

One way to compute the curvature of an ellipse

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage <http://squash.1gainesville.com/>
23 September, 2017 (at 15:19)

[NB: We use angle-brackets, $\langle \mathbf{v}, \mathbf{a} \rangle$, to mean the inner-product (the dot-product) of vectors \mathbf{v} and \mathbf{a} .]

We want to compute the curvature function of the ellipse, \mathcal{E} , whose axes-of-symmetry are the coordinate axes and whose semi-axis lengths are A and B ; so $A, B > 0$. In cartesian coordinates, \mathcal{E} is the solution set of the equation

$$1: \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1^2.$$

If $A = B$ then this ellipse is circle. If $A > B$ then the major-axis direction is horizontal, etc.

The first **issue** in computing the curvature function of \mathcal{E} is to find a *representation* of \mathcal{E} for which we have a corresponding curvature formula. We don't yet have a formula which applies to (1); so far, our only curvature formulæ apply to either the graph of a 1-variable function *or* to the image of a parameterized curve.

Now \mathcal{E} is not the graph of a single function since it doesn't pass the vertical line test. We could try to represent parts of \mathcal{E} as graphs of functions, but it is probably better to represent \mathcal{E} as a parameterized curve.

How do we do that? Well, we *already know* a parameterization of one ellipse, namely, the unit circle \mathcal{C} .

\mathcal{C} is the range of the vector-valued function
 $M_{\mathcal{C}}(t) := \cos(t)\hat{\mathbf{i}} + \sin(t)\hat{\mathbf{j}}$.

We know that by a linear stretching we can change the circle into any desired ellipse. In our case, we can stretch horizontally by a factor A and stretch vertically by a factor B to get that \mathcal{E} is the range of "moving point"

$$\mathcal{E}: \quad M(t) := A\cos(t)\hat{\mathbf{i}} + B\sin(t)\hat{\mathbf{j}}.$$

We will henceforth suppress the variable "t" and use "c" to abbreviate $\cos(t)$ and use "s" to abbreviate $\sin(t)$. In this notation, then,

$$2: \quad M = Ac\hat{\mathbf{i}} + Bs\hat{\mathbf{j}}$$

and it is a triviality to check that the x - and y - coordinates of M fulfill equation (1).

What formula to use?

Let \mathcal{K} denote the curvature function of \mathcal{E} . Now that we have a parameterization for \mathcal{E} , what formula for \mathcal{K} is likely to be easy to use? Well, there is no difficulty in computing the first and second derivatives of M and so perhaps a reasonable formula to employ is

$$\mathcal{K} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}, \quad \text{where } \mathbf{v} := M' \text{ and } \mathbf{a} := \mathbf{v}',$$

from our textbook. *However*, there is one issue that we must think about. A *cross-product* is defined between two vectors in \mathbb{R}^3 ; it is not defined for vectors in \mathbb{R}^2 , although that is where \mathbf{v} and \mathbf{a} live. But for this particular problem, we are not interested in the *vector* $\mathbf{v} \times \mathbf{a}$ —we are merely interested in its *length*. So for our situation, we can think of the plane containing \mathcal{E} as being embedded as the x - y -plane inside of \mathbb{R}^3 , and so we can interpret \mathbf{v} and \mathbf{a} as having a $\hat{\mathbf{k}}$ -component which is zero.

Let's compute.

$$3: \quad \mathbf{v} = -As\hat{\mathbf{i}} + Bc\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{a} = -[Ac\hat{\mathbf{i}} + Bs\hat{\mathbf{j}}].$$

Thus $\mathbf{v} \times \mathbf{a} = [(-As) \cdot (-Bs) - [Bc] \cdot [-Ac]]\hat{\mathbf{k}}$, which equals $[ABs^2 + ABc^2]\hat{\mathbf{k}}$. Consequently

$$\|\mathbf{v} \times \mathbf{a}\| = |AB| \cdot |s^2 + c^2| = |AB| = AB,$$

since A and B are each positive. Also, $\|\mathbf{v}\|^3$ equals $\langle \mathbf{v}, \mathbf{v} \rangle^{3/2}$ which equals $[A^2s^2 + B^2c^2]^{3/2}$. Thus,

$$4: \quad \mathcal{K} = \mathcal{K}_{A,B} = \frac{AB}{[A^2s^2 + B^2c^2]^{3/2}}.$$

We could write this out in full as

$$\mathcal{K}_{A,B}(t) = \frac{AB}{[A^2\sin(t)^2 + B^2\cos(t)^2]^{3/2}}.$$

Is our result, (4), reasonable?

Take the special case of a circle $A = B = \rho$. Then

$$\mathcal{K}_{\rho,\rho} = \frac{\rho\rho}{[\rho^2 s^2 + \rho^2 c^2]^{3/2}} = \frac{\rho^2}{\rho^3 [s^2 + c^2]^{3/2}} = \frac{1}{\rho},$$

which is indeed the curvature of a circle of radius ρ .

Here is a second test: Hold B fixed and send $A \searrow 0$. The ellipse flattens out (draw a picture!) and we see that at the two points of \mathcal{E} on the x -axis, the curvature should go to zero. Now, these two points of \mathcal{E} are the points when $(c, s) = (\pm 1, 0)$. At these points, the curvature equals

$$\begin{aligned} \mathcal{K}_{A,B} &= \frac{AB}{[A^2 \cdot 0^2 + B^2 \cdot [\pm 1]^2]^{3/2}} = \frac{AB}{[B^2]^{3/2}} \\ &= \frac{A}{B^2}. \end{aligned}$$

And $\lim_{A \rightarrow 0} \frac{A}{B^2}$ does indeed equal zero.

Similarly, holding B fixed and sending $A \nearrow \infty$ stretches the ellipse horizontally, and should send the curvature at the “ x -axis points of \mathcal{E} ” to infinity. Is our formula, (4), consistent with that? Yes, since $\lim_{A \rightarrow \infty} \frac{A}{B^2} = \infty$. Do similar tests at the “ y -axis points”.

Exercise

Suppose $P = (x, y)$ is a point on \mathcal{E} . Show that

$$5: \quad \mathcal{K}_{A,B}(P) = \frac{A^4 B^4}{[B^4 x^2 + A^4 y^2]^{3/2}}.$$

Filename: `Classwork/GeneralHandouts/ellipsecurvature.tex`
 As of: *Tuesday 21Feb2006*. Typeset: *23Sep2017 at 15:19*.