The Euler-line of a triangle

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Nomenclature. Fix points B and C.
Use $\overline{BC}$ for the *line-segment* with those endpts, and use $\text{Len}(\overline{BC})$ or just BC for its length.
Use $\overrightarrow{BC}$ for the *ray* starting at B and traversing C.
Use $\overleftrightarrow{BC}$ for the *line* that B and C determine.
As sets, then, $\overleftrightarrow{BC} \supset \overrightarrow{BC} \supset \overline{BC}$. And $C \in \overline{BC}$.
[If $B = C$ then the segment, ray and line are degenerate.]

Defn. The "W***-altitude*** of $\mathbf{T} := \triangle UVW$ " is the line through W which is orthogonal to edge $\overline{UV}$. Three particular points associated with $\mathbf{T}$ are:

$$\begin{aligned} \text{CircumCenter}(\mathbf{T}) &:= \bigcap \{\text{Perp-bisectors of } \mathbf{T}\}\,; \\ \text{Centroid}(\mathbf{T}) &:= \bigcap \{\text{Medians of } \mathbf{T}\}\,; \\ \text{OrthoCenter}(\mathbf{T}) &:= \bigcap \{\text{Altitudes of } \mathbf{T}\}\,. \end{aligned}$$ □

1: Euler-line Theorem. *For triangle $\mathbf{T} := \triangle UVW$, let*

$$\textbf{1a:}\qquad \begin{aligned} \mathsf{P} &:= \text{CircumCenter}(\mathbf{T})\,, \\ \mathsf{Q} &:= \text{Centroid}(\mathbf{T}) \qquad \textit{and} \\ \mathsf{R} &:= \text{OrthoCenter}(\mathbf{T})\,. \end{aligned}$$

Then this triple is colinear in that order, and

$$\textbf{1b:}\qquad \text{Dist}(\mathsf{R},\mathsf{Q}) = 2\cdot\text{Dist}(\mathsf{Q},\mathsf{P})\,.$$

If $\mathbf{T}$ is equilateral, then points $\mathsf{P},\mathsf{Q},\mathsf{R}$ coincide; otherwise, no two of $\mathsf{P},\mathsf{Q},\mathsf{R}$ coincide. ◊

Vectors. Take an arbitrary point P in the plane. If we regard P as the origin, then we can view the plane as a vectorspace. How? Well, for each point X, interpret X as the vector from-P-to-X. Write the (Euclidean) length of this vector as $\|X\|$; this is $\text{Len}(\overline{PX})$. □

2: Lemma. *Fix a line-segment $\overline{BC}$. Consider a point P on the perp-bisector of $\overline{BC}$. Viewing P as the origin of a vectorspace, vector $B + C$ is orthogonal*[♡1] *to $\overline{BC}$.* ◊

[♡1] If P is also *on* $\overline{BC}$, then $B = -C$, i.e $B + C$ is the zero-vector. The conclusion remains true, as the zero-vector is orthogonal to all vectors.

Proof. Since $\|B\| = \|C\|$ [because $P \in \text{PerpBisect}(\overline{BC})$] points, $P, B, B{+}C, C$ form the vertices of a rhombus. Thus point $B{+}C$ is the reflection of P across $\overleftrightarrow{BC}$. ♦

Pf of (1). View P, the circumcenter of $\mathbf{T}$, as the origin of a vectorspace. Define the vector sum

$$H \quad := \quad U + V + W\,.$$

Since P is on $\text{PerpBisect}(\overline{UV})$, vector $U + V$ is orthogonal to $\overline{UV}$. Thus $W + [U + V]$ is on the line through W perpendicular to $\overline{UV}$. IOWords, H is on the W-altitude of $\mathbf{T}$.
Since vector-addition is commutative and associative, we can write H as

$$V + [W + U] \quad \text{and as} \quad U + [V + W]\,.$$

Hence H also lies on the V and U-altitudes of $\mathbf{T}$. Thus $H = \mathsf{R}$.

Colinearity. The centroid of $\mathbf{T}$ is the average[♡2] of $\mathbf{T}$'s vertices. The upshot: With P viewed as the origin of a vectorspace, we have that

$$\begin{aligned} \mathsf{R} &= 1\cdot[U+V+W]\,; \\ \mathsf{Q} &= \tfrac{1}{3}\cdot[U+V+W]\,; \\ \mathsf{P} &= 0\cdot[U+V+W]\,. \end{aligned}$$

These points are multiples of a single vector, hence form a colinear triple [in the given order, the order of their scalars] satisfying (1b).

When $\mathsf{R} = \mathsf{P}$ [***i.e, the*** 3 ***points coincide***]. The U-altitude is $\text{PerpBisect}(\overline{VW})$, so $\triangle VUW$ is isosceles. Similarly, $\triangle UWV$ is isosceles. Thus $\mathbf{T}$ is equilateral. ♦

2nd proof of (1). Use (1a). Let $\mathbf{s} := \triangle uvw$ be the rev-medial triangle of $\mathbf{T}$; so U is $\text{Midpt}(\overline{vw})$, etc..
Using similar triangles [perhaps the Reader can provide the Picture?] each $\mathbf{s}$-median is a $\mathbf{T}$-median. Hence

$$\text{Centroid}(\mathbf{s}) \;=\; \text{Centroid}(\mathbf{T}) \stackrel{\text{def}}{=} \mathsf{Q}\,.$$

[♡2] Averaging is an origin-invariant notion, BTWay.

Let φ:Plane→Plane spin the plane about Q by 180°, then dilate by a factor of two[♡3]. So φ sends lines through Q to themselves, reversing their orientation, and dilating by two. Thus $\overleftrightarrow{\mathsf{PQ}}$ is sent to itself, the line $\overleftrightarrow{\varphi(\mathsf{P})\,\mathsf{Q}}$. And

$$*: \qquad \mathrm{Dist}(\varphi(\mathsf{P}),\mathsf{Q}) \quad = \quad 2\cdot\mathrm{Dist}(\mathsf{P},\mathsf{Q})\,.$$

Note that φ carries $\mathbf{T}$ to $\mathbf{s}$, hence carries P to CircumCenter($\mathbf{s}$). But CircumCenter($\mathbf{s}$) equals OrthoCenter($\mathbf{T}$). I.e, $\varphi(\mathsf{P}) = \mathsf{R}$, thus $\overleftrightarrow{\mathsf{PQ}} = \overleftrightarrow{\mathsf{RQ}}$, so $\mathsf{P},\mathsf{Q},\mathsf{R}$ are colinear. And $(*)$ is a restatement of (1b). ♦

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[♡3]With $(0,0) \coloneqq \mathsf{Q}$ the origin, this is the $(x,y) \mapsto (-2x,-2y)$ map.