Ending the semester in Style

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B1: Suppose $(\Omega, \mathcal{T})$ is a topological space and $\mathcal{C}$ is a prebase for $\mathcal{T}$. Say that an open cover (of Ω) is "good" if it has a finite subcover. i Suppose $\mathcal{T}$ is not compact. Use AC or Zorn's lemma to prove that there exists a *maximal* bad $\mathcal{T}$-cover $\mathcal{M}$. That is, every open cover $\mathcal{M}' \supsetneq \mathcal{M}$ is necessarily good.

ii Prove that if $\mathcal{M}$ is a maximal bad $\mathcal{T}$-cover, then $\mathcal{M} \cap \mathcal{C}$ is a cover of Ω.

iii Use (i) and (ii) to to prove the Alexander Prebase Lemma: *If every $\mathcal{C}$-cover is good then every $\mathcal{T}$-cover is good, i.e, $\mathcal{T}$ is compact.*

B2: a Property "Stavros": *For every point p and neighborhood N of p there exists a nbhd V of p whose closure $\overline{V} \subset N$.* Is "Stavros" equivalent to one of the separation (T_0–T_4, regular, normal) properties? Prove your result.

b On $\mathbb{R}$, give an example of two distinct topologies, $\mathcal{T} \neq \mathcal{B}$, which are sequentially-equivalent, $\mathcal{T} \asymp \mathcal{B}$.

B3: Let $\mathbf{J} := [0,1]$. You may use, without proof, the Schröder-Bernstein thm and the following.

$\mathbf{a_1}$: $\mathbb{R} \asymp \{0,1\}^{\mathbb{N}}$. $\mathbf{a_2}$: $\mathbb{N}\times\mathbb{R} \asymp \mathbb{R}$.

$\mathbf{a_3}$: *For each three sets Ω, B, D: $\Omega^{B\times D} \asymp [\Omega^B]^D$.*

$\mathbf{a_4}$: *The set $S := \mathbb{Q} \cap \mathbf{J}$ is countable.*

Prove that $\mathbf{C}(\mathbf{J})$, the set of continuous functions $\mathbf{J}\to\mathbb{R}$, is bijective with $\mathbb{R}$. Cite each $(\mathbf{a}_i)$ where you use it. Specify what Ω, B, D are, when you apply $(\mathbf{a_3})$. [*Note:* Does your proof split into easily-understood lemmas?]

B4: Let Ω be the half-plane $[0,\infty) \times \mathbb{R}$, let $\mathcal{T}$ be the tangent-disk topology on Ω and let $\mathcal{U}$ be the usual (metric) topology on the half-plane.

Prove or provide (with proof) a CEX: *If $K \subset \Omega$ is $\mathcal{T}$-closed then K is $\mathcal{U}$-closed.*

B5: Let $X := \bigotimes_{j=1}^{\infty} Y_j$, where $Y_j := [0,1]$. Equip X with the *box topology* $\mathcal{B}$. a Prove or disprove: $(X, \mathcal{B})$ *is metrizable.*

b Show that $(X, \mathcal{B})$ is *not* sequentially compact by giving an explicit sequence $\vec{\mathbf{x}} := (x_n)_{n=1}^{\infty} \subset X$ and proving that $\vec{\mathbf{x}}$ has no convergent subsequence.

c Prove or disprove: *The box space $(X, \mathcal{B})$ is compact.*