Fibonacci sequences

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squatsh@ufl.edu
Webpage http://people.clas.ufl.edu/squash/
23 September, 2017 (at 15:50)

See also Problems/Algebra/LinearAlg/linear-recurr.latex

Prolegomenon. The famous Fibonacci sequence \( \overline{f} := (f_n)_{n=-\infty}^\infty \) is defined by \( f_0 := 0, f_1 := 1 \) and
\[
f_{n+1} = f_n + f_{n-1}.
\]
Let \( \alpha \) and \( \beta \) be the positive and negative roots of
\[
\text{Fib}(x) := x^2 - x - 1 \quad \text{note} \quad [x - \alpha][x - \beta].
\]
So
\[
\alpha + \beta = 1 \quad \text{and} \quad \alpha \cdot \beta = -1.
\]
Moreover,
\[
\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1.
\]
One checks that
\[
\alpha > 1 > |\beta|.
\]

Formula. Let \( \rho := \sqrt{5} \) and \( b := \frac{1}{\sqrt{5}} \). Our \( \overline{f} \) is some linear combination \( x[n \mapsto \alpha^n] + y[n \mapsto \beta^n] \). Easily,
\[
x = b \quad \text{and} \quad y = -b,
\]
yielding
\[
\forall n \in \mathbb{Z} : \quad f_n = b \cdot [\alpha^n - \beta^n],
\]

since this formula gives correct values for \( f_0 \) and \( f_1 \).

5: Theorem. For each integer \( N \):
\[
5a: \quad [f_N]^2 + [f_{N-1}]^2 = f_{2N-1}.
\]

5b: \[
\begin{align*}
R &:= \rho^2 \cdot [f_{2N-1}] & \text{note} \quad \rho^2 \cdot [\alpha^{2N-1} - \beta^{2N-1}] \\
L &:= \rho^2 \cdot ([f_N]^2 + [f_{N-1}]^2).
\end{align*}
\]

Leftside. By (4), \( \rho \cdot f_N = \alpha^N - \beta^N \). So \( \rho^2 \cdot f_N^2 \) equals
\[
\alpha^{2N} + \beta^{2N} - 2[\alpha \beta]^N.
\]
But \( \alpha \beta = -1 \), and \( N \) and \( N-1 \) have opposite parities. Thus \( L \) equals
\[
\alpha^{2N} + \beta^{2N} + \alpha^{2[N-1]} + \beta^{2[N-1]}
\]
\[
= \alpha^{2[N-1]}[\alpha^2 + 1] + \cdots;
\]
where the “\( \cdots \)” represents a copy of all the \( \alpha \)-terms to its left, but with “\( \alpha \)” replaced by “\( \beta \)”.

By (2), note, \( \alpha^2 + 1 = \alpha + 2 \). Thus
\[
L = \alpha^{2[N-1]}[\alpha^2 + 2] + \beta^{2[N-1]}[\beta^2 + 2].
\]

Squaring \( L \) will give twice this cross-term:
\[
[\alpha \beta]^{2[N-1]}[\alpha + 2][\beta + 2] = [\alpha + 2][\beta + 2] = \alpha \beta + 2[\alpha + \beta] + 4
\]
\[
= -1 + 2 + 4 = 5.
\]

Also note \( [\alpha + 2]^2 = \alpha^2 + 4[\alpha + 1] = 5\alpha^2 \). Thus
\[
L^2 = \alpha^{4[N-1]} \cdot 5\alpha^2 + \cdots + 5 \cdot 2.
\]
Consequently
\[
5c: \quad \frac{1}{5} \cdot L^2 = \alpha^{4N-2} + \beta^{4N-2} + 2.
\]

Rightside of (5a). Square \( R \) and divide by 5. Since \( 5 = \rho^2 \),
\[
\frac{1}{5} \cdot R^2 = [\alpha^{2N-1} - \beta^{2N-1}]^2
\]
The cross-term is \( -2[\alpha \beta]^{2N-1} = -2 \cdot [-1]^{2N-1} = 2 \), since \( 2N-1 \) is odd. We have thus shown that
\[
5d: \quad \frac{1}{5} \cdot R^2 = \alpha^{4N-2} + \beta^{4N-2} + 2
\]
And this equals RhS(5c), as desired.

Remark. It is a proof, but what follows is a prettier proof, and of a stronger result.

\[\square\]

Dot-product proof
Fix three integers \( S = j + k \). Evidently the dot-product
\[
f_{j+1} \cdot f_k + f_j \cdot f_{k-1} = f_{j+1} [f_k + f_{k-1}] + f_j f_{k-1}
\]
\[
= [f_{j+1} + f_j] f_{k-1} + f_{j+1} f_{k-2} = f_{j+2} f_{k-1} + f_{j+1} f_{k-2}.
\]
This last dot-product is the same as the first, but with “\( j+k = S \)” replaced by “\( j+1+[k-1] = S \)”.
Thus, for all \( j \in \mathbb{Z} \), the expression \( f_{j+1} \cdot f_k + f_j \cdot f_{k-1} \) depends only on \( S \).
6: Theorem. For all triples of integers $S = j + k$:

6a: \( f_{j+1} \cdot f_k + f_j \cdot f_{k-1} = f_S \).

In particular, (5a) holds. ♦

Proof. Setting $j := 0$ in LhS(6a) results in

\[
f_1f_S + f_0f_{S-1} = 1 \cdot f_S + 0 \cdot f_{S-1} = f_S.
\]

Hence (6a). To obtain (5a) from (6a), set $S := 2N - 1$ and $j := N - 1$. ♦