List of some topology problems

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8 Dec., 1995: Almost all of the problems on the final will come from this list; of course, this list is much much . . . much longer than the final will be!

Take a look at the first page of "World's Longest Proof of Tychonoff's Theorem" for an example of a compact space which is not sequentially-compact.

Use AC for the Axiom of Choice. Use CMS for complete metric space.

Cardinalities

B1: State and prove the Schröder-Bernstein theorem.

B2: `bij-CantorDiag.prove.tex` Cantor Diagonalization Thm: *For each set B, there does* <u>*not*</u> *exist a surjection* $B \twoheadrightarrow \mathcal{P}(B)$.

B3: (a) Give an example where the two sets $A^{B\times C}$ and $A^{[B^C]}$ are *not* bijective. On the other hand, for every three sets A, B, C prove that $A^{B\times C} \asymp \left[A^B\right]^C$.

(b) Prove that $\mathbb{R} \asymp \{0,1\}^{\mathbb{N}}$.

(c) Prove that $\mathbb{R}^2 \asymp \mathbb{R}$. Prove that $\mathbb{R}^{\mathbb{N}} \asymp \mathbb{R}$.

B4: `bij-Cts>reals.E.tex` Let $\mathbf{J} := [0,1]$.

Prove that $\mathbf{C}(\mathbf{J})$, *the set of continuous functions* $\mathbf{J}\to\mathbb{R}$, *is bijective with* $\mathbb{R}$. Cite each $(\mathbf{a}_i)$ <u>where</u> you <u>use</u> it. Specify what Ω, B, D are, when you <u>apply</u> $(\mathbf{a_3})$. [*Note:* Does your proof split into easily-understood lemmas?]

Metric spaces

B5: Prove or give a counter-example: *If (Y, m) is a metric space then Y is normal.*

B6: Prove or give a counter-example: *If (X, d) is a CMS of finite diameter then X is sequentially-compact.*

B7: Prove or give a counter-example: *Every subset of* $[0,1]$ (*usual topology*) *is either residual or meager.*

B8: `closed-is-gdelta.tex` In metric space (X, d) prove that: (a) Every closed $K \subset X$ is a $\mathcal{G}_\delta$-set.

(b) Let $\mathcal{T}$ be the co-finite topology on $X := \mathbb{R}$. Prove that assertion (a) is false here.

B9: Let (X, d) be a metric space. Then there exists a bounded metric ρ on X which is Cauchy-equivalent to d.

B10: A ***completion*** of a metric space (X, d) is a diagram $f{:}(X, d)\to(\Omega, \rho)$ where (Ω, ρ) is a CMS, and $f{:}\,,\hookrightarrow X, \Omega.$ is an into-isometry with dense range.

(a) Show that every metric space (X, d) has a completion.

(b) If (Ω_0, ρ_0) and (Ω_1, ρ_1) are completions of X, show that Ω_1 is isometric with Ω_0.

B11: (a) Suppose (Y, m) is a CMS and $\mathcal{D} \subset Y$ is a finite set. Construct a metric m' on $Y' := Y \smallsetminus \mathcal{D}$ with m' topologically-equivalent to $m\lfloor_{Y'}$, so that (Y', m') is complete.

(b) Do the same but where $\mathcal{D}$ is a denumerable subset of Y.

B12: `sup-metr-is-complete.tex` From a metric space (X, m), construct the metric space $\Omega := \mathbf{C}_{\mathrm{Bnd}}(X, \mathbb{R})$ with the supremum metric

$$d(f, g) \;:=\; \|f - g\|_{\sup}\,.$$

(Notes, P. 19; the set of *continuous* and *bounded* fncs.)

Prove that Ω is complete by first showing, for each d-Cauchy sequence $(f_n)_{n=1}^{\infty}$, that for all x the limit

$$h(x) \;:=\; \lim_{n\to\infty} f_n(x)$$

exists in $\mathbb{R}$, by using the completeness of $\mathbb{R}$. Next show that h is continuous; don't just cite uniform-convergence –give a *proof*. Finally, show that h is bnded and that $d(f_n, h) \to 0$.

B13: Prove Baire's theorem: *In a CMS (X, μ), every residual subset $R \subset X$ is dense.*

B14: Let Ω be the metric space $\mathbf{C}_{\mathrm{Bnd}}\big((0,1)\big)$ with the supremum metric $\mathsf{d}(f, g) := \|f - g\|_{\sup}$. Let $R \subset \Omega$ be the collection of nowhere differentiable functions. Show that R is a residual subset of Ω.

B15: Let G be the set of real numbers α such that for all positive ε and positive integers D: There exists a rational p/q in lowest terms with

$$\left|\alpha - \frac{p}{q}\right| \;<\; \frac{1}{q^D} \;<\; \varepsilon\,.$$

Determine, with proof, whether G is residual, meager, or neither.

General topology

B16: `bufferable1.tex` a Property "Sam": *For every point p and neighborhood N of p there exists a nbhd V of p whose closure $\overline{V} \subset N$.* Is "Sam" equivalent to one of the separation (T_0–T_4, regular, normal) properties? Prove your result.

b On $\mathbb{R}$, give an example of two distinct topologies, $\mathcal{T} \neq \mathcal{B}$, which are sequentially-equivalent, $\mathcal{T} \asymp \mathcal{B}$.

B17: a Give an example of a space $(X, \mathcal{T})$ which is *not* Locally Countably Generated.

b On $\mathbb{R}$, give an example of two topologies $\mathcal{T} \neq \mathcal{B}$ which are sequentially-equivalent, $\mathcal{T} \asymp \mathcal{B}$.

B18: Suppose $C_1 \supset C_2 \supset C_3 \ldots$ are non-empty, closed, compact subsets of a space X. Prove that $\bigcap_{n=1}^{\infty} C_n$ is non-empty.

B19: `alexander-prebase.tych.tex` Suppose $(\Omega, \mathcal{T})$ is a topological space and $\mathcal{C}$ is a prebase for $\mathcal{T}$. Say that an open cover (of Ω) is "good" if it has a finite subcover.

i Suppose $\mathcal{T}$ is not compact. Use AC or Zorn's lemma to prove that there exists a *maximal* bad $\mathcal{T}$-cover $\mathcal{M}$. That is, every open cover $\mathcal{M}' \supsetneqq \mathcal{M}$ is necessarily good.

ii Prove that if $\mathcal{M}$ is a maximal bad $\mathcal{T}$-cover, then $\mathcal{M} \cap \mathcal{C}$ is a cover of Ω.

iii Use (i) and (ii) to to prove the Alexander Prebase Lemma: *If every $\mathcal{C}$-cover is good then every $\mathcal{T}$-cover is good, i.e, $\mathcal{T}$ is compact.*

iv State Tychonoff's theorem. Now prove this theorem using some form of AC together with the Alexander Prebase lemma.

B20: Give, with proof, an example of a compact space which is not sequentially-compact.