

## Fibonacci sequences

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See also [Problems/Algebra/LinearAlg/linear-recurr.tex](#)

**Prolegomenon.** The famous *Fibonacci sequence*  $\vec{f} := (f_n)_{n=-\infty}^{\infty}$  is defined by  $f_0 := 0$ ,  $f_1 := 1$  and

$$1: \quad f_{n+1} = f_n + f_{n-1}.$$

Let  $\alpha$  and  $\beta$  be the positive and negative roots of

$$\text{Fib}(x) := x^2 - x - 1 \stackrel{\text{note}}{=} [x - \alpha][x - \beta]. \text{ So}$$

$$2: \quad \alpha + \beta = 1 \quad \text{and} \quad \alpha \cdot \beta = -1. \text{ Moreover,}$$

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1.$$

One checks that

$$3: \quad \alpha > 1 > |\beta|.$$

**Formula.** Let  $\rho := \sqrt{5}$  and  $b := \frac{1}{\sqrt{5}}$ . Our  $\vec{f}$  is some linear combination  $x \cdot [n \mapsto \alpha^n] + y \cdot [n \mapsto \beta^n]$ . Easily,  $x = b$  and  $y = -b$ , yielding

$$4: \quad \forall n \in \mathbb{Z}: \quad f_n = b \cdot [\alpha^n - \beta^n],$$

since this formula gives correct values for  $f_0$  and  $f_1$ .

**5: Theorem.** *For each integer  $N$ :*

$$5a: \quad [f_N]^2 + [f_{N-1}]^2 = f_{2N-1}. \quad \diamond$$

**Proof.** Always, LhS(5a) is non-negative. And RhS(5a) is non-negative, even when  $N \in \mathbb{Z}_-$ , since  $f_{\text{odd}}$  is always non-negative. So ISTProve that the squares of LhS(5a) and RhS(5a) are equal. To this end, define

$$5b: \quad R := \rho^2 \cdot [f_{2N-1}] \stackrel{\text{note}}{=} \rho \cdot [\alpha^{2N-1} - \beta^{2N-1}] \text{ and}$$

$$L := \rho^2 \cdot [[f_N]^2 + [f_{N-1}]^2].$$

**Leftside.** By (4),  $\rho \cdot f_N = \alpha^N - \beta^N$ . So  $\rho^2 \cdot f_N^2$  equals

$$\alpha^{2N} + \beta^{2N} - 2[\alpha\beta]^N.$$

But  $\alpha\beta = -1$ , and  $N$  and  $N-1$  have opposite parities. Thus  $L$  equals

$$\alpha^{2N} + \beta^{2N} + \alpha^{2[N-1]} + \beta^{2[N-1]}$$

$$= \alpha^{2[N-1]}[\alpha^2 + 1] + \dots,$$

where the “ $\dots$ ” represents a copy of all the  $\alpha$ -terms to its left, but with “ $\alpha$ ” replaced by “ $\beta$ ”.

By (2), note,  $\alpha^2 + 1 = \alpha + 2$ . Thus

$$L = \alpha^{2[N-1]}[\alpha + 2] + \beta^{2[N-1]}[\beta + 2].$$

Squaring  $L$  will give twice this cross-term:

$$[\alpha\beta]^{2[N-1]}[\alpha + 2][\beta + 2] = [\alpha + 2][\beta + 2]$$

$$= \alpha\beta + 2[\alpha + \beta] + 4$$

$$= -1 + 2 + 4 = 5.$$

Also note  $[\alpha + 2]^2 = \alpha^2 + 4[\alpha + 1] = 5\alpha^2$ . Thus

$$L^2 = \alpha^{4[N-1]} \cdot 5\alpha^2 + \dots + 5 \cdot 2.$$

Consequently

$$5c: \quad \frac{1}{5} \cdot L^2 = \alpha^{4N-2} + \beta^{4N-2} + 2.$$

**Rightside of (5a).** Square  $R$  and divide by 5. Since  $5 = \rho^2$ ,

$$\frac{1}{5} \cdot R^2 = [\alpha^{2N-1} - \beta^{2N-1}]^2$$

The cross-term is  $-2[\alpha\beta]^{2N-1} = -2 \cdot [-1]^{2N-1} = 2$ , since  $2N-1$  is odd. We have thus shown that

$$5d: \quad \frac{1}{5} \cdot R^2 = \alpha^{4N-2} + \beta^{4N-2} + 2.$$

And this equals RhS(5c), as desired.  $\blacklozenge$

**Remark.** It is a proof, but what follows is a prettier proof, and of a stronger result.  $\square$

### Dot-product proof

Fix three integers  $S = j + k$ . Evidently the dot-product

$$f_{j+1}f_k + f_jf_{k-1} = f_{j+1}[f_{k-1} + f_{k-2}] + f_jf_{k-1}$$

$$= [f_{j+1} + f_j]f_{k-1} + f_{j+1}f_{k-2}$$

$$= f_{j+2}f_{k-1} + f_{j+1}f_{k-2}.$$

This last dot-product is the same as the first, but with “ $j+k = S$ ” replaced by “ $[j+1]+[k-1] = S$ ”. Thus, for all  $j \in \mathbb{Z}$ , the expression  $f_{j+1} \cdot f_k + f_j \cdot f_{k-1}$  depends only on  $S$ .

**6: Theorem.** For all triples of integers  $S = j + k$ :

$$6a: \quad f_{j+1} \cdot f_k + f_j \cdot f_{k-1} = f_S.$$

In particular, (5a) holds.  $\diamond$

*Proof.* Setting  $j := 0$  in LhS(6a) results in

$$f_1 f_S + f_0 f_{S-1} = 1 \cdot f_S + 0 \cdot f_{S-1} = f_S.$$

Hence (6a). To obtain (5a) from (6a), set  $S := 2N - 1$  and  $j := N - 1$ .  $\blacklozenge$

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