The Euler-line of a triangle

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**Nomenclature.** Fix points $B$ and $C$.
Use $BC$ for the line-segment with those endpts, and use $\text{Len}(BC)$ or just $BC$ for its length.
Use $\overrightarrow{BC}$ for the ray starting at $B$ and traversing $C$.
Use $\overline{BC}$ for the line that $B$ and $C$ determine.
As sets, then, $\overline{BC} \supseteq \overrightarrow{BC} \supseteq BC$. And $C \in BC$.
[If $B = C$ then the segment, ray and line are degenerate.]

**Defn.** The “$W$-altitude” of $T := \triangle UV W$” is the line through $W$ which is orthogonal to edge $\overrightarrow{UV}$. Three particular points associated with $T$ are:

$$\text{CircumCenter}(T) := \bigcap \{\text{Perp-bisectors of } T \};$$
$$\text{Centroid}(T) := \bigcap \{\text{Medians of } T \};$$
$$\text{OrthoCenter}(T) := \bigcap \{\text{Altitudes of } T \}. \quad \square$$

1: Euler-line Theorem. For triangle $T := \triangle UV W$, let

$$P := \text{CircumCenter}(T),$$
$$Q := \text{Centroid}(T) \quad \text{and}$$
$$R := \text{OrthoCenter}(T).$$

Then this triple is collinear in that order, and

1b: $\text{Dist}(R, Q) = 2 \cdot \text{Dist}(Q, P)$. \quad \square

If $T$ is equilateral, then points $P, Q, R$ coincide; otherwise, no two of $P, Q, R$ coincide. \quad \diamond

**Vectors.** Take an arbitrary point $P$ in the plane. If we regard $P$ as the origin, then we can view the plane as a vectorspace. How? Well, for each point $X$, interpret $X$ as the vector from-$P$-to-$X$. Write the (Euclidean) length of this vector as $\|X\|$; this is $\text{Len}(PX)$.

2: Lemma. Fix a line-segment $BC$. Consider a point $P$ on the perp-bisector of $BC$. Viewing $P$ as the origin of a vectorspace, vector $B + C$ is orthogonal\textsuperscript{1} to $BC$.\diamond

\textsuperscript{1}If $P$ is also on $BC$, then $B = -C$, i.e $B + C$ is the zero-vector. The conclusion remains true, as the zero-vector is orthogonal to all vectors.

Proof. Since $\|B\| = \|C\|$ [because $P \in \text{PerpBisect}(BC)$] points, $P, B, B+C, C$ form the vertices of a rhombus. Thus point $B+C$ is the reflection of $P$ across $BC$. \diamond

**Pf of (1).** View $P$, the circumcenter of $T$, as the origin of a vectorspace. Define the vector sum

$$H := U + V + W.$$ Since $P$ is on $\text{PerpBisect}(UV)$, vector $U + V$ is orthogonal to $\overrightarrow{UV}$. Thus $W + [U + V]$ is on the line through $W$ perpendicular to $\overrightarrow{UV}$. IOWords, $H$ is on the $W$-altitude of $T$.

Since vector-addition is commutative and associative, we can write $H$ as

$$V + [W + U] \quad \text{and as } U + [V + W].$$ Hence $H$ also lies on the $V$ and $U$-altitudes of $T$. Thus $H = R$. \quad \square

**Collinearity.** The centroid of $T$ is the average\textsuperscript{2} of $T$’s vertices. The upshot: With $P$ viewed as the origin of a vectorspace, we have that

$$R = 1 \cdot [U + V + W];$$
$$Q = \frac{1}{3} \cdot [U + V + W];$$
$$P = 0 \cdot [U + V + W].$$

These points are multiples of a single vector, hence form a collinear triple [in the given order, the order of their scalars] satisfying (1b).

\textbf{When $R = P$ [i.e, the 3 points coincide].} The $U$-altitude is $\text{PerpBisect}(VW)$, so $\triangle UV W$ is isosceles. Similarly, $\triangle UW V$ is isosceles. Thus $T$ is equilateral.\diamond

2\textsuperscript{nd} proof of (1). Use (1a). Let $s := \triangle uvw$ be the rev-medial triangle of $T$; so $U$ is $\text{Midt}(\overrightarrow{vw})$, etc.. Using similar triangles [perhaps the Reader can provide the Picture?] each $s$-median is a $T$-median. Hence

$$\text{Centroid}(s) = \text{Centroid}(T) \overset{\text{def}}{=} Q. \quad \square$$

Let $\varphi$: Plane$\rightarrow$Plane spin the plane about $Q$ by 180°, then dilate by a factor of two\textsuperscript{3}. So $\varphi$ sends lines

\textsuperscript{2}Averaging is an origin-invariant notion, BTWay.
\textsuperscript{3}With $(0,0) := Q$ the origin, this is the $(x, y) \mapsto (\text{-}2x, \text{-}2y)$ map.
through \( Q \) to themselves, reversing their orientation, and dilating by two. Thus \( \overrightarrow{PQ} \) is sent to itself, the line \( \overrightarrow{\varphi(P)Q} \). And

\[
\ast: \quad \text{Dist}(\varphi(P), Q) = 2 \cdot \text{Dist}(P, Q).
\]

Note that \( \varphi \) carries \( T \) to \( s \), hence carries \( P \) to \( \text{CircumCenter}(s) \). But \( \text{CircumCenter}(s) \) equals \( \text{OrthoCenter}(T) \). I.e, \( \varphi(P) = R \), thus \( \overrightarrow{PQ} = \overrightarrow{RQ} \), so \( P, Q, R \) are collinear. And (\( \ast \)) is a restatement of (1b). ☻