One way to compute the curvature of an ellipse

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[*NB:* We use angle-brackets, $\langle \mathbf{v}, \mathbf{a} \rangle$, to mean the inner-product (the dot-product) of vectors $\mathbf{v}$ and $\mathbf{a}$.]

We want to compute the curvature function of the ellipse, $\boldsymbol{\mathcal{E}}$, whose axes-of-symmetry are the coordinate axes and whose semi-axis lengths are A and B; so $A,B > 0$. In cartesian coordinates, $\boldsymbol{\mathcal{E}}$ is the solution set of the equation

1: $$\frac{x^2}{A^2} + \frac{y^2}{B^2} \;=\; 1^2 \,.$$

If $A = B$ then this ellipse is circle. If $A > B$ then the major-axis direction is horizontal, etc.

The first **issue** in computing the curvature function of $\boldsymbol{\mathcal{E}}$ is to find a *representation* of $\boldsymbol{\mathcal{E}}$ for which we have a corresponding curvature formula. We don't yet have a formula which applies to (1); so far, our only curvature formulæ apply to either the graph of a 1-variable function *or* to the image of a parameterized curve.

Now $\boldsymbol{\mathcal{E}}$ is not the graph of a single function since it doesn't pass the vertical line test. We could try to represent parts of $\boldsymbol{\mathcal{E}}$ as graphs of functions, but it is probably better to represent $\boldsymbol{\mathcal{E}}$ as a parameterized curve.

How do we do that? Well, we *already know* a parameterization of one ellipse, namely, the unit circle $\mathfrak{C}$.

> $\mathfrak{C}$ *is the range of the vector-valued function*
> $M_{\mathfrak{C}}(t) \;:=\; \cos(t)\widehat{\boldsymbol{\imath}} + \sin(t)\widehat{\boldsymbol{\jmath}}\,.$

We know that by a linear stretching we can change the circle into any desired ellipse. In our case, we can stretch horizontally by a factor A and stretch vertically by a factor B to get that $\boldsymbol{\mathcal{E}}$ is the range of "moving point"

$$\boldsymbol{\mathcal{E}}: \qquad M(t) \;:=\; A\cos(t)\widehat{\boldsymbol{\imath}} \,+\, B\sin(t)\widehat{\boldsymbol{\jmath}}\,.$$

We will henceforth suppress the variable "t" and use "c" to abbreviate $\cos(t)$ and use "s" to abbreviate $\sin(t)$. In this notation, then,

2: $$M \;=\; Ac\widehat{\boldsymbol{\imath}} + Bs\widehat{\boldsymbol{\jmath}}$$

and it is a triviality to check that the x- and y- coordinates of M fulfill equation (1).

What formula to use?

Let $\mathcal{K}$ denote the curvature function of $\boldsymbol{\mathcal{E}}$. Now that we have a parameterization for $\boldsymbol{\mathcal{E}}$, what formula for $\mathcal{K}$ is likely to be easy to use? Well, there is no difficulty in computing the first and second derivatives of M and so perhaps a reasonable formula to employ is

$$\mathcal{K} \;=\; \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}\,, \quad \text{where } \mathbf{v} := M' \text{ and } \mathbf{a} := \mathbf{v}',$$

from our textbook. <u>*However*</u>, there is one issue that we must think about. A *cross-product* is defined between two vectors <u>in $\mathbb{R}^3$</u>; it is not defined for vectors in $\mathbb{R}^2$, although that is where $\mathbf{v}$ and $\mathbf{a}$ live. But for this particular problem, we are not interested in the *vector* $\mathbf{v}\times\mathbf{a}$ —we are merely interested in its *length*. So for our situation, we can think of the plane containing $\boldsymbol{\mathcal{E}}$ as being embedded as the x-y-plane inside of $\mathbb{R}^3$, and so we can interpret $\mathbf{v}$ and $\mathbf{a}$ as having a $\widehat{\mathbf{k}}$-component which is zero.

Let's compute.

3: $$\mathbf{v} \;=\; -As\widehat{\boldsymbol{\imath}} + Bc\widehat{\boldsymbol{\jmath}} \quad\text{and}\quad \mathbf{a} \;=\; -[Ac\widehat{\boldsymbol{\imath}} + Bs\widehat{\boldsymbol{\jmath}}]\,.$$

Thus $\mathbf{v}\times\mathbf{a} \;=\; \Big[[-As]\cdot[-Bs] \;-\; [Bc]\cdot[-Ac]\Big]\widehat{\mathbf{k}}$, which equals $[ABs^2 + ABc^2]\widehat{\mathbf{k}}$. Consequently

$$\|\mathbf{v}\times\mathbf{a}\| \;=\; |AB|\cdot|s^2+c^2| \;=\; |AB| = AB\,,$$

since A and B are each positive. Also, $\|\mathbf{v}\|^3$ equals $\langle \mathbf{v},\mathbf{v}\rangle^{3/2}$ which equals $[A^2s^2 + B^2c^2]^{3/2}$. Thus,

4: $$\mathcal{K} \;=\; \mathcal{K}_{A,B} \;=\; \frac{AB}{\left[A^2s^2 + B^2c^2\right]^{3/2}}\,.$$

We could write this out in full as

$$\mathcal{K}_{A,B}(t) \;=\; \frac{AB}{\left[A^2\sin(t)^2 + B^2\cos(t)^2\right]^{3/2}}\,.$$

Is our result, (4), reasonable?

Take the special case of a circle $A = B = \rho$. Then

$$\mathcal{K}_{\rho,\rho} = \frac{\rho\rho}{\left[\rho^2 s^2 + \rho^2 c^2\right]^{3/2}} \;=\; \frac{\rho^2}{\rho^3\left[s^2 + c^2\right]^{3/2}} \;=\; \frac{1}{\rho}\,,$$

which is indeed the curvature of a circle of radius ρ.

Here is a second test: Hold B fixed and send $A\searrow 0$. The ellipse flattens out (draw a picture!) and we see that at the two points of $\boldsymbol{\mathcal{E}}$ on the x-axis, the curvature should go to zero. Now, these two points of $\boldsymbol{\mathcal{E}}$ are the points when $\mathbf{(}c, s\mathbf{)} = \mathbf{(}\pm 1, 0\mathbf{)}$. At these points, the curvature equals

$$\begin{aligned}\mathcal{K}_{A,B} \;=\; \frac{AB}{\left[A^2 \cdot 0^2 \;+\; B^2 \cdot [\pm 1]^2\right]^{3/2}} &= \frac{AB}{\left[B^2\right]^{3/2}} \\ &= \frac{A}{B^2}\,.\end{aligned}$$

And $\lim\limits_{A\to 0} \frac{A}{B^2}$ does indeed equal zero.

Similarly, holding B fixed and sending $A\nearrow\infty$ stretches the ellipse horizontally, and should send the curvature at the "x-axis points of $\boldsymbol{\mathcal{E}}$" to infinity. Is our formula, (4), consistent with that? Yes, since $\lim\limits_{A\to\infty} \frac{A}{B^2} = \infty$. Do similar tests at the "y-axis points".

Exercise

Suppose $P = \mathbf{(}x, y\mathbf{)}$ is a point on $\boldsymbol{\mathcal{E}}$. Show that

5: $$\mathcal{K}_{A,B}(P) \;=\; \frac{A^4\, B^4}{\left[B^4 x^2 \;+\; A^4 y^2\right]^{3/2}}\,.$$

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