γ and $\boldsymbol{\Gamma}()$

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EM-number

Recall that the ***harmonic numbers*** $\mathsf{H}_N := \sum_{k=1}^{N} \frac{1}{k}$ upper/lower bound $\log()$, in that

$$\mathsf{H}_{N-1} \;>\; \log(N) \;>\; \mathsf{H}_N - 1\,.$$

The ***Euler-Mascheroni*** number[♡1] is defined as the asymptotic discrepancy,

$$\text{1:}\quad \begin{aligned} \boldsymbol{\gamma} \;&:=\; \lim_{N\nearrow\infty} \left[\mathsf{H}_{N-1} \;-\; \log(N)\right] \\ &= \lim_{N\nearrow\infty} \left[\mathsf{H}_N \;-\; \log(N)\right] \;=\; \int_1^\infty \Big[\frac{1}{\lfloor t\rfloor} \;-\; \frac{1}{t}\Big]\,\mathrm{d}t. \end{aligned}$$

Although estimate $\boldsymbol{\gamma} \approx 0.578^-$ does not help with rationality, it is reassuring to know.

1a: Lemma. *The N^{th} harmonic number satisfies*

$$\ast1:\qquad \mathsf{H}_N \;=\; \sum_{k=1}^{\infty} \frac{N}{k[k+N]}\,.$$

Also

$$\ast2:\qquad \frac{N}{k[k+N]} \;=\; \int_0^1 \frac{t^{k/N}}{k}\,\mathrm{d}t\,.$$

Consequently,

$$\ast3:\qquad \mathsf{H}_N \;=\; -\int_0^1 \log(1 \;-\; t^{1/N})\,\mathrm{d}t\,. \qquad \diamondsuit$$

Pf $(\ast1)$. Sum $S_\ell := \sum_{k=1}^{\ell} \frac{N}{k[k+N]} = \sum_{k=1}^{\ell} [\frac{1}{k} - \frac{1}{k+N}]$ telescopes. Once $\ell \geq N$, then,

$$S_\ell \;=\; \Big[\sum_{k=1}^{N} \frac{1}{k}\Big] \;-\; \Big[\sum_{k=1}^{N} \frac{1}{\ell+k}\Big]\,.$$

The righthand-sum goes to zero, as $\ell\nearrow\infty$. ♦

Pf of $(\ast2)$. Observe $\int t^{\frac{k}{N}}\,\mathrm{d}t \;=\; \frac{N}{k+N} t^{\frac{k+N}{N}}$. Hence RhS$(\ast2)$ equals

$$\frac{N}{k[k+N]} \cdot t^{\frac{k+N}{N}}\Big|_{t=0}^{t=1} \;=\; \text{LhS}(\ast2)\,. \quad ♦$$

[♡1] As of Nov.2017, it is unknown if the ***EM-number*** $\boldsymbol{\gamma}$ is rational or irrational.

Pf of $(\ast3)$. One Taylor-series expansion for log is

$$-\log(1-x) \;\overset{\text{for } |x|<1}{=\!=\!=}\; \sum_{k=1}^{\infty} \frac{x^k}{k}\,.$$

For $0<t<1$, then,

$$-\log(1 \;-\; t^{1/N}) \;=\; \sum_{k=1}^{\infty} \frac{t^{\frac{k}{N}}}{k}\,.$$

It is valid to commute $\int_0^1$ with the sum, as all the summands have the same sign. Thus

$$\begin{aligned} -\int_0^1 \log(1 \;-\; t^{1/N})\,\mathrm{d}t \;&=\; \sum_{k=1}^{\infty} \int_0^1 \frac{t^{\frac{k}{N}}}{k}\,\mathrm{d}t \\ &= \sum_{k=1}^{\infty} \frac{N}{k[k+N]} \;=\; \mathsf{H}_N\,. \quad ♦ \end{aligned}$$

1b: Lemma. *For all t with $0<t<1$,*

$$\dagger:\qquad \lim_{N\nearrow\infty} N\cdot[1 \;-\; t^{1/N}] \;=\; \text{-}\log(t)\,.$$

$$\ddagger:\qquad \lim_{N\nearrow\infty} \log(N\cdot[1 \;-\; t^{1/N}]) \;=\; \log(\text{-}\log(t))\,. \qquad \diamondsuit$$

Proof. Setting $h := \frac{1}{N}$, we can rewrite limit $(\dagger)$ as

$$-\lim_{h\searrow 0} \frac{t^h - 1}{h} \;\overset{\text{note}}{=\!=}\; -f'(0)\,,$$

where $f(h) := t^h$. By definition, $f(h) = \exp(\log(t)\cdot h)$. So $f'(h) = \log(t)\cdot f(h)$. Thus $f'(0) = \log(t)$.

Lastly, $(\ddagger)$ holds, since [the outer] log is continuous. ♦

1c: $\boldsymbol{\gamma}$-$\boldsymbol{\Gamma}$ Thm. *Using fnc $\boldsymbol{\Gamma}()$ from the next section,*

$$\boldsymbol{\gamma} \;=\; \int_0^1 \log(\text{-}\log(t))\,\mathrm{d}t \;=\; \text{-}\boldsymbol{\Gamma}'(1)\,. \qquad \diamondsuit$$

Proof. Eqns (1) and $(\ast3)$ say $\boldsymbol{\gamma}$ is the limit of

$$\begin{aligned} \mathsf{H}_N \;-\; \log(N) \;&=\; \Big[\text{-}\int_0^1 \log(1 \;-\; t^{1/N})\,\mathrm{d}t\Big] \;-\; \int_0^1 \log(N)\,\mathrm{d}t \\ &= \text{-}\int_0^1 \Big[\log(1 \;-\; t^{1/N}) \;+\; \log(N)\Big]\,\mathrm{d}t \\ &= \text{-}\int_0^1 \log\Big(N\cdot[1 \;-\; t^{1/N}]\Big)\,\mathrm{d}t\,. \end{aligned}$$

Skipping the justification needed to pass $\lim_{N\nearrow\infty}$ through the integral sign,

$$\boldsymbol{\gamma} = -\int_0^1 \Big[\lim_{N\nearrow\infty} \log\Big(N\cdot[1 - t^{1/N}]\Big)\Big]\,dt \stackrel{\text{by (‡)}}{=\!=} -\int_0^1 \log(-\log(t))\,dt\,.$$

Exponential CoV. To compute this last integral, we use CoV $u = -\log(t)$. I.e, $t = e^{-u}$, so $\frac{dt}{du} = -e^{-u}$ and thus $dt = -e^{-u}\,du$.

As $0\nearrow t\nearrow 1$, remark that $\infty\searrow u\searrow 0$. Hence,

$$\int_0^1 \log(-\log(t))\,dt \stackrel{\text{CoV}}{=\!=} \int_\infty^0 \log(u)\,\overbrace{[-e^{-u}]\,du}^{dt} = \int_0^\infty \log(u)\,e^{-u}\,du\,.$$

And this last equals $\boldsymbol{\Gamma}'(1)$, courtesy (3) ♦

Gamma function

The ***Gamma fnc*** arises in volumes of N-dimensional balls, and in Laplace transforms. Leaving motivation for later, define the Gamma fnc by

2: $$\boldsymbol{\Gamma}(z) := \int_0^\infty t^{z-1}\,e^{-t}\,dt\,, \quad \textit{for } z\in\mathbb{C} \textit{ with } \mathrm{Re}(z) > 0\,,$$

When $\mathrm{Re}(w)>0$, note $\lim_{t\searrow 0}[t^w e^{-t}]$ is zero. Thus

$$t^w[-e^{-t}]\Big|_{t=0}^{t=\infty} = 0\,.$$

For $\mathrm{Re}(z)>1$, then, integration by parts produces

$$\boldsymbol{\Gamma}(z) = t^{z-1}\cdot[-e^{-t}]\Big|_{t=0}^{t=\infty} - \int_0^\infty [z-1]t^{z-2}\cdot[-e^{-t}]\,dt. \ \textit{So}$$

2a: $$\boldsymbol{\Gamma}(z) = [z-1]\cdot\boldsymbol{\Gamma}(z-1)\,, \quad \textit{for } \mathrm{Re}(z) > 1.$$

Since $\boldsymbol{\Gamma}(1) = 1$,

2b: $$\boldsymbol{\Gamma}(n) = [n-1]!\,, \quad \textit{for } n\in\mathbb{Z}_+.$$

As a consequence, binomial/multinomial coefficients can be generalized using $\boldsymbol{\Gamma}$, and many of the identities extend.

Analytic continuation. Writing $w := z-1$,

2c: $$\boldsymbol{\Gamma}(w) = \tfrac{1}{w}\cdot\boldsymbol{\Gamma}(w+1)$$

from (2a). As (3) will show, our $\boldsymbol{\Gamma}()$ is ∞ly differentiable, hence is *analytic* courtesy the Cauchy-Goursat theorem. Use (2c) to iteratively extend $\boldsymbol{\Gamma}()$ to the complex plane. This $\boldsymbol{\Gamma}()$ is analytic on $\mathbb{C}\smallsetminus\{0,-1,-2,-3,\dots\}$. Further, $\boldsymbol{\Gamma}()$ is meromorphic on $\mathbb{C}$, with simple poles at $0,-1,-2,\dots$.

Why simple? Equality $\boldsymbol{\Gamma}(1) = \int_0^\infty e^{-t}\,dt = 1$, together with (2c) imply $\lim_{w\to 0} w\cdot\boldsymbol{\Gamma}(w) = 1$. This is not zero, hence $\boldsymbol{\Gamma}()$ has a simple pole at 0. Then (2c) iteratively shows that the other poles are simple.

Residues. For $N\in\mathbb{N}$, we compute the $\boldsymbol{\Gamma}$-residue at $-N$. Iterating (2c),

$*$: $$\boldsymbol{\Gamma}(z) = \tfrac{1}{z}\cdot\tfrac{1}{z+1}\cdot\tfrac{1}{z+2}\cdots\tfrac{1}{z+N-1}\cdot\tfrac{1}{z+N}\cdot\boldsymbol{\Gamma}(z+N+1)\,.$$

So $\operatorname{Res}_{z=-N}(\boldsymbol{\Gamma}(z)) = \lim_{z\to -N}[z+N]\cdot\boldsymbol{\Gamma}(z)$. By ($*$), then,

2d: $$\operatorname{Res}_{z=-N}(\boldsymbol{\Gamma}(z)) = \frac{[-1]^N}{N!}\,.$$

Calculus. Differentiating under the integral sign is valid in (2). For $k = 0,1,\dots$, applying $\frac{d}{dz}$ gives

3: $$\boldsymbol{\Gamma}^{(k)}(z) = \int_0^\infty t^{z-1}e^{-t}[\log(t)]^k\,dt, \quad \textit{when } \mathrm{Re}(z)>0.$$

$$\textit{In particular, } \boldsymbol{\Gamma}'(1) = \int_0^\infty e^{-t}\log(t)\,dt \stackrel{\text{by (1c)}}{=\!=} -\boldsymbol{\gamma}\,.$$

Lap. Use $\widehat{f}$ for the ***Laplace transform*** of f, where

$$\widehat{f}(s) = [\mathcal{L}(f)](s) := \int_0^\infty e^{-st}\cdot f(t)\cdot dt\,. \quad \square$$

4: Theorem. *The Laplace transform of logarithm, for $s>0$, is*

†: $$\widehat{\log}(s) = \frac{1}{s}[\boldsymbol{\Gamma}'(1) - \log(s)] = \frac{-1}{s}[\boldsymbol{\gamma} + \log(s)]\,.$$

Fix z with $\mathrm{Re}(z) > -1$. For $t>0$, define $P(t) := t^z$. Then

‡: $$\widehat{P}(s) = \frac{\boldsymbol{\Gamma}(z+1)}{s^{z+1}}, \quad \textit{for } s>0. \quad \diamondsuit$$

Pf of (†). Fix an $s>0$. With $u := st$, then, $\mathrm{d}t = \frac{1}{s}\,\mathrm{d}u$. As $0\nearrow t\nearrow\infty$, our $0\nearrow u\nearrow\infty$, since $s>0$. Thus $\widehat{\log}(s)$ equals

$$\int_0^\infty \mathrm{e}^{-st}\log(t)\cdot\mathrm{d}t \stackrel{\text{CoV}}{=\!=} \int_0^\infty \mathrm{e}^{-u}\overbrace{\log(\tfrac{u}{s})}^{\log(t)}\cdot\overbrace{\tfrac{1}{s}\,\mathrm{d}u}^{\mathrm{d}t}. \quad \textit{So,}$$

$$\begin{aligned} s\cdot\widehat{\log}(s) &= \int_0^\infty \mathrm{e}^{-u}[\log(u)-\log(s)]\,\mathrm{d}u \\ &= \boldsymbol{\Gamma}'(1) - \log(s)\boldsymbol{\Gamma}(1) = \boldsymbol{\Gamma}'(1) - \log(s), \end{aligned}$$

as claimed. ♦

Pf of (‡). With $u := st$ as before, $\mathrm{d}t = \frac{1}{s}\,\mathrm{d}u$. So $\widehat{P}(s)$ equals

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-st}\,P(t)\cdot\mathrm{d}t &\stackrel{\text{since } s>0}{=\!=\!=} \int_0^\infty \mathrm{e}^{-u}\,[\tfrac{u}{s}]^z\cdot\overbrace{\tfrac{1}{s}\,\mathrm{d}u}^{\mathrm{d}t} \\ &= \frac{1}{s^{z+1}}\int_0^\infty \mathrm{e}^{-u}\,u^z\,\mathrm{d}u\,. \end{aligned}$$ ♦

As our last fact in this introduction to $\boldsymbol{\Gamma}()$, let's show that

5: $$\boldsymbol{\Gamma}(\tfrac{1}{2}) = \sqrt{\boldsymbol{\pi}}\,.$$

We employ substitution $t = x^2$. Thus $\mathrm{d}t = 2x\,\mathrm{d}x$. As $0\nearrow t\nearrow\infty$, note $0\nearrow x\nearrow\infty$. Recall $\boldsymbol{\Gamma}(\frac{1}{2})$ equals

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-t}\,t^{\frac{1}{2}-1}\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-t}\,\tfrac{1}{t^{1/2}}\cdot\mathrm{d}t \\ &\stackrel{\text{CoV}}{=\!=} \int_0^\infty \mathrm{e}^{-[x^2]}\,\tfrac{1}{x}\cdot 2x\,\mathrm{d}x \\ &= 2\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x\,. \end{aligned}$$

This last integral equals $\sqrt{\boldsymbol{\pi}}$, as shown in the next section by means of the famous Polar-Coordinate Trick.

Polar-coordinate Trick

Let $J := \int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x$. We use the ***PCT*** ("polar coordinate trick") to show that $J = \sqrt{\boldsymbol{\pi}}$. We integrate the cartesian-square of the integrand to conclude that

6.1: $$\begin{aligned} J^2 &= \left[\int_{-\infty}^{+\infty}\mathrm{e}^{-[x^2]}\,\mathrm{d}x\right]\cdot\left[\int_{-\infty}^{+\infty}\mathrm{e}^{-[y^2]}\,\mathrm{d}y\right] \\ &= \int_{-\infty}^{+\infty}\mathrm{e}^{-[x^2+y^2]}\cdot\mathrm{d}\boldsymbol{(}x,y\boldsymbol{)} \\ &= \int_0^{+\infty}\mathrm{e}^{-r^2}\cdot\underbrace{2\boldsymbol{\pi}r\cdot\mathrm{d}r}_{\substack{\text{Area of radius-}r\text{ annulus}\\ \text{of thickness }\mathrm{d}r.}}\,. \end{aligned}$$

Hence $J^2 = \boldsymbol{\pi}\cdot[-\mathrm{e}^{-r^2}]\big|_{r=0}^{r=+\infty} = \boldsymbol{\pi}$. Since J is *the integral of a non-negative fnc*, nec. $J\geq 0$. Thus

6.2: $$\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\boldsymbol{\pi}}\,.$$

Further $\mathbf{\Gamma}$ results

This section will always be "in progress".

It follows from the defn of $\mathbf{\Gamma}$ that $\mathbf{\Gamma}(\overline{z}) = \overline{\mathbf{\Gamma}(z)}$.

7: Prop. *For N a natnum,*

$$\mathbf{\Gamma}'(N+1) \;=\; N! \cdot [\mathsf{H}_N - \boldsymbol{\gamma}]\,. \qquad \Diamond$$

Proof. Set $g(N+1) := N! \cdot [\mathsf{H}_N - \boldsymbol{\gamma}]$. For $N \geq 1$, note, $g(N+1) - N \cdot g(N)$ equals

$$N! \cdot [\mathsf{H}_N - \boldsymbol{\gamma}] \;-\; N! \cdot [\mathsf{H}_{N-1} - \boldsymbol{\gamma}] \;=\; N! \cdot [\mathsf{H}_N - \mathsf{H}_{N-1}]\,.$$

Restated,

$$\dagger: \qquad g(N+1) \;-\; N \cdot g(N) \;=\; [N-1]!\,.$$

Recall $\mathbf{\Gamma}(w+1) = w \cdot \mathbf{\Gamma}(w)$ from (2c). Differentiating,

$$\mathbf{\Gamma}'(w+1) \;=\; 1 \cdot \mathbf{\Gamma}(w) \;+\; w\mathbf{\Gamma}'(w)\,.$$

Hence

$$\ddagger: \quad \mathbf{\Gamma}'(N+1) \;-\; N \cdot \mathbf{\Gamma}'(N) \;=\; \mathbf{\Gamma}(N) \overset{\text{by (2b)}}{=\!=} [N-1]!\,.$$

So $\mathbf{\Gamma}'$ and g satisfy the same recurrence relation. And

$$\mathbf{\Gamma}'(1) \overset{\text{by (2b)}}{=\!=} -\boldsymbol{\gamma} \overset{\text{def}}{=} g(0+1)$$

handles the base case. ♦

8: Euler's reflection formula. *For all $z \in \mathbb{C}$:*

$$\mathbf{\Gamma}(z) \cdot \mathbf{\Gamma}(1-z) \;=\; \frac{\boldsymbol{\pi}}{\sin(\boldsymbol{\pi} z)}\,. \qquad \Diamond$$

Proof. Challenging... [Courtesy analytic-continuation, IT-SProve (8) for a seq. of z converging to a point of analyticity. We will prove (8) for real $0 < Z < 1$.] ● ● ● ♦

9: Lemma. *For real Z with $0 < Z < 1$:*

$$\mathbf{\Gamma}(z) \cdot \mathbf{\Gamma}(1-z) \;=\; \int_0^\infty \frac{\rho^{Z-1}}{\rho+1}\,\mathrm{d}\rho\,. \qquad \Diamond$$

Proof. Difference $1 - Z$ is in $(0,1)$, so $\mathbf{\Gamma}(Z) \cdot \mathbf{\Gamma}(1-Z)$ equals

$$\begin{aligned}\mathbf{I} \;&:=\; \Big[\int_0^\infty x^{Z-1}\mathsf{e}^{-x}\,\mathrm{d}x\Big]\int_0^\infty t^{[1-Z]-1}\mathsf{e}^{-t}\,\mathrm{d}t \\ &=\; \int_0^\infty\int_0^\infty \mathsf{e}^{-[x+t]}\,x^{Z-1}\,t^{-Z}\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned}$$

We use CoV [sum] $\sigma = t + x$, and [ratio] $\rho = \frac{x}{t}$. For a fixed positive sum $\sigma = t + x$, note that x can be made arbitrarily close to zero, or to σ. Hence the ratio $\rho = \frac{x}{t}$ varies over $\mathbb{R}_+$. Thus as (t, x) varies over $\mathbb{R}_+ \times \mathbb{R}_+$, so does (σ, ρ); i.e, after the CoV, the limits-of-integration will be $\int_0^\infty \int_0^\infty$.

Jacobian. The Jacobian matrix is

$$\begin{bmatrix}\partial\sigma/\partial t & \partial\rho/\partial t \\ \partial\sigma/\partial x & \partial\rho/\partial x\end{bmatrix} \;=\; \begin{bmatrix}1 & -x/t^2 \\ 1 & 1/t\end{bmatrix}.$$

Its determinant is thus

$$\frac{\partial(\sigma,\rho)}{\partial(t,x)} \;:=\; \frac{1}{t} \;-\; \frac{-x}{t^2} \overset{\text{note}}{=\!=} \frac{\sigma}{t^2}\,.$$

Viewing σ and ρ as functions of t and x, we rewrite $\mathbf{I}$ as

$$*: \qquad \int_0^\infty\int_0^\infty \mathsf{e}^{-[t+x]} \cdot x^{Z-1}\,t^{-Z}\,\frac{t^2}{\sigma} \cdot \overbrace{\frac{\sigma}{t^2}\,\mathrm{d}t\,\mathrm{d}x}^{\frac{\partial(\sigma,\rho)}{\partial(t,x)}\,\mathrm{d}t\,\mathrm{d}x}$$

Now $\sigma = x + t = \rho t + t = [\rho+1]t$, so $\frac{t}{\sigma} = \frac{1}{\rho+1}$. And $x^{Z-1}\,t^{-Z}\,\frac{t^2}{\sigma} = [\rho t]^{Z-1}\,t^{-Z}\,\frac{t^2}{\sigma} = \rho^{Z-1}\,\frac{t}{\sigma} = \frac{\rho^{Z-1}}{\rho+1}$.

Our CoV applied to $(*)$ says $\mathbf{\Gamma}(Z) \cdot \mathbf{\Gamma}(1-Z)$ equals

$$10: \qquad \int_0^\infty\int_0^\infty \mathsf{e}^{-\sigma} \cdot \frac{\rho^{Z-1}}{\rho+1} \cdot \mathrm{d}\sigma\,\mathrm{d}\rho \;=\; \int_0^\infty \frac{\rho^{Z-1}}{\rho+1}\,\mathrm{d}\rho\,,$$

since $\int_0^\infty \mathsf{e}^{-\sigma}\mathrm{d}\sigma$ equals 1. ♦