One way to compute the curvature of an ellipse

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[NB: We use angle-brackets, $\langle v, a \rangle$, to mean the inner-product (the dot-product) of vectors $v$ and $a$.]

We want to compute the curvature function of the ellipse, $\mathcal{E}$, whose axes-of-symmetry are the coordinate axes and whose semi-axis lengths are $A$ and $B$; so $A,B > 0$. In cartesian coordinates, $\mathcal{E}$ is the solution set of the equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1^2.$$  

If $A = B$ then this ellipse is circle. If $A > B$ then the major-axis direction is horizontal, etc.

The first issue in computing the curvature function of $\mathcal{E}$ is to find a representation of $\mathcal{E}$ for which we have a corresponding curvature formula. We don’t yet have a formula which applies to (1); so far, our only curvature formulæ apply to either the graph of a 1-variable function or to the image of a parameterized curve.

Now $\mathcal{E}$ is not the graph of a single function since it doesn’t pass the vertical line test. We could try to represent parts of $\mathcal{E}$ as graphs of functions, but it is probably better to represent $\mathcal{E}$ as a parameterized curve.

How do we do that? Well, we already know a parameterization of one ellipse, namely, the unit circle $\mathcal{C}$.

$$\mathcal{E} \text{ is the range of the vector-valued function }$$

$$M_\mathcal{E}(t) := \cos(t)\hat{i} + \sin(t)\hat{j}.$$  

We know that by a linear stretching we can change the circle into any desired ellipse. In our case, we can stretch horizontally by a factor $A$ and stretch vertically by a factor $B$ to get that $\mathcal{E}$ is the range of “moving point”

$$\mathcal{E} : \quad M(t) := Acos(t)\hat{i} + Bsin(t)\hat{j}.$$  

We will henceforth suppress the variable “$t$” and use “$c$” to abbreviate $\cos(t)$ and use “$s$” to abbreviate $\sin(t)$. In this notation, then,

$$2: \quad M = Ac\hat{i} + Bs\hat{j}$$

and it is a triviality to check that the $x$- and $y$-coordinates of $M$ fulfill equation (1).

What formula to use?

Let $\mathcal{K}$ denote the curvature function of $\mathcal{E}$. Now that we have a parameterization for $\mathcal{E}$, what formula for $\mathcal{K}$ is likely to be easy to use? Well, there is no difficulty in computing the first and second derivatives of $M$ and so perhaps a reasonable formula to employ is

$$\mathcal{K} = \frac{||v \times a||}{||v||^3}, \quad \text{where } v := M' \text{ and } a := v',$$

from our textbook. However, there is one issue that we must think about. A cross-product is defined between two vectors in $\mathbb{R}^3$; it is not defined for vectors in $\mathbb{R}^2$, although that is where $v$ and $a$ live. But for this particular problem, we are not interested in the vector $v \times a$ — we are merely interested in its length.

So for our situation, we can think of the plane containing $\mathcal{E}$ as being embedded as the $x$-$y$-plane inside of $\mathbb{R}^3$, and so we can interpret $v$ and $a$ as having a $k$-component which is zero.

Let’s compute.

$$3: \quad v = -As\hat{i} + Bs\hat{j} \quad \text{and} \quad a = -[Ac\hat{i} + Bs\hat{j}].$$

Thus $v \times a = \left[[-As][-Bs] - [-Bc][-Ac]\right]\hat{k}$, which equals $[ABs^2 + ABC^2]\hat{k}$. Consequently

$$||v \times a|| = |AB| \cdot |s^2 + c^2| = |AB| = AB,$$

since $A$ and $B$ are each positive. Also, $||v||^3$ equals $\langle v, v \rangle^{3/2}$ which equals $[A^2s^2 + B^2c^2]^{3/2}$. Thus,

$$4: \quad \mathcal{K} = \mathcal{K}_{A,B} = \frac{AB}{[A^2s^2 + B^2c^2]^{3/2}}.$$  

We could write this out in full as

$$\mathcal{K}_{A,B}(t) = \frac{AB}{[A^2\sin(t)^2 + B^2\cos(t)^2]^{3/2}}.$$
Is our result, (4), reasonable?

Take the special case of a circle \( A = B = \rho \). Then

\[
K_{\rho, \rho} = \frac{\rho \rho}{\left[ \rho^2 s^2 + \rho^2 c^2 \right]^{3/2}} = \frac{\rho^2}{\rho^3 [s^2 + c^2]^{3/2}} = \frac{1}{\rho},
\]

which is indeed the curvature of a circle of radius \( \rho \).

Here is a second test: Hold \( B \) fixed and send \( A \downarrow 0 \). The ellipse flattens out (draw a picture!) and we see that at the two points of \( \mathcal{E} \) on the \( x \)-axis, the curvature should go to zero. Now, these two points of \( \mathcal{E} \) are the points when \( (c, s) = (\pm 1, 0) \). At these points, the curvature equals

\[
K_{A, B} = \frac{AB}{[A^2 \cdot 0^2 + B^2 \cdot [\pm 1]^2]^{3/2}} = \frac{AB}{[B^2]^{3/2}} = \frac{A}{B^2}.
\]

And \( \lim_{A \to 0} \frac{A}{B^2} \) does indeed equal zero.

Similarly, holding \( B \) fixed and sending \( A \uparrow \infty \) stretches the ellipse horizontally, and should send the curvature at the "\( x \)-axis points of \( \mathcal{E} \)" to infinity. Is our formula, (4), consistent with that? Yes, since \( \lim_{A \to \infty} \frac{A}{B^2} = \infty \). Do similar tests at the "\( y \)-axis points".

Exercise

Suppose \( P = (x, y) \) is a point on \( \mathcal{E} \). Show that

5: \( K_{A, B}(P) = \frac{A^4 B^4}{[B^4 x^2 + A^4 y^2]^{3/2}}. \)