Brief answers to *Ending the semester in Style*

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11 Dec., 1995. Here are some answers, but without proofs.

B1: Suppose $(\Omega, \mathcal{T})$ is a topological space and $\mathcal{C}$ is a prebase for $\mathcal{T}$. Say that an open cover (of Ω) is **"good"** if it has a finite subcover...

An answer: You can find this in Folland's **Real Analysis**.

B2: [a] Property "Stavros": *For every point p and neighborhood N of p there exists a nbhd V of p whose closure $\overline{V} \subset N$.* Is "Stavros" equivalent to one of the separation (T_0–T_4, regular, normal) properties? Prove your result.

An answer: "Stavros" is equivalent to T_3; a closed set can be separated from a singleton via disjoint open sets.

[b] On $\mathbb{R}$, give an example of two distinct topologies, $\mathcal{T} \neq \mathcal{B}$, which are sequentially-equivalent, $\mathcal{T} \asymp \mathcal{B}$.

An answer: On $\mathbb{R}$, for both the co-countable topology and the discrete topology, the only convergent sequences are the eventually-constant sequences. (The co-finite topology has other convergent sequences.)

B3: Let $\mathbf{J} := [0,1]$. You may use, without proof, the Schröder-Bernstein thm and the following.

$\mathbf{a_1}$: $\mathbb{R} \asymp \{0,1\}^{\mathbb{N}}$. $\mathbf{a_2}$: $\mathbb{N}\times\mathbb{R} \asymp \mathbb{R}$.

$\mathbf{a_3}$: *For each three sets Ω,B,D: $\Omega^{B\times D} \asymp [\Omega^B]^D$.*

$\mathbf{a_4}$: *The set $S := \mathbb{Q} \cap \mathbf{J}$ is countable.*

Prove that $\mathbf{C}(\mathbf{J})$, the set of continuous functions $\mathbf{J}\to\mathbb{R}$, is bijective with $\mathbb{R}$. Cite each $(\mathbf{a}_i)$ <u>where</u> you <u>use</u> it. Specify what Ω, B, D are, when you <u>apply</u> $(\mathbf{a_3})$. [*Note:* Does your proof split into easily-understood lemmas?]

B4: Let Ω be the half-plane $[0,\infty) \times \mathbb{R}$, let $\mathcal{T}$ be the tangent-disk topology on Ω and let $\mathcal{U}$ be the usual (metric) topology on the half-plane.

Prove or provide (with proof) a CEX: *If $K \subset \Omega$ is $\mathcal{T}$-closed then K is $\mathcal{U}$-closed.*

An answer: The implication <u>fails</u>. Let $p = (17,0)$ on the x-axis. Let B be the radius-5 $\mathcal{U}$-openBall centered at $(17,5)$; so $B' := B \cup \{p\}$ is $\mathcal{T}$-open. Thus $K := \Omega \smallsetminus B'$ is $\mathcal{T}$-closed; so p is *not* in its $\mathcal{T}$-closure. Yet the $\mathcal{U}$-closure of K <u>does</u> own p.

B5: Let $X := \bigotimes_{j=1}^{\infty} Y_j$, where $Y_j := [0,1]$. Equip X with the *box topology* $\mathcal{B}$. [a] Prove or disprove: *$(X,\mathcal{B})$ is metrizable.*

An answer: This space is not metrizable. You can use a Cantor Diagonalization argument to show that the space fails to be Locally Countably Generated.

[b] Show that $(X,\mathcal{B})$ is *not* sequentially compact by giving an explicit sequence $\vec{\mathbf{x}} := (x_n)_{n=1}^{\infty} \subset X$ and proving that $\vec{\mathbf{x}}$ has no convergent subsequence.

An answer: Let x_n be the tuple $(1,1,\overset{n}{\dots},1,0,0,\dots)$ in X.

[c] Prove or disprove: *The box space $(X,\mathcal{B})$ is compact.*

An answer: This space is not compact. Let K be the collection $\{x_n\}_{n=1}^{\infty}$ defined in part (b). This set K is $\mathcal{B}$-closed and so, were X compact, would be compact. But the relative topology on K is discrete. Thus, since K is infinite, K is not compact.