Fibonacci sequences

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://www.math.ufl.edu/~squash/
27 October, 2010 (at *23:03*)

See also *Problems/Algebra/LinearAlg/linear-recurr.latex*

Prolegomenon. The famous ***Fibonacci sequence*** $\vec{\mathsf{f}} := (\mathsf{f}_n)_{n=-\infty}^{\infty}$ is defined by $\mathsf{f}_0 := 0$, $\mathsf{f}_1 := 1$ and

1: $$\mathsf{f}_{n+1} = \mathsf{f}_n + \mathsf{f}_{n-1}.$$

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be the positive and negative roots of

2: $$\mathrm{Fib}(x) := x^2 - x - 1 \stackrel{\text{note}}{=\!=} [x-\boldsymbol{\alpha}][x-\boldsymbol{\beta}]. \text{ So}$$
$$\boldsymbol{\alpha}+\boldsymbol{\beta} = 1 \quad\text{and}\quad \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \text{-}1. \text{ Moreover,}$$
$$\boldsymbol{\alpha}^2 = \boldsymbol{\alpha}+1 \quad\text{and}\quad \boldsymbol{\beta}^2 = \boldsymbol{\beta}+1.$$

One checks that

3: $$\boldsymbol{\alpha} > 1 > |\boldsymbol{\beta}|.$$

Formula. Let $\rho := \sqrt{5}$ and $b := \frac{1}{\sqrt{5}}$. Our $\vec{\mathsf{f}}$ is some linear combination $x\cdot[n \mapsto \boldsymbol{\alpha}^n] + y\cdot[n \mapsto \boldsymbol{\beta}^n]$. Easily, $x = b$ and $y = \text{-}b$, yielding

4: $$\forall n \in \mathbb{Z}: \quad \mathsf{f}_n = b\cdot[\boldsymbol{\alpha}^n - \boldsymbol{\beta}^n],$$

since this formula gives correct values for f_0 and f_1.

5: Theorem. *For each integer N:*

5a: $$[\mathsf{f}_N]^2 + [\mathsf{f}_{N-1}]^2 = \mathsf{f}_{2N-1}. \qquad \diamondsuit$$

Proof. Always, LhS($5a$) is non-negative. And RhS($5a$) is non-negative, even when $N \in \mathbb{Z}_-$, since $\mathsf{f}_{\mathrm{Odd}}$ is always non-negative. So ISTProve that the squares of LhS($5a$) and RhS($5a$) are equal. To this end, define

5b: $$R := \rho^2\cdot[\mathsf{f}_{2N-1}] \stackrel{\text{note}}{=\!=} \rho\cdot[\boldsymbol{\alpha}^{2N-1} - \boldsymbol{\beta}^{2N-1}] \text{ and}$$
$$L := \rho^2\cdot\big[[\mathsf{f}_N]^2 + [\mathsf{f}_{N-1}]^2\big].$$

Leftside. By (4), $\rho\cdot\mathsf{f}_N = \boldsymbol{\alpha}^N - \boldsymbol{\beta}^N$. So $\rho^2\cdot{\mathsf{f}_N}^2$ equals

$$\boldsymbol{\alpha}^{2N} + \boldsymbol{\beta}^{2N} - 2[\boldsymbol{\alpha}\boldsymbol{\beta}]^N.$$

But $\boldsymbol{\alpha}\boldsymbol{\beta} = \text{-}1$, and N and $N-1$ have opposite parities. Thus L equals

$$\boldsymbol{\alpha}^{2N} + \boldsymbol{\beta}^{2N} + \boldsymbol{\alpha}^{2[N-1]} + \boldsymbol{\beta}^{2[N-1]}$$
$$= \boldsymbol{\alpha}^{2[N-1]}[\boldsymbol{\alpha}^2+1] + \bullet\bullet\bullet,$$

where the "$\bullet\bullet\bullet$" represents a copy of all the $\boldsymbol{\alpha}$-terms to its left, but with "$\boldsymbol{\alpha}$" replaced by "$\boldsymbol{\beta}$".

By (2), note, $\boldsymbol{\alpha}^2 + 1 = \boldsymbol{\alpha}+2$. Thus

$$L = \boldsymbol{\alpha}^{2[N-1]}[\boldsymbol{\alpha}+2] + \boldsymbol{\beta}^{2[N-1]}[\boldsymbol{\beta}+2].$$

Squaring L will give twice this cross-term:

$$\begin{aligned}[\boldsymbol{\alpha}\boldsymbol{\beta}]^{2[N-1]}[\boldsymbol{\alpha}+2][\boldsymbol{\beta}+2] &= [\boldsymbol{\alpha}+2][\boldsymbol{\beta}+2] \\ &= \boldsymbol{\alpha}\boldsymbol{\beta} + 2[\boldsymbol{\alpha}+\boldsymbol{\beta}] + 4 \\ &= \text{-}1 + 2 + 4 = 5.\end{aligned}$$

Also note $[\boldsymbol{\alpha}+2]^2 = \boldsymbol{\alpha}^2 + 4[\boldsymbol{\alpha}+1] = 5\boldsymbol{\alpha}^2$. Thus

$$L^2 = \boldsymbol{\alpha}^{4[N-1]}\cdot 5\boldsymbol{\alpha}^2 + \bullet\bullet\bullet + 5\cdot 2.$$

Consequently

5c: $$\tfrac{1}{5}\cdot L^2 = \boldsymbol{\alpha}^{4N-2} + \boldsymbol{\beta}^{4N-2} + 2.$$

Rightside of (5a). Square R and divide by 5. Since $5 = \rho^2$,

$$\tfrac{1}{5}\cdot R^2 = [\boldsymbol{\alpha}^{2N-1} - \boldsymbol{\beta}^{2N-1}]^2$$

The cross-term is $\text{-}2[\boldsymbol{\alpha}\boldsymbol{\beta}]^{2N-1} = \text{-}2\cdot[\text{-}1]^{2N-1} = 2$, since $2N-1$ is odd. We have thus shown that

5d: $$\tfrac{1}{5}\cdot R^2 = \boldsymbol{\alpha}^{4N-2} + \boldsymbol{\beta}^{4N-2} + 2.$$

And this equals RhS($5c$), as desired. ♦

Remark. It is a proof, but what follows is a prettier proof, and of a stronger result. □

Dot-product proof

Fix three integers $S = j + k$. Evidently the dot-product

$$\begin{aligned}\mathsf{f}_{j+1}\mathsf{f}_k + \mathsf{f}_j\mathsf{f}_{k-1} &= \mathsf{f}_{j+1}[\mathsf{f}_{k-1} + \mathsf{f}_{k-2}] + \mathsf{f}_j\mathsf{f}_{k-1} \\ &= [\mathsf{f}_{j+1} + \mathsf{f}_j]\mathsf{f}_{k-1} + \mathsf{f}_{j+1}\mathsf{f}_{k-2} \\ &= \mathsf{f}_{j+2}\mathsf{f}_{k-1} + \mathsf{f}_{j+1}\mathsf{f}_{k-2}.\end{aligned}$$

This last dot-product is the same as the first, but with "$j + k = S$" replaced by "$[j+1] + [k-1] = S$". Thus, for all $j \in \mathbb{Z}$, the expression $\mathsf{f}_{j+1}\cdot\mathsf{f}_k + \mathsf{f}_j\cdot\mathsf{f}_{k-1}$ depends only on S.

6**:** Theorem. *For all triples of integers* $S = j + k$*:*

6a**:** $$\mathsf{f}_{j+1} \cdot \mathsf{f}_k + \mathsf{f}_j \cdot \mathsf{f}_{k-1} = \mathsf{f}_S\,.$$

In particular, (5a) *holds.* ♢

Proof. Setting $j := 0$ in LhS($6a$) results in

$$\mathsf{f}_1\mathsf{f}_S + \mathsf{f}_0\mathsf{f}_{S-1} = 1{\cdot}\mathsf{f}_S + 0{\cdot}\mathsf{f}_{S-1} = \mathsf{f}_S\,.$$

Hence (6a). To obtain (5a) from (6a), set $S := 2N - 1$ and $j := N-1$. ♦

Filename: `Problems/NumberTheory/fibonacci.latex`
As of: *Sunday 30Apr2006.* Typeset: *27Oct2010 at 23:03.*