e is irrational

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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From the web.

1: Lemma. *Consider positive reals $a_0 \geq a_1 \geq a_2 \geq \ldots$ such that $a_n \to 0$. Then*

$$L := a_0 - a_1 + a_2 - a_3 + a_4 - \ldots$$

exists in $\mathbb{R}$. Moreover, for each even posint N,

$$\Big[\sum_{k=0}^{N-1} [-1]^k a_k\Big] + a_N > L > \sum_{k=0}^{N-1} [-1]^k a_k .$$

Proof. Exercise. ◊

2: Theorem. *The number e is irrational* ◊

Proof. FTSOC, suppose $1/\mathsf{e}$ equals p/q, for some posints p,q. Recall that $\mathsf{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus

$$\frac{1}{\mathsf{e}} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots .$$

For N even, then,

$$*: \quad \Big[\sum_{k=0}^{N-1} \frac{[-1]^k}{k!}\Big] + \frac{1}{N!} > \frac{p}{q} > \sum_{k=0}^{N-1} \frac{[-1]^k}{k!} .$$

Multiplying this by $N!$ gives

$$S+1 > \tfrac{N!}{q} \cdot p > S ,$$

where $S := N! \cdot \mathrm{RhS}(*)$ is an integer.

Taking an $N \geq q$ makes $\frac{N!}{q}$ an integer, yielding contradiction that $S+1 > \textit{Integer} > S$. ♦

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