Flat stacks, Joining-Closure and Genericity

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://www.math.ufl.edu/~squash/

13 June, 2013 (at *08:10*)

ABSTRACT: In the usual Halmos topology on the group of transformations ($\mathbb{Z}$-actions), we show that a joinings version of the Weak-Closure Theorem holds for the "generic T" (that is, for a residual set of T). Specifically, the closure of the set of off-diagonal self-joinings of T is the *full* simplex of self-joinings. This also holds for infinite-dimensional self-joinings.

An ingredient in the proof is showing that if T has "flat stacks", then the off-diagonal self-joinings are dense in the set of *ergodic* self-joinings. A second ingredient is a combinatorial lemma, fancifully called the "Chameleon lemma", involving words and blockings.

§A Introduction

In the past two decades it has become evident that properties of the self-joinings (henceforth just called "joinings") of a transformation provide important tools-of-classification of such maps. My purpose here, *Strong Joining-Closure*, theorem 4, is to show that a certain "joinings extension" of the *Weak-Closure Theorem* holds generically.

Section A and §B transmogrify the problem to a discrete combinatorial statement, the *Chameleon Lemma* of §C, which is dispatched in §D. Here is a bird's eye view.

§Outline

AMS Subject Classification: Primary: 28D05. Secondary: 54E52. *Key words and phrases:* Joinings, Baire category, Flat stacks, Weak closure theorem.

A ***measure*** will mean "Lebesgue probability measure". We will work on a non-atomic measure space $(X, \mathscr{X}, \mu)$ with a (measure-preserving) map $T{:}X\circlearrowleft$ which is invertible[♡1]. Let $\mathbf{\Omega}{=}\mathbf{\Omega}(\mu)$ denote the group, under composition, of such transformations. We have subgroups $\mathbf{\Omega} \supset \mathbf{C}(T) \supset \mathbb{Z}(T)$. Here $\mathbf{C}(T)$, the ***commutant*** of T, comprises those transformations S such that $ST = TS$, and $\mathbb{Z}(T)$ means the set $\{T^e\}_{e\in\mathbb{Z}}$ of powers of T.

[♡1] It turns out that the maps which commute with a rank-1 must necessarily be invertible, [**?**]. For this reason, there will be no loss of generality in considering only invertible transformations.

Idiosyncrasies. Expressions "$d =: s$" and "$s := d$" each mean that d is the definition of symbol s.

For real numbers, let "b exceeds c" mean $b > c$, whereas the weaker $b \geqslant c$ condition is "b dominates c". Say that b is ***δ-close*** to c, written $b \overset{\delta}{\approx} c$, if $|b - c| < \delta$. For vectors $\mathbf{b}$ and $\mathbf{c}$ over a common index set, say $\mathbf{b}$ is δ-close to $\mathbf{c}$ if each $b_i \overset{\delta}{\approx} c_i$. For equal length words B and C, let $\mathsf{B} \overset{\delta}{\approx} \mathsf{C}$ mean that $\bar{d}(\mathsf{B}, \mathsf{C}) < \delta$. ("Word" and "$\bar{d}$-distance" are defined below.)

An interval of integers $[a \mathinner{..} b)$ means $[a, b) \cap \mathbb{Z}$, with analogous notation for open/closed intervals. Use $\mathbb{N}$ for $[0 \mathinner{..} \infty)$ and use $[0 \mathinner{..} \infty]$ for $\mathbb{N} \cup \{\infty\}$. For G a set, let $\#G$ mean its cardinality; e.g $\#\text{Stooges} = 3$.

Employ notation $G^{\times K}$ to mean the K-fold cartesian power of a set, partition, field (σ-algebra), or transformation. Let Id be the identity transformation.

I will first state three theorems, defining later the terms *rank*-1, *flat stacks*, and the set $\mathscr{J}(\mu)$ of *joinings*. Also postponed are the definitions of the standard topologies on $\mathbf{\Omega}$ and $\mathscr{J}(\mu)$. Agree to $\text{Cl}(\cdot)$ for the closure operator on each of these two spaces.

1: Weak-Closure Theorem ([?, P. 365]). *Suppose that T is rank-1. Then*

$$\heartsuit 1: \qquad \mathbf{C}(T) \subset \text{Cl}(\mathbb{Z}(T)),$$

where Cl() *is the closure operator on* $\mathbf{\Omega}$. ◊

An analog of the sets $\mathbf{C}(T)$ and $\mathbb{Z}(T)$ are the sets $\mathscr{J}^2_{\mathsf{Gra}}(T)$ and $\mathscr{J}^2_{\mathsf{Dia}}(T)$ of 2-fold "graph" and "diagonal" joinings of T. In the language of joinings, then, ($\heartsuit$1) can be written as

$$\heartsuit 1': \qquad \mathscr{J}^2_{\mathsf{Gra}}(T) \subset \text{Cl}(\mathscr{J}^2_{\mathsf{Dia}}(T)).$$

The graph joinings are a special case of ergodic joinings, suggesting a question raised in [?] a decade and a half ago.

2: Question. *If T is rank-1, must*

$$\mathscr{J}^2_{\mathsf{Erg}}(T) \subset \text{Cl}(\mathscr{J}^2_{\mathsf{Dia}}(T))\,? \qquad ◊$$

While still unresolved, the conclusion *is* known for the "flat stacks" maps, a subset of RANK-1. Indeed for flat stacks it holds for ∞-order.

3: EJCl Theorem (Ergodic Joining-Closure). *Suppose that T has flat stacks. Then*

$$\heartsuit 2: \qquad \mathscr{J}^\infty_{\mathsf{Erg}}(T) \subset \text{Cl}(\mathscr{J}^\infty_{\mathsf{Dia}}(T)).$$

I.e, each ergodic ∞-order self-joining is a limit of diagonal joinings. ◊

Ergodic joinings comprise the extreme points of the simplex of *all* joinings. By beefing up $\mathscr{J}^\infty_{\mathsf{Erg}}(T)$ to the full simplex, the theorem below strengthens ($\heartsuit$2), but at a cost. The price paid is to conclude this, alas, no longer for all flat-stack maps, but only for a residual set of such T.

4: SJCl Theorem (Strong Joining-Closure). *The generic T has the inclusion*

$$\heartsuit 3: \qquad \mathscr{J}_\infty(T) \subset \text{Cl}(\mathscr{J}^\infty_{\mathsf{Dia}}(T)).$$

Each ∞-order self-joining η (not necessarily ergodic) is a limit of diagonal joinings. ◊

For a dimension $\mathfrak{D} \in [2 \mathinner{..} \infty]$, if T satisfies

$$\heartsuit 2': \qquad \mathscr{J}^{\mathfrak{D}}_{\mathsf{Erg}}(T) \subset \text{Cl}(\mathscr{J}^{\mathfrak{D}}_{\mathsf{Dia}}(T)), \text{ respectively,}$$

$$\heartsuit 3': \qquad \mathscr{J}^{\mathfrak{D}}(T) \subset \text{Cl}(\mathscr{J}^{\mathfrak{D}}_{\mathsf{Dia}}(T)),$$

then say that T is ***$\mathfrak{D}$-ergodic-dense***, respectively, ***$\mathfrak{D}$-simplex-dense***.

Words & Blockings

Over a (finite) alphabet P, an L-word H is a sequence

$$\mathsf{H} = h_0 h_1 \cdots h_i \cdots h_{L-1}$$

of letters $h_i \in \mathsf{P}$. Use $\mathsf{H}(i) = h_i$ for the i^{th} letter in a word. Notation $\mathsf{H}[3 \mathinner{..} 6) = h_3 h_4 h_5$ indicates a substring. The length $\text{Len}(\mathsf{H})$, above, is L. An expression such as $\mathsf{H} \overset{\text{Len}}{\leqslant} 5$ indicates that $\text{Len}(\mathsf{H}) \leqslant 5$.

For H a letter or word, let $\mathsf{H}^{\otimes\mathfrak{R}}$ be the word formed by concatenating $\mathfrak{R}$ copies of H, e.g, $[\mathtt{zb}]^{\otimes 3}$ is `zbzbzb`. Also, in characterizing flat-stacks and in the words arguments of §C, §D, the symbol "$\mathfrak{R}$" will always denote a positive integer. It is called a ***repetition number***.

Let "$\mathsf{H}\oplus e$" denote H shifted (i.e, rotated) by e positions. Define this by example for an L-word H:

$$\mathsf{H}\oplus 3 \quad = \quad h_3h_4h_5\dots h_{L-2}h_{L-1}h_0h_1h_2\,.$$

Evidently $\mathsf{H}\oplus[e{+}L]$ equals $\mathsf{H}\oplus e$.

Joint-words. Given L-words B and C, let $\begin{Vmatrix}\mathsf{B}\\ \mathsf{C}\end{Vmatrix}$ denote the ***joint-word*** over alphabet $\mathsf{P}^{\times 2}$ consisting of letter-pairs. As a 3-dimensional example, if H is `abcde12345` then

$$\begin{Vmatrix}\mathsf{H}\\ \mathsf{H}\oplus 18\\ \mathsf{H}\oplus 1\end{Vmatrix} = \begin{matrix}\mathtt{abcde12345}\\ \mathtt{45abcde123}\\ \mathtt{bcde12345a}\end{matrix}$$

over the alphabet $\mathsf{P}^{\times 3}$. A $\mathfrak{D}$-tuple $\vec{e}$ of integers $(e_0,e_1,\dots,e_{\mathfrak{D}-1})$ will be called a ***shift***. A word H and shift $\vec{e}$ determine a ***shift-word***,

5: $$\mathsf{H}^{\vec{e}} \;:=\; \begin{Vmatrix}\mathsf{H}\oplus e_0\\ \mathsf{H}\oplus e_1\\ \vdots\\ \mathsf{H}\oplus e_{\mathfrak{D}-1}\end{Vmatrix}.$$

Length Convention. For words we use symbols H, A, B, C, D. A script version, $\mathscr{H},\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$, automatically denotes the length of the corresponding word. Use L as a general length.

To avoid trivialities, all word-lengths are positive. (Some definitions fail for the empty word.)

The d-bar metric. The ***$\bar{d}$-distance*** between two L-words is $\bar{d}(\mathsf{B},\mathsf{C}) := \frac{\#G}{L}$, where G is the set of indices i with $b_i \neq c_i$.

Blockings. Consider a set $G\subset\mathbb{N}$. Given a length $L\in\mathbb{N}$, let "the density of G in $[0\,..\,L)$" mean

$$\mathrm{Den}_L(G) \;:=\; \#(G\cap[0\,..\,L))/L\,.$$

Write $\underline{\mathrm{Den}}(G)$ for the ***lower density*** of G in $\mathbb{N}$; it is

$$\liminf_{L\to\infty}\mathrm{Den}_L(G)\,.$$

Define ***upper density*** $\overline{\mathrm{Den}}(G)$ analogously. If $\overline{\mathrm{Den}}(G)$ equals $\underline{\mathrm{Den}}(G)$ then use $\mathrm{Den}(G)$ to denote the common density.

A ***blocking*** $\Upsilon := ([\ell_i\,..\,r_i))_{i=1}^{I}$ of an interval $[0\,..\,L)$ is a list of intervals with $\ell_i, r_i \in\mathbb{N}$ and $\ell_i < r_i \leqslant \ell_{i+1}$ and $\bigcup_1^I[\ell_i\,..\,r_i)\subset[0\,..\,L)$. Usually both L and I are ∞; define the general case now for use in §C. This Υ is an ***ε-blocking*** of $[0\,..\,L)$ if the lower density of $\bigcup_1^I[\ell_i\,..\,r_i)$ dominates $1{-}\varepsilon$. An ε-blocking is also called a $[1{-}\varepsilon]$***-cover***.

Consider an $\mathscr{H}$-word H and a name (an infinite string) $x = x[0\,..\,\infty)$. A blocking $\Upsilon := ([\ell_i\,..\,\ell_i{+}\mathscr{H}))_{i=1}^{\infty}$ is a "H-blocking" of x if

$*$: $$x[\ell_i\,..\,\ell_i{+}\mathscr{H}) \;=\; \mathsf{H}$$

for each i. Naturally, if $\underline{\mathrm{Den}}(\bigcup\Upsilon)\geqslant 1-\varepsilon$ then Υ is called an "ε-H-blocking". Lastly, retaining the density condition but weakening $(*)$ to

$$x[\ell_i\,..\,\ell_i{+}\mathscr{H}) \;\overset{\varepsilon}{\approx}\; \mathsf{H}$$

yields a "$\bar{d}$-ε-H-blocking" of x.

Rank-1 and Flat Stacks

Given a Rohlin stack Ξ with $\mathscr{H}$ many levels, let $\mathsf{P}{=}\mathsf{P}_\Xi$ be the partition of X into the levels of Ξ and the stack complement $X\setminus\Xi$; so P has $\mathscr{H}{+}1$ atoms. A map T is ***rank-1*** if: *For each partition* Q *and* $\varepsilon>0$ *there exists a T-stack Ξ whose partition ε-refines* Q, *i.e*

6: $$\mathsf{P} \;\succcurlyeq_{\varepsilon}\; \mathsf{Q}\,.$$