{x}{[1-x^2]^2}.
\]

Convolving this sequence with

\[
(1, 1, 1, \ldots) \stackrel{\text{OGF}}{\leftrightarrow} \frac{1}{1-x}
\]

forms the partial sums of \( (0, 1, 0, 2, \ldots) \). Hence (6.2).

Finally, \( Q(x) \) is the EGF of \( \vec{s} \), whence (6.3). ♦

---

#### 2nd Soln.

This problem is a fancier version of #14P176, the bookshelf. Make a line of books by choosing one of the \( n! \) orderings. Let \( s_n \) be the number of ways of:

- Taking some even number from the left of the line, and putting them on shelf “E”, then taking some oddly many from the left of what remains, and putting that on shelf “D”.

Hence (6.3), and we just need to compute \( s_n \).

In the plane, consider the triangle of lattice-points,

\[
\Omega_n := \left\{ (x, y) \in \mathbb{N} \times \mathbb{N} \mid x+y \leq n \right\}.
\]

Interpret a point \((\varepsilon, \delta) \in \Omega_n\) as putting \( \varepsilon \) many books on “E”, then \( \delta \) many books on “D”, finally \( n-|\varepsilon+\delta| \) many books on the “Arbitrary” shelf. If \( \varepsilon \) is even and \( \delta \) odd, then this is a valid placement. Thus

\[
6.5: \quad \text{Our } s_n = \text{ the number of (Even, Odd) points in the } \Omega_n \text{ lattice-triangle.}
\]

Approximately half have even \( x \)-coordinate, about half have odd \( y \), and these two events are more-or-less independent. Conclusion: \( s_n = \text{ approx. } \frac{2^n}{4} \). Also, a valid \((\varepsilon, \delta)\) has \( \varepsilon+\delta \) odd. E.g., the line of lattice-points \((x, y)\) with \( x+y = 6 \) has no valid points. Hence \( s_6 = s_5 \). More generally, \( n \text{ odd } \Rightarrow s_n = s_{n+1} \). Of course, our (6.2) implies both of the circled stmts.

---

#### Counting valid pts.

Fix an odd \( n = 2k - 1 \), and consider those \((x, y) \in \mathbb{N} \times \mathbb{N} \) with \( x+y = n \). Certainly \( x \) is even IFF \( y \) is odd, so precisely \( k \) of those points are valid. It follows that \( \vec{s} \) comprises the partial sums of sequence \((0, 1, 0, 2, \ldots)\), as \( n \) takes on values \( 0, 1, 2, 3, \ldots \). So we have again derived (6.2). ♦

---

#### 3rd Soln.

Back to GFs!

The partial-fraction decom of Rhs(6.4) is

\[
\frac{-1}{16} \cdot \left[ \frac{1}{1-x} + \frac{1}{1+x} + \frac{4}{1-x^3} + \frac{2}{1+x^2} \right].
\]

This looks arduous to do directly, so let’s finesse things. The “\( 1-x \)” and “\( 1+x \)” will cause even/odd index terms to behave differently, so let’s separate them. Write \( \sum_{n=0}^{\infty} s_n x^n := Q(x) = \varepsilon(x) + \delta(x) \), where

\[
\varepsilon(x) := \sum_{n \text{ even}} s_n x^n \quad \text{and} \quad \delta(x) := \sum_{n \text{ odd}} s_n x^n.
\]
Now \(2\varepsilon(x) = Q(x) + Q(-x)\), which equals

\[
\dagger: \quad \frac{1}{[1 - x^2]_2} \cdot \left[ \frac{x}{1 - x^2} + \frac{-x}{1 - x^2} \right] = \frac{2x^2}{[1 - x^2]^3}.
\]

So \(\varepsilon(x) = f(x^2)\), where \(f(y) = \frac{y}{1 - y^3}\). By (1.3),

\[
f(y) = y \cdot \sum_{i=0}^{\infty} (i+2) y^i \Leftrightarrow \sum_{k=1}^{\infty} \tau_k \cdot y^k,
\]

by setting \(k = i+1\). And \(\tau_0 = 0\), so

\[
\varepsilon(x) = \sum_{k=1}^{\infty} \tau_k \cdot x^{2k};
\]

this justifies the even-indexed terms in (6.2).

For the odd-index terms, \(2\delta(x) = Q(x) - Q(-x)\) which equals

\[
\ddagger: \quad \frac{1}{[1 - x^2]_2} \cdot \left[ \frac{x}{1 - x^2} - \frac{-x}{1 - x^2} \right] = \frac{2x}{[1 - x^2]^3}.
\]

Comparing with (\(\dagger\)), then, \(x \cdot \delta(x) = \varepsilon(x)\). I.e, when \(n\) is an even index, then \(s_{n-1} = s_n\). Hence (6.2).

### 7.1: \#44P176.

From \(n\) people, select a committee of oddly many. From the committee, select a council of evenly many [allowing the value zero]. Get a closed-formula for \(r_n\), the number of ways of doing this. ♦

**Prelim.** View this as splitting the people into

- Group E, the council with evenly many.
- Group D with oddly many, where the committee is \(E \cup D\).
- Group A, with arbitrarily many; those that remain.

While superficially similar to \#43P176, the lack of ordering makes a difference. A seat-of-the-pants growth estimate is

\[
7.2: \quad 3^n \geq r_n \geq 2^{n-1} - 1.
\]

The first follows by removing the even/odd restrictions, so each elt of \([1..n]\) admits 3 colors.

The lower bnd holds for \(n=0\); what about \(n \geq 1\)? Well, \([1..n]\) has \(2^{n-1}\) even-cardinality subsets. Hence there are at least \(2^{n-1} - 1\) even subsets that are not all of \([1..n]\); and so we can pick one element of the complement to make a singleton D.

**Soln.** With \(R(x) \overset{\text{Egf}}{\longleftrightarrow} \overline{r}\), we wish to define EGFs so that \(R(x) = E(x)D(x)A(x)\). So \(E(x) \overset{\text{Egf}}{\longleftrightarrow} \overline{e}\), where \(e_k\) is the number of ways making an even-sized council using all \(k\) people. I.e, \(\overline{e} = \{1, 0, 1, 0, \ldots\}\). Thus

\[
E(x) = \sum_{k \text{ even}} \frac{x^k}{k!} = \frac{1}{2} [e^x + e^{-x}]. \quad \text{Similarly,}
\]

\[
D(x) = \sum_{k \text{ odd}} \frac{x^k}{k!} = \frac{1}{2} [e^x - e^{-x}].
\]

So \(E(x)D(x) = \frac{1}{4} [e^{2x} - e^{-2x}]\). Since \(A(x) = e^x\), our \(R(x) = \frac{1}{4} [e^{2x} - e^{-2x}]\). Thus for each natnum \(n\),

\[
7.3: \quad r_n = \frac{1}{4} \cdot [3^n - [-1]^n].
\]

BTWay, \(3^n - [-1]^n \equiv_4 [-1]^n - [-1]^n = 0\), as it must.
7.4: **Rem.** Curiously, (7.3) is a sum of exponentials, thus satisfies a 2-term linear recurrence. The two bases, 3 and -1, are roots of the polynomial

\[ f(x) := [x - 3][x + 1] \overset{\text{note}}{=} x^2 - 2x - 3. \]

Each base \( b \in \{3, -1\} \) satisfies \( f(b) = 0 \), i.e. \( b^2 = 2b + 3 \), and thus \( b^{n+2} = 2b^{n+1} + 3b^n \), for each \( n \). Consequently, \( \vec{r} \) satisfies recurrence

\[ r_{n+2} = 2r_{n+1} + 3r_n. \]

**Exer:** Find a bijective proof of (7.5). \( \square \)

8.1: **#38** Ex. 176. Let \( g_n \) be the number of ways of selecting a permutation of \([1..n]\), then marking a particular cycle in the permutation. Obtain a formula for \( g_n \). \( \diamond \)

**Rem.** Bona’s Ex **8.25** Ex. 171 asked how many ways to seat \( n \) people around circular tables. The answer was \( n! \) [reminding us of the Canonical cycle notation], derived from Thm 8.24. Equivalent, is to use Thm **8.28** Ex. 172 but with the outer seq trivial, the constant-1 sequence, because given \( j \) cycles, there is only one way to take all of them.

In the current problem, our outer sequence is \((0, 1, 2, \ldots)\), since there are \( j \) ways to pick one cycle from \( j \). \( \square \)

**Soln.** With \( c_0 := 0 \), henceforth let \( c_k \) be the number of cyclic-permutations of a \( k \)-set; so \( c_k = [k-1]! \). The EGF of \( \vec{c} \) is thus

\[ C(x) := \sum_{k=1}^{\infty} \frac{[k-1]!}{k!} x^k = \sum_{k=1}^{\infty} \frac{1}{k} \cdot x^k = \log\left(\frac{1}{1-x}\right), \]

by (1.4). The EGF of picking one object from \( j \) is

\[ B(x) := \sum_{j=1}^{\infty} \frac{j}{j!} \cdot x^j \overset{\text{note}}{=} x \cdot e^x. \]

**Thm** **8.28** Ex. 172 now says \( \vec{g} \) has EGF

\[ B(C(x)) = \log\left(\frac{1}{1-x}\right) \cdot \frac{1}{1-x} \]

\[ \overset{\text{OGF}}{\leftrightarrow} (0, 1, 1/2, 1/3, \ldots) \otimes (1, 1, \ldots). \]

The partial-sum seq of \((0, 1, 1/2, 1/3, \ldots)\) is the harmonic number seq. \( (H_0, H_1, H_2, \ldots) \), where

\[ H_n := \sum_{\ell=1}^{n} 1/\ell. \]

Consequently,

\[ g_n = n! \cdot H_n. \]

**Evidence?** Let’s compute \( g_3 \) by hand.

\( i \): From three 1-cycles, pick one; \( 3 \) choices.

\( ii \): There are \( \binom{3}{1} = 3 \) ways to split into a 2-cycle and a 1-cycle. For each, we have 2 ways to select a cycle, giving \( 3 \cdot 2 = 6 \) choices.

\( iii \): We can just have a single 3-cycle, and we must choose it. This gives 1 choice.
Thus $g_3 = 3 + 6 + 1 = 10$. We now use (8.3) to reassure ourselves, by computing $3! \cdot \left[ 1 + \frac{1}{2} + \frac{1}{3} \right] = 6 + 3 + 2$, which equals...  

Eleven?!  

Oh no! –Mathematics is inconsistent! -oh, woe is me, oh... 

Wait a darn minute!; $n$ objects can have many different cyclic permutations. For cycles of length 1 or 2, the cyclic ordering is unique. But in case (iii), above, there are 2 cyclic permutations of three objects. So my computation should have been: $g_3 = 3 + 6 + 2$, which –whew!– indeed equals eleven.

---

9.1: #23$^P$176. Find a formula for $a_n$, where $a_0 : = 1$ and $a_{n+1} = 3a_n + 2^n$.

**Soln.** Set $A(x) \rightarrow \mathbf{a}$. Multiply the recurrence by $x^{n+1}$ and sum, to get that $A(x) - 1 = A(x) - a_0$ equals $3xA(x) + x \cdot \sum_{n=0}^{\infty} [2x]^n$. Hence

$$[1 - 3x] \cdot A(x) = 1 + \frac{x}{1 - 2x} \overset{\text{note}}{=} \frac{1 - x}{1 - 2x}.$$ 

So

$$A(x) = \frac{1 - x}{[1 - 3x][1 - 2x]} = \frac{2}{[1 - 3x]} - \frac{1}{[1 - 2x]} = \sum_{n=0}^{\infty} (2 \cdot 3^n - 2^n) \cdot x^n.$$ 

Hence $a_n = [2 \cdot 3^n - 2^n]$, for each natnum $n$.  


10.1: #14P.176 (Books). Let \( t_n \) be the number of ways of placing \( n \) books, some on shelf A, some on shelf B, with at least one book on each shelf. Obtain a “closed formula” for \( t_n \).

\[ \text{Soln.} \text{ Since the books can be split arbitrarily between the two shelves, we’d like to take a product of EGFs. Let } A(x) \text{ be the EGF of seq } \{0, 1, 2!, \ldots\}, \text{ arranging books at least one on shelf A; so } \]

\[ A(x) = \sum_{j=1}^{\infty} \frac{j^1}{j!}x^j \text{ note } x = \frac{1}{1-x}. \]

Similarly, the EGF of arranging books on shelf B is \( B(x) = \frac{x}{1-x} \). With \( T(x) \) the EGF of \( \vec{b} \), then,

\[ T(x) = A(x)B(x) = x^2 \cdot \frac{1}{[1-x]^2} = x^2 \sum_{\ell=0}^{\infty} \frac{(\ell+1)x^\ell}{1-x} = \sum_{n=2}^{\infty} [n-1]x^n. \]

For \( n \geq 2 \), then, \( t_n/n! = n-1 \). Consequently, \( t_0 = t_1 = 0 \), and then \( t_n = n! \cdot [n-1] \cdot [n-1] \). ♦

10.2: Remark. Now we have \( \text{(10.2)} \), we can see a direct argument. Pick one of \( n! \) orderings of all the books, then put a separator at any one of the \( n-1 \) junctures between adjacent books. Those on the separator’s left, go on shelf A.

The partition function. Let \( \varphi(n) \) be the number of partitions of integer \( n \). E.g., the five partitions of 4 are \( 1+1+1+1, 1+1+2, 1+3, 2+2, 4 \); so \( \varphi(4) = 5 \). For \( n \) negative, \( \varphi(n) = 0 \). And \( \varphi(0) = 1 \). For the partition \( 1+1+3+4 \) of nine, the summands \( 1, 1, 3, 4 \) are called the \textbf{parts} of the partition.

Recall from class [or pages 98–101 of Bona] the \textbf{Ferrers diagram} of a ptn, and the \textbf{conjugate} (I also call it the transpose) of a partition.

Interpret picking the \( k \text{th-term from sum } 1 + x^2 + [x^3]^2 + [x^3]^3 + \ldots + [x^3]^k + \ldots \text{ note } \frac{1}{1-x^3}, \] as having \( k \) copies of the part 3. Consequently,

\[ P(x) := \prod_{j=1}^{\infty} \frac{1}{1-x^j} \]

is the OGF of \( [n \mapsto \varphi(n)] \). More generally, fix a subset \( S \subset \mathbb{Z}_+ \) and let \( \varphi_S(n) \) be the number of ptns of \( n \) using only parts from \( S \). Then

\[ P_S(x) := \prod_{j \in S} \frac{1}{1-x^j} \xrightarrow{\text{OGF}} [n \mapsto \varphi_S(n)]. \]

12.1: #10P.174 (LargestPart=4). Let \( b_n \) be the number of \( n \)-partitions whose largest part is 4. Compute the OGF, \( B(x) \), of \( \vec{b} \).

\[ \text{Soln.} \text{ Picking only size-4 parts, and at least one such, has OGF } [x^4 + x^8 + x^{12} + \ldots], \text{ which is } \frac{1}{1-x^4}. \text{ Hence } B(x) = x^4 \cdot \prod_{k=1}^{4} \frac{1}{1-x^k}. \]
13.1: \# $11^P.105$ (Equal-largest). Let $e_n$ be the number of $n$-ptns whose two largest parts are equal. [So $e_1 = 0$, $e_2 = 1$, $e_3 = 1$, $e_4 = 2$] Prove that

\[ e_n = \varphi(n) - \varphi(n-1). \]

\[ \text{\bf Soln.} \ 	ext{Rather than the injection argument of Thm 5.20, let's use GFs. Since } \varphi(-1) = 0, \text{ the OGF of } [n \mapsto \varphi(n-1)] \text{ is } C(x) = \sum_{n=1}^{\infty} \frac{\varphi(n-1) \cdot x^n}{n!} = x \cdot P(x). \]

Courtesy (11a), then,

\[ P(x) - C(x) = [1 - x] \cdot P(x) = \prod_{j=2}^{\infty} \frac{1}{1 - x^j}. \]

By (11b), this is $P_S(x)$ where $S := [2..\infty]$. And the transpose of an $S$-ptn is a ptn that either has no parts [i.e, $n = 0$] or its largest two parts are equal. Defining $e_0 := 1$, then, we’ve shown that $P(x) - C(x)$ equals the OGF of $\bar{e}$. Hence (*)..

14.1: \# $40^P.176$ (Derangements). Let $d_n$ be the number of derangements of $[1..n]$. Compute $D(x)$, the EGF of $\bar{d}$.

\[ \text{Defn. A derangement of a set, is a fixed-point–free permutation of the set. So the above } \bar{d} \text{ has } d_0 = 1, d_1 = 0, d_2 = 1 \text{ and } d_3 = 2. \]

\[ \text{\bf Soln.} \ 	ext{A permutation is a derangement IFF each cycle has length} \geq 2. \text{ Set } c_0 = c_1 = 0 \text{ and, for } k \geq 2, \text{ let } c_k \text{ be the number of cyclic-permutations of a } k-\text{set; so } c_k = [k-1]!. \text{ The EGF of } \bar{c} \text{ is thus } C(x) = \sum_{k=2}^{\infty} \frac{[k-1]!}{k!} x^k = \sum_{k=1}^{\infty} \frac{1}{k} x^k - x. \]

By (1.4), then, $C(x) = \log \left( \frac{1}{1-x} \right) - x$. Our Exponential Thm [Thm 8.24] now implies that

\[ \text{14.2: } D(x) = \frac{1}{1-x} \cdot e^x. \]

14.3: \# $41^P.176$ (More derangements). For $n \in \mathbb{Z}_+$, prove that

\[ 14.4: \ d_n - [n \cdot d_{n-1}] = [1]^n. \]

\[ \text{\bf Soln.} \ 	ext{Set } b_k := k \cdot d_{k-1}; \text{ so } b_0 := 0. \text{ Set } a_n := [1]^n. \text{ Let } B(x) \text{ and } A(x) \text{ be the EGFs of } \bar{b} \text{ and } \bar{a}. \text{ Thus } (14.4) \text{ will follow from } \]

\[ \text{14.5: } D(x) - B(x) \overset{?}{=} A(x). \]

\[ \text{Computing. So } A(x) = \sum_{n=0}^{\infty} \frac{[1]^n}{n!} x^n = e^{x}. \text{ And } \]

\[ B(x) = \sum_{k=1}^{\infty} \frac{k \cdot d_{k-1}}{k!} x^k \overset{\text{note}}{=} x \cdot \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n = x \cdot D(x). \]

Thus $D(x) - B(x)$ equals $[1 - x]D(x)$ which, courtesy (14.2), equals $e^{-x}$. \q
15.1: **Ex. 8.26**. Let \( t_n \) be the number of partitions of an \( n \)-set into atoms, each of cardinality 3. Get a closed formula for \( t_n \).

\[
\text{Rem. When } n \not\div 3, \text{ then } t_n = 0. \text{ With } T(x) \xrightarrow{\text{Egf}} \bar{t}, \text{ then,}
\]
\[
\begin{align*}
\star: & \quad T(x) = \sum_{k=0}^{\infty} \frac{c_k}{[3k]!} \cdot x^{3k}, \\
\text{where } c_k & := t_{3k}. 
\end{align*}
\]

1st Soln. The neat soln in Bona’s text: Let \( b_n \) be the number of ptns of an \( n \)-set, using a single atom of cardinality 3. So \( b_3 = 1 \), and every other \( b_n \) is zero. Thus \( \bar{b} \xrightarrow{\text{Egf}} x^3/3! =: B(x) \). So our Thm 8.24P 171 says \( T(x) \) equals
\[
\sum_{k=0}^{\infty} \frac{1}{k!} \cdot B(x)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{x^{3k}}{[3!]^k}.
\]
Equating terms with (\( \star \)) yields
\[
15.2: \quad c_k = \frac{[3k]!}{k! \cdot [3!]^k}.
\]

Amusingly, it is not even evident that the RhS is an integer...

“Bare hands” Soln. For a \( k > 0 \), consider a valid ptn of \( [1..n] \), where \( n := 3k \). For the other two members of the atom owning \( n \), there are \( \binom{n-1}{2} \) choices. Consequently,
\[
c_k = \binom{n-1}{2} \cdot c_{k-1}.
\]
Since \( c_0 = 1 \), iterating gives a product of \( k \) terms,
\[
c_k = \binom{n-1}{2} \cdot \binom{n-4}{2} \cdot \binom{n-7}{2} \cdots \binom{5}{2} \cdot \binom{2}{2}.
\]
So \( [2!]^k \cdot c_k \) equals
\[
1 \cdot [n-1] \cdot [n-2] \cdots 1 \cdot [n-4] \cdot [n-5] \cdot 1 \cdot [n-7] \cdot [n-8] \\
\cdots 1 \cdot 5 \cdot 4 \cdot 1 \cdot 2 \cdot 1,
\]
where I have put an italic-1 in front of each group. Replacing these 1s successively by \( n, n-3, n-6, \ldots, 6, 3 \) multiplies this product by \( 3^k \cdot k! \), and thus:
\[
3^k \cdot k! \cdot [2!]^k \cdot c_k = n! \xrightarrow{\text{note}} [3k]!.
\]

Since \( 3^k \) times \([2!]^k\) is \([3!]^k\), we can rewrite this as
\[
k! \cdot [3!]^k \cdot c_k = [3k]!.
\]
Solving for \( c_k \) now gives (15.2). But, \( Oy! \), this was so much more work....