$\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}()$

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22 November, 2017 (at *19:47*)

EM-number

Recall that the ***harmonic numbers*** $\mathcal{H}_N := \sum_{k=1}^{N} \frac{1}{k}$ upper/lower bound log(), in that

$$\mathcal{H}_{N-1} \;>\; \log(N) \;>\; \mathcal{H}_N - 1\,.$$

The ***Euler-Mascheroni*** number[♡1] is defined as the asymptotic discrepancy,

$$\text{1:}\quad \begin{aligned}\gamma \;&:=\; \lim_{N\nearrow\infty}[\mathcal{H}_{N-1} \,-\, \log(N)] \\ &=\; \lim_{N\nearrow\infty}[\mathcal{H}_N \,-\, \log(N)] \;=\; \int_1^\infty \Big[\frac{1}{\lfloor t\rfloor} \,-\, \frac{1}{t}\Big]\,\mathrm{d}t.\end{aligned}$$

Although estimate $\gamma \approx 0.578^-$ does not help with rationality, it is reassuring to know.

1a: Lemma. *The N^{th} harmonic number satisfies*

$$*1:\qquad \mathcal{H}_N \;=\; \sum_{k=1}^{\infty} \frac{N}{k[k+N]}\,.$$

Also

$$*2:\qquad \frac{N}{k[k+N]} \;=\; \int_0^1 \frac{t^{k/N}}{k}\,\mathrm{d}t\,.$$

Consequently,

$$*3:\qquad \mathcal{H}_N \;=\; -\!\int_0^1 \log(1 \,-\, t^{1/N})\,\mathrm{d}t\,. \qquad \diamondsuit$$

Pf $(*1)$. Sum $S_\ell := \sum_{k=1}^{\ell} \frac{N}{k[k+N]} = \sum_{k=1}^{\ell}\big[\frac{1}{k} - \frac{1}{k+N}\big]$ telescopes. Once $\ell \geqslant N$, then,

$$S_\ell \;=\; \Big[\sum_{k=1}^{N} \frac{1}{k}\Big] \;-\; \Big[\sum_{k=1}^{N} \frac{1}{\ell+k}\Big]\,.$$

The righthand-sum goes to zero, as $\ell\nearrow\infty$. ♦

Pf of $(*2)$. Observe $\int t^{\frac{k}{N}}\,\mathrm{d}t \;=\; \frac{N}{k+N} t^{\frac{k+N}{N}}$. Hence RhS$(*2)$ equals

$$\frac{N}{k[k+N]} \cdot t^{\frac{k+N}{N}}\Big|_{t=0}^{t=1} \;=\; \text{LhS}(*2)\,. \quad ♦$$

Pf of $(*3)$. One Taylor-series expansion for log is

$$-\log(1-x) \;\overset{\text{for } |x|<1}{=\!=\!=}\; \sum_{k=1}^{\infty} \frac{x^k}{k}\,.$$

For $0 < t < 1$, then,

$$-\log(1 \,-\, t^{1/N}) \;=\; \sum_{k=1}^{\infty} \frac{t^{\frac{k}{N}}}{k}\,.$$

It is valid to commute $\int_0^1$ with the sum, as all the summands have the same sign. Thus

$$\begin{aligned}-\!\int_0^1 \log(1 \,-\, t^{1/N})\,\mathrm{d}t \;&=\; \sum_{k=1}^{\infty} \int_0^1 \frac{t^{\frac{k}{N}}}{k}\,\mathrm{d}t \\ &=\; \sum_{k=1}^{\infty} \frac{N}{k[k+N]} \;=\; \mathcal{H}_N\,. \quad ♦\end{aligned}$$

1b: Lemma. *For all t with $0 < t < 1$,*

$$\dagger:\qquad \lim_{N\nearrow\infty} N\cdot[1 \,-\, t^{1/N}] \;=\; \text{-}\log(t)\,.$$

$$\ddagger:\qquad \lim_{N\nearrow\infty} \log(N\cdot[1 \,-\, t^{1/N}]) \;=\; \log(\text{-}\log(t))\,. \qquad \diamondsuit$$

Proof. Setting $h := \frac{1}{N}$, we can rewrite limit $(\dagger)$ as

$$-\lim_{h\searrow 0} \frac{t^h - 1}{h} \;\overset{\text{note}}{=\!=\!=}\; -f'(0)\,,$$

where $f(h) := t^h$. By definition, $f(h) = \exp(\log(t)\cdot h)$. So $f'(h) = \log(t)\cdot f(h)$. Thus $f'(0) = \log(t)$.

Lastly, $(\ddagger)$ holds, since [the outer] log is continuous. ♦

1c: $\boldsymbol{\gamma}$-$\boldsymbol{\Gamma}$ Thm. *Using fnc $\boldsymbol{\Gamma}()$ from the next section,*

$$\gamma \;=\; \int_0^1 \log(\text{-}\log(t))\,\mathrm{d}t \;=\; \text{-}\boldsymbol{\Gamma}'(1)\,. \qquad \diamondsuit$$

Proof. Eqns (1) and $(*3)$ say γ is the limit of

$$\begin{aligned}\mathcal{H}_N \,-\, \log(N) \;&=\; \Big[\text{-}\!\int_0^1 \log(1 \,-\, t^{1/N})\,\mathrm{d}t\Big] \;-\; \int_0^1 \log(N)\,\mathrm{d}t \\ &=\; \text{-}\!\int_0^1 \Big[\log(1 \,-\, t^{1/N}) \,+\, \log(N)\Big]\,\mathrm{d}t \\ &=\; \text{-}\!\int_0^1 \log\Big(N\cdot[1 \,-\, t^{1/N}]\Big)\,\mathrm{d}t\,.\end{aligned}$$

Skipping the justification needed to pass $\lim_{N\nearrow\infty}$ through the integral sign,

$$\begin{aligned}\gamma \;&=\; \text{-}\!\int_0^1 \Big[\lim_{N\nearrow\infty} \log\Big(N\cdot[1 \,-\, t^{1/N}]\Big)\Big]\,\mathrm{d}t \\ &\overset{\text{by } (\ddagger)}{=\!=\!=}\; \text{-}\!\int_0^1 \log(\text{-}\log(t))\,\mathrm{d}t\,.\end{aligned}$$

[♡1] As of Nov.2017, it is unknown if the ***EM-number*** γ is rational or irrational.

Exponential CoV. To compute this last integral, we use CoV $u = -\log(t)$. I.e, $t = e^{-u}$, so $\frac{dt}{du} = -e^{-u}$ and thus $dt = -e^{-u}\,du$.

As $0 \nearrow t \nearrow 1$, remark that $\infty \searrow u \searrow 0$. Hence,

$$\int_0^1 \log(-\log(t))\,dt \overset{\text{CoV}}{=\!=} \int_\infty^0 \log(u)\,\overbrace{[-e^{-u}]\,du}^{dt}$$
$$= \int_0^\infty \log(u)\; e^{-u}\,du\,.$$

And this last equals $\Gamma'(1)$, courtesy (2d) ♦

Gamma function

The ***Gamma fnc*** arises in volumes of N-dimensional balls, and in Laplace transforms. Leaving motivation for later, define the Gamma fnc by

2: $$\Gamma(z) := \int_0^\infty t^{z-1}\,e^{-t}\,dt\,, \quad \textit{for } z\in\mathbb{C} \textit{ with } \operatorname{Re}(z)>0\,,$$

When $\operatorname{Re}(w)>0$, note $\lim_{t\searrow 0}[t^w e^{-t}]$ is zero. Thus

$$t^w[-e^{-t}]\Big|_{t=0}^{t=\infty} \quad = \quad 0\,.$$

For $\operatorname{Re}(z)>1$, then, integration by parts produces

$$\Gamma(z) = t^{z-1}\cdot[-e^{-t}]\Big|_{t=0}^{t=\infty} - \int_0^\infty [z-1]t^{z-2}\cdot[-e^{-t}]\,dt. \quad \textit{So}$$

2a: $$\Gamma(z) \;=\; [z-1]\cdot\Gamma(z-1)\,, \quad \textit{for } \operatorname{Re}(z)>1.$$

Analytic continuation. Writing $w := z-1$,

2b: $$\Gamma(w) \;=\; \tfrac{1}{w}\cdot\Gamma(w+1)$$

from (2a). As (2d) will show, our $\Gamma()$ is ∞ly differentiable, hence is *analytic* courtesy the Cauchy-Goursat theorem. Use (2b) to iteratively extend $\Gamma()$ to the complex plane. This $\Gamma()$ is analytic on $\mathbb{C} \smallsetminus \{0,-1,-2,-3,\dots\}$. Further, $\Gamma()$ is meromorphic on $\mathbb{C}$, with simple poles at $0,-1,-2,\dots$.

To see that $\Gamma()$ has a simple pole at 0, it suffices to note that $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$ is non-zero.

Since $\Gamma(1)=1$, our (2a) implies

2c: $$\Gamma(n) \;=\; [n-1]!\,, \quad \textit{for } n\in\mathbb{Z}_+.$$

As a consequence, binomial/multinomial coefficients can be generalized using Γ, and many of the identities extend.

Calculus. Differentiating under integral sign is valid in (2). For $k=0,1,\dots$, applying $\frac{d}{dz}$ gives

2d: $$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1}e^{-t}[\log(t)]^k\,dt, \quad \textit{when } \operatorname{Re}(z)>0.$$
In particular, $$\Gamma'(1) \;=\; \int_0^\infty e^{-t}\log(t)\,dt \overset{\text{by (1c)}}{=\!=} -\gamma\,.$$

Use $\widehat{f}$ to mean the ***Laplace transform*** of f, where

$$\widehat{f}(s) \;=\; [\mathcal{L}(f)](s) \;:=\; \int_0^\infty e^{-st}\cdot f(t)\cdot dt\,.$$

2e: Theorem. *The Laplace transform of logarithm, for $s>0$, is*

†: $$\widehat{\log}(s) \;=\; \frac{1}{s}\,[\Gamma'(1) \;-\; \log(s)]$$
$$= \frac{-1}{s}[\gamma + \log(s)]\,.$$

Fix z with $\operatorname{Re}(z)>-1$. For $t>0$, define $P(t) := t^z$. Then

‡: $$\widehat{P}(s) \;=\; \frac{\Gamma(z+1)}{s^{z+1}}, \quad \textit{for } s>0.$$ ♢

Pf of (†). Fix an $s>0$. With $u := st$, then, $dt = \frac{1}{s}\,du$. As $0 \nearrow t \nearrow \infty$, our $0 \nearrow u \nearrow \infty$, since $s>0$. Thus $\widehat{\log}(s)$ equals

$$\int_0^\infty e^{-st}\log(t)\cdot dt \overset{\text{CoV}}{=\!=} \int_0^\infty e^{-u}\,\overbrace{\log(\tfrac{u}{s})}^{\log(t)}\cdot\overbrace{\tfrac{1}{s}\,du}^{dt}. \quad \textit{So,}$$

$$s\cdot\widehat{\log}(s) \;=\; \int_0^\infty e^{-u}[\log(u) - \log(s)]\,du$$
$$= \Gamma'(1) \;-\; \log(s)\Gamma(1) \;=\; \Gamma'(1) \;-\; \log(s),$$

as claimed. ♦

Pf of (‡). With $u := st$, as before $dt = \frac{1}{s}\,du$. So $\widehat{P}(s)$ equals

$$\int_0^\infty e^{-st}\,P(t)\cdot dt \overset{\text{since } s>0}{=\!=} \int_0^\infty e^{-u}\,[\tfrac{u}{s}]^z\cdot\overbrace{\tfrac{1}{s}\,du}^{dt}$$
$$= \frac{1}{s^{z+1}}\int_0^\infty e^{-u}\,u^z\,du\,.$$ ♦

As our last fact in this introduction to $\Gamma()$, let's show that

2f: $$\mathbf{\Gamma}(\tfrac{1}{2}) = \sqrt{\boldsymbol{\pi}}\,.$$

We employ substitution $t = x^2$. Thus $\mathrm{d}t = 2x\,\mathrm{d}x$. As $0\nearrow t\nearrow\infty$, note $0\nearrow x\nearrow\infty$. Recall $\mathbf{\Gamma}(\frac{1}{2})$ equals

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-t}\, t^{\frac{1}{2}-1}\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-t}\,\tfrac{1}{t^{1/2}}\cdot \mathrm{d}t \\ &\stackrel{\text{CoV}}{=\!=} \int_0^\infty \mathrm{e}^{-x^2}\,\tfrac{1}{x}\cdot 2x\,\mathrm{d}x \\ &= 2\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x\,.\end{aligned}$$

This last integral equals $\sqrt{\boldsymbol{\pi}}$, as shown in the next section by means of the famous Polar-Coordinate Trick.

Polar-coordinate Trick

Let $J := \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$. We use the ***PCT*** ("polar coordinate trick") to show that $J = \sqrt{\boldsymbol{\pi}}$. We integrate the cartesian-square of the integrand to conclude that

3.1: $$\begin{aligned}J^2 &= \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2]}\,\mathrm{d}x\right]\cdot\left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[y^2]}\,\mathrm{d}y\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2+y^2]}\cdot \mathrm{d}(x,y) \\ &= \int_0^{+\infty} \mathrm{e}^{-r^2}\cdot \underbrace{2\boldsymbol{\pi} r\cdot \mathrm{d}r}_{\substack{\text{Area of radius-}r\text{ annulus}\\ \text{of thickness } \mathrm{d}r.}}\,.\end{aligned}$$

Hence $J^2 = \boldsymbol{\pi}\cdot[-\mathrm{e}^{-r^2}]\big|_{r=0}^{r=+\infty} = \boldsymbol{\pi}$. Since J is the integral of a non-negative fnc, nec. $J \geqslant 0$. Thus

3.2: $$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\boldsymbol{\pi}}\,.$$

Further $\mathbf{\Gamma}$ results

This section will always be "in progress".

It follows from the defn of $\mathbf{\Gamma}$ that $\mathbf{\Gamma}(\overline{z}) = \overline{\mathbf{\Gamma}(z)}$.

4: Prop. *For N a natnum,*

$$\mathbf{\Gamma}'(N{+}1) = N!\cdot[\mathcal{H}_N - \boldsymbol{\gamma}]\,. \qquad \diamondsuit$$

Proof. Set $g(N{+}1) := N!\cdot[\mathcal{H}_N - \boldsymbol{\gamma}]$. For $N \geqslant 1$, note, $g(N{+}1) - N{\cdot}g(N)$ equals

$$N!{\cdot}[\mathcal{H}_N - \boldsymbol{\gamma}] - N!{\cdot}[\mathcal{H}_{N-1} - \boldsymbol{\gamma}] = N!{\cdot}[\mathcal{H}_N - \mathcal{H}_{N-1}]\,.$$

Restated,

†: $$g(N{+}1) - N\cdot g(N) = [N{-}1]!\,.$$

Recall $\mathbf{\Gamma}(w{+}1) = w\cdot\mathbf{\Gamma}(w)$ from (2b). Differentiating,

$$\mathbf{\Gamma}'(w{+}1) = 1{\cdot}\mathbf{\Gamma}(w) + w\mathbf{\Gamma}'(w)\,.$$

Hence

‡: $$\mathbf{\Gamma}'(N{+}1) - N{\cdot}\mathbf{\Gamma}'(N) = \mathbf{\Gamma}(N) \stackrel{\text{by (2c)}}{=\!=} [N{-}1]!\,.$$

So $\mathbf{\Gamma}'$ and g satisfy the same recurrence relation. And

$$\mathbf{\Gamma}'(1) \stackrel{\text{by (2c)}}{=\!=} -\boldsymbol{\gamma} \stackrel{\text{def}}{=\!=} g(0+1)$$

handles the base case. ♦

5: Euler's reflection formula. *For all $z \in \mathbb{C}$:*

$$\mathbf{\Gamma}(z)\cdot\mathbf{\Gamma}(1-z) = \frac{\boldsymbol{\pi}}{\sin(\boldsymbol{\pi} z)}\,. \qquad \diamondsuit$$

Proof. Challenging... [Courtesy analytic-continuation, ITSProve (5) for a seq. of z converging to a point of analyticity. We will prove (5) for real $0 < Z < 1$.] • • • ♦

6: Lemma. *For real Z with $0 < Z < 1$:*

$$\mathbf{\Gamma}(z)\cdot\mathbf{\Gamma}(1-z) = \int_0^\infty \frac{\rho^{Z-1}}{\rho+1}\,\mathrm{d}\rho\,. \qquad \diamondsuit$$

Proof. Difference $1-Z$ is in $(0,1)$, so $\mathbf{\Gamma}(Z)\cdot\mathbf{\Gamma}(1-Z)$ equals

$$\begin{aligned}\mathbf{I} &:= \left[\int_0^\infty x^{Z-1}\mathrm{e}^{-x}\,\mathrm{d}x\right]\int_0^\infty t^{[1-Z]-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \int_0^\infty\!\!\int_0^\infty \mathrm{e}^{-[x+t]}\,x^{Z-1}\,t^{-Z}\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned}$$

We use CoV [sum] $\sigma = t + x$, and [ratio] $\rho = \frac{x}{t}$. For a fixed positive sum $\sigma = t + x$, note that x can be made arbitrarily close to zero, or to σ. Hence the ratio $\rho = \frac{x}{t}$ varies over $\mathbb{R}_+$. Thus as (t, x) varies over $\mathbb{R}_+\times\mathbb{R}_+$, so does (σ,ρ); i.e, after the CoV, the limits-of-integration will be $\int_0^\infty\int_0^\infty$.

Jacobian. The Jacobian matrix is

$$\begin{bmatrix} \partial\sigma/\partial t & \partial\rho/\partial t \\ \partial\sigma/\partial x & \partial\rho/\partial x \end{bmatrix} = \begin{bmatrix} 1 & \text{-}x/t^2 \\ 1 & 1/t \end{bmatrix}.$$

Its determinant is thus

$$\frac{\partial(\sigma,\rho)}{\partial(t,x)} := \frac{1}{t} - \frac{-x}{t^2} \overset{\text{note}}{=\!=} \frac{\sigma}{t^2}\,.$$

Viewing σ and ρ as functions of t and x, we rewrite $\mathbf{I}$ as

$$*\text{:} \qquad \int_0^\infty\!\!\int_0^\infty \mathsf{e}^{\text{-}[t+x]} \cdot x^{Z-1}\,t^{\text{-}Z}\,\frac{t^2}{\sigma} \cdot \overbrace{\frac{\sigma}{t^2}\,\mathrm{d}t\,\mathrm{d}x}^{\frac{\partial(\sigma,\rho)}{\partial(t,x)}\,\mathrm{d}t\,\mathrm{d}x}$$

Now $\sigma = x + t = \rho t + t = [\rho+1]t$, so $\frac{t}{\sigma} = \frac{1}{\rho+1}$. And $x^{Z-1}\,t^{\text{-}Z}\,\frac{t^2}{\sigma} = [\rho t]^{Z-1}\,t^{\text{-}Z}\,\frac{t^2}{\sigma} = \rho^{Z-1}\frac{t}{\sigma} = \frac{\rho^{Z-1}}{\rho+1}$.

Our CoV applied to $(*)$ says $\mathbf{\Gamma}(Z)\cdot\mathbf{\Gamma}(1-Z)$ equals

$$7\text{:} \qquad \int_0^\infty\!\!\int_0^\infty \mathsf{e}^{\text{-}\sigma}\cdot\frac{\rho^{Z-1}}{\rho+1}\cdot\mathrm{d}\sigma\,\mathrm{d}\rho \;=\; \int_0^\infty \frac{\rho^{Z-1}}{\rho+1}\,\mathrm{d}\rho\,,$$

since $\int_0^\infty \mathsf{e}^{\text{-}\sigma}\mathrm{d}\sigma$ equals 1. ♦