

## Existence and uniqueness of solution to an ODE

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**ABSTRACT:** Exposition of the standard contraction-mapping proof for a 1<sup>st</sup>-order DE. Shows how to apply this to higher-order DEs.

(The Fundamental Thm of ODEs, abbreviated FTODE: Wikipedia calls this either Picard-Lindelöf thm or Picard's existence thm or Cauchy-Lipschitz thm.)

**Entrance.** Consider<sup>♥1</sup> an initial-condition-DE

$$1: \quad \mathbf{x}'(t) = \mathbf{K}(t; \mathbf{x}(t)),$$

where  $\mathbf{x}(5) = \mathbf{0}$ .

Here  $\mathbf{x}$  is the *unknown function*. Putting sufficient conditions on  $\mathbf{K}()$ , the *kernel*, will guarantee a local solution; a *unique* local soln.

### The Setting

Fix a Banach space  $\mathbb{H}$  with  $\mathbf{0}$  its zero vector; use  $[\cdot]$  for its norm. Let  $\mathbf{B} \subset \mathbb{H}$  be a *closed* ball centered at  $\mathbf{0}$ , with  $\text{Radius}(\mathbf{B}) > 0$ . (Possibly  $\mathbf{B}$  will have infinite-radius and be all of  $\mathbb{H}$ ; in that instance, step (6), below, is superfluous, and consequently condition (A2) is unnecessary.) Typically,  $\mathbb{H}$  is a finite-dim'nal Euclidean space  $\mathbb{R}^N$ . Then, if  $\text{Radius}(\mathbf{B})$  is finite, our  $\mathbf{B}$  will automatically be compact.

**The “contraction space”  $\Omega$ .** Consider a compact interval

$$J_w := [5-w, 5+w],$$

<sup>♥1</sup>**Phrases:** WLOG: ‘Without loss of generality’. TFAE: ‘The following are equivalent’. ITOF: ‘In Terms Of’. OTForm: ‘of the form’. FTSOC: ‘For the sake of contradiction’. Use iff: ‘if and only if’.

IST: ‘It Suffices to’ as in ISTShow, ISTExhibit.

Use w.r.t: ‘with respect to’ and s.t: ‘such that’.

**Latin:** e.g: *exempli gratia*, ‘for example’. i.e: *id est*, ‘that is’. N.B: *Nota bene*, ‘Note well’. QED: *quod erat demonstrandum*, meaning “end of proof”.

where posreal  $w$  is what we’ll call its “width”. Let  $\Omega_w$  be the space of **continuous** functions

$$\mathbf{z}: J_w \rightarrow \mathbf{B} \quad \text{such that} \quad \mathbf{z}(5) = \mathbf{0}.$$

Our goal is to find a differentiable fnc  $\mathbf{x} \in \Omega_w$  fulfilling (1) for all  $t \in J_w$ . Henceforth write these two sets as  $J$  and  $\Omega$ ; the dependency on the width was made explicit, above, since *we will later shrink  $w$* . Use  $[\cdot]$  to indicate the sup-norm,

$$2a: \quad [\mathbf{z}] := \sup_{t \in J} [\mathbf{z}(t)],$$

for functions mapping  $J \rightarrow \mathbb{H}$ . For each  $\mathbf{z} \in \Omega$ , since  $J$  is compact,  $[\mathbf{z}] < \infty$ .

Convergence in  $[\cdot]$  is uniform-convergence. The usual “three- $\varepsilon$  argument” shows that the uniform-limit of continuous fncs (exists and) is cts. Thus

2b: *Metric space  $(\Omega, [\cdot])$  is complete.*

(This used that  $\mathbf{B}$  is closed.) Each contraction mapping  $\Omega \rightarrow \Omega$  has a (unique) fixed-point. Our goal:

2c: *Produce a contraction mapping  $\mathbf{z} \mapsto \hat{\mathbf{z}}$  whose unique fixed-point is a soln to (1).*

### Conditions on $\mathbf{K}$

Our kernel  $\mathbf{K}$  is a function

$$A0: \quad \mathbf{K}: J \times \mathbf{B} \rightarrow \mathbb{H}$$

satisfying these three conditions:

A1:  $\mathbf{K}()$  is continuous.

A2:  $\mathbf{K}()$  is bounded. I.e, its sup-norm  $\|\mathbf{K}\|_{J \times \mathbf{B}}$  is finite. [This boundedness is not needed if  $\mathbf{B}$  is all of  $\mathbb{H}$ .]

A3:  $\mathbf{K}()$  is  $\mathbb{H}$ -wise Lipschitz.

This last means we have a real number  $\mathcal{U} < \infty$ , where  $\mathcal{U}$  is the supremum of

$$\frac{[\mathbf{K}(t; \mathbf{z}) - \mathbf{K}(t; \mathbf{y})]}{[\mathbf{z} - \mathbf{y}]}$$

taken over all  $t \in J$  and all distinct-point pairs  $\mathbf{z}, \mathbf{y} \in \mathbf{B}$ . In particular

$$\forall t \in J, \forall \mathbf{z}, \mathbf{y} \in \mathbf{B} : \|\mathbf{K}(t; \mathbf{z}) - \mathbf{K}(t; \mathbf{y})\| \leq \mathcal{U} \cdot \|\mathbf{z} - \mathbf{y}\|.$$

Evidently

If  $\mathbf{B}$  is compact then  $J \times \mathbf{B}$  is cpt, hence (A2).

A4: And automatically (A3), when  $\mathbf{B}$  is compact and the partial derivative  $\frac{d\mathbf{K}}{d\mathbf{y}}(t; \mathbf{y})$  exists, and is a continuous function of  $(t, \mathbf{y}) \in J \times \mathbf{B}$ .

Each function  $\mathbf{z} \in \Omega$  has a “**K-image**”, which is a fnc  $f_{\mathbf{z}}: J \rightarrow \mathbb{H}$ . It is

$$4a: \quad f_{\mathbf{z}}(t) := \mathbf{K}(t; \mathbf{z}(t)).$$

Our Lipschitz constant  $\mathcal{U}$  yields this:

For all functions  $\mathbf{z}, \mathbf{y} \in \Omega$ :

$$4b: \quad \|f_{\mathbf{z}} - f_{\mathbf{y}}\| \leq \mathcal{U} \cdot \|\mathbf{z} - \mathbf{y}\|.$$

### The operator

Each  $f_{\mathbf{z}}$  is cts, courtesy the cty of  $\mathbf{K}$ . Thus (1), with DE  $\boxed{\mathbf{x}' = f_{\mathbf{x}}}$ , is equivalent to assertion

$$1': \quad \mathbf{x}(t) \stackrel{?}{=} \int_5^t f_{\mathbf{x}} \stackrel{\text{def}}{=} \int_5^t f_{\mathbf{x}}(\tau) \cdot d\tau, \quad \text{for every time } t \in J.$$

This, courtesy the Fund.thm of Calculus, since  $f_{\mathbf{x}}$  is cts so RhS(1') is defined and differentiable.

Each fnc  $\mathbf{z} \in \Omega$  yields a new fnc  $\hat{\mathbf{z}}: J \rightarrow \mathbb{H}$ ,

$$5: \quad \hat{\mathbf{z}} := \left[ t \mapsto \int_5^t f_{\mathbf{z}} \right].$$

A solution  $\mathbf{x}$  to (1') is a **fixed-point** of the mapping  $\mathbf{z} \mapsto \hat{\mathbf{z}}$ . Each  $\hat{\mathbf{z}}$  maps into  $\mathbb{H}$ ; as a first step, we need to guarantee that  $\hat{\mathbf{z}}$  maps into  $\mathbf{B}$ . So we need to arrange that the norm of  $\hat{\mathbf{z}}$  is  $\leq \text{Radius}(\mathbf{B})$ . From (5) we have, since  $w$  is the radius of  $J$ , that

$$\|\hat{\mathbf{z}}\| \leq w \cdot \|f_{\mathbf{z}}\| \stackrel{\text{note}}{\leq} w \cdot \|\mathbf{K}\|_{J \times \mathbf{B}}.$$

Courtesy (A2), our  $\mathbf{K}()$  is bounded. So we can simply shrink<sup>♥1</sup> width  $w$  until the product

$$6: \quad w \cdot \|\mathbf{K}\| \text{ is dominated by } \text{Radius}(\mathbf{B}).$$

This arranges that the  $[\mathbf{z} \mapsto \hat{\mathbf{z}}]$  mapping indeed maps  $\Omega$  into  $\Omega$ .

<sup>♥1</sup>Shrinking  $w$  shrinks interval  $J$  and so might decrease norm  $\|\mathbf{K}\|_{J \times \mathbf{B}}$ . Lipschitz constant  $\mathcal{U}$  might also decrease.

### The Contraction

For each two functions  $\mathbf{z}, \mathbf{y} \in \Omega$  observe that

$$7: \quad \begin{aligned} \|\hat{\mathbf{z}} - \hat{\mathbf{y}}\| &\leq w \cdot \|f_{\mathbf{z}} - f_{\mathbf{y}}\| \\ &\leq w \cdot \mathcal{U} \cdot \|\mathbf{z} - \mathbf{y}\|. \end{aligned}$$

Again<sup>♥1</sup> shrink  $w$ , this time so that

$$6': \quad w \cdot \mathcal{U} < 1.$$

Now, finally,  $\mathbf{z} \mapsto \hat{\mathbf{z}}$  is a contraction-map on  $\Omega$  and thus has a unique fixed-point.  $\diamond$

**A general initial-condition.** If we instead require that  $\mathbf{y}(5) = \mathbf{P}$  for some particular pt  $\mathbf{P} \in \mathbb{H}$ , then just center ball  $\mathbf{B}$  at  $\mathbf{P}$  and replace (5) by

$$5': \quad \hat{\mathbf{z}} := \left[ t \mapsto \mathbf{P} + \int_5^t f_{\mathbf{z}}(\tau) d\tau \right].$$

## Higher-order DEs

We first give an *example* of “re-coding” a higher-order DE to a 1<sup>st</sup>-order DE.

**2<sup>nd</sup>-order to 1<sup>st</sup>-order.** Over an interval  $J \subset \mathbb{R}$ , suppose we seek a soln  $h: J \rightarrow \mathbb{R}$  to DE

$$8a: \quad h''(t) = \sin(t \cdot h'(t)) + h(t).$$

A fnc  $\mathbf{x}: J \rightarrow \mathbb{R}^2$  can be written in components as

$$\mathbf{x}(t) = \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix}.$$

If each  $h_j$  is differentiable then so is  $\mathbf{x}$ , and  $\mathbf{x}'$  equals  $\begin{bmatrix} h_1 \\ h_0 \end{bmatrix}' \stackrel{\text{note}}{=} \begin{bmatrix} h_1' \\ h_0' \end{bmatrix}$ . Consider DE (\*)

$$8b: \quad \mathbf{x}'(t) \stackrel{\text{def}}{=} \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix}' \stackrel{*}{=} \begin{bmatrix} \sin(t \cdot h_1(t)) + h_0(t) \\ h_1(t) \end{bmatrix}.$$

The 0<sup>th</sup>-component,  $h_0$ , of a soln  $\mathbf{x}$  to (8b) is automatically a soln to (8a). Conversely, a soln  $h$  to (8a) yields a soln  $\begin{bmatrix} h_1(0) \\ h_0(0) \end{bmatrix}$  to (8b), by setting  $h_0 := h$  and  $h_1 := h'$ .

Lastly, defining the kernel  $\mathbf{K}: J \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$8c: \quad \mathbf{K}\left(t; \begin{bmatrix} h_1 \\ h_0 \end{bmatrix}\right) := \begin{bmatrix} \sin(t \cdot h_1) + h_0 \\ h_1 \end{bmatrix},$$

makes the vector-valued DE  $\mathbf{x}'(t) = \mathbf{K}(t; \mathbf{x}(t))$  a restatement of (8b), hence of (8a).

**Re-coding done more generally**

Now consider a third-order DE OTForm

9: 
$$h''' = G(t; h, h', h''),$$
 with  $h^{(k)}(5) = P_k$ , for  $k = 0, 1, 2$ .

Here  $h: J \rightarrow \mathbb{H}$  and  $G: J \times \mathbb{H}^3 \rightarrow \mathbb{H}$  and each  $P_k$  is a point in  $\mathbb{H}$ . By letting  $h_k$  denote the  $k^{\text{th}}$  derivative  $h^{(k)}$ , we can restate the DE part of (9) as a vector equation

$$\begin{bmatrix} h_2' \\ h_1' \\ h_0' \end{bmatrix} = \begin{bmatrix} G(t; h_0, h_1, h_2) \\ h_2 \\ h_1 \end{bmatrix}.$$

**Reduction.** In order to write (9) as a *first-order* eqn, consider a fnc  $\mathbf{x}: J \rightarrow \mathbb{H}^3$  written as a column vector  $\mathbf{x}(t) = \begin{bmatrix} h_2(t) \\ h_1(t) \\ h_0(t) \end{bmatrix}$ , where each  $h_k$  maps  $J \rightarrow \mathbb{H}$ . This notation hands us a function

10: 
$$\mathbf{K}: J \times \mathbb{H}^3 \rightarrow \mathbb{H}^3 \text{ defined by}$$

$$\mathbf{K}\left(t; \begin{bmatrix} h_2 \\ h_1 \\ h_0 \end{bmatrix}\right) := \begin{bmatrix} G(t; h_0, h_1, h_2) \\ h_2 \\ h_1 \end{bmatrix}.$$

This allows us to rewrite (9) as

9': 
$$\forall t \in J: \mathbf{x}'(t) = \mathbf{K}(t; \mathbf{x}(t)),$$
 with  $\mathbf{x}(5) = \mathbf{P}$ ,

where  $\mathbf{P}$  denotes the point  $\begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix}$  in  $\mathbb{H}^3$ . ◇

*Commentary.* Suppose  $\mathbf{K}$  is, say, 7-times continuously-differentiable. Then a fixed-pt of  $\mathbf{z} \mapsto \hat{\mathbf{z}}$ , where

$$\hat{\mathbf{z}}(t) := \mathbf{P} + \int_5^t \mathbf{K}(\tau; \mathbf{z}(\tau)) \, d\tau,$$

is (at least) 8-times continuously-diff'able.

When  $\mathbf{K}()$  comes from (10) then the level of differentiability of  $\mathbf{K}()$  is that of  $G()$ . □

**When our hypotheses fail.** Consider the following  $\mathbb{H} := \mathbb{R}$  case. For  $C$  a real constant, define

11a: 
$$\mathbf{x}(t) = \mathbf{x}_C(t) := \frac{1}{1 + Ct}.$$

Note  $\mathbf{x}(t) - 1 = \frac{-Ct}{1+Ct}$ , so

$$\mathbf{x}(t) \cdot [\mathbf{x}(t) - 1] = \frac{-Ct}{[1 + Ct]^2}.$$

But this latter equals  $t \cdot \mathbf{x}'(t)$ . hence  $\mathbf{x}_C()$  satisfies initial-value problem

11b: 
$$t \cdot \mathbf{x}'(t) = \mathbf{x}(t) \cdot [\mathbf{x}(t) - 1], \text{ with}$$

$$\mathbf{x}(0) = 1.$$

For each  $C \in [0, 3)$  our  $\mathbf{x}_C()$  is well-defined on interval  $J := (-\frac{1}{3}, \infty)$ . Since  $J \ni 0$  [our initial-cond is at 0], we thus have an infinite family of solns to IVP (11b) on  $J$ ; one soln for each  $C \in [0, 3)$ .

Since the conclusion to FTODE is false, it must be that some FTODE-hypothesis failed.

Writing (11b) in form (1), our kernel is

11c: 
$$\mathbf{K}(t; x) := \frac{1}{t} \cdot [x^2 - x].$$

But this is **not** well-defined at  $t=0$ , which is where our initial-condition takes place. Also, no matter how small a  $t_0 > 0$  is take, the RhS(11c) is *not*  $\mathbb{H}$ -wise Lipschitz as  $t$  ranges over interval  $(0, t_0)$ . □