Existence and uniqueness of solution to an ODE

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ABSTRACT: Exposition of the standard contraction-mapping proof for a 1st-order DE. Shows how to apply this to higher-order DEs.

(The Fundamental Thm of ODEs, abbreviated FTODE: Wikipedia calls this either Picard-Lindelöf thm or Picard’s existence thm or Cauchy-Lipschitz thm.)

Entrance. Consider an initial-condition-DE

\[ x'(t) = K(t; x(t)), \]
where \( x(5) = 0 \).

Here \( x \) is the unknown function. Putting sufficient conditions on \( K() \), the kernel, will guarantee a local solution; a unique local soln.

The Setting

Fix a Banach space \( \mathbb{H} \) with 0 its zero vector; use \([\cdot]\) for its norm. Let \( B \subset \mathbb{H} \) be a closed ball centered at 0, with \( \text{Radius}(B) > 0 \). (Possibly \( B \) will have infinite-radius and be all of \( \mathbb{H} \); in that instance, step (6), below, is superfluous, and consequently condition (A2) is unnecessary.) Typically, \( \mathbb{H} \) is a finite-dim’al Euclidean space \( \mathbb{R}^n \). Then, if \( \text{Radius}(B) \) is finite, our \( B \) will automatically be compact.

The “contraction space” \( \Omega \). Consider a compact interval

\[ J_w := [5-w, 5+w], \]

where posreal \( w \) is what we’ll call its “width”. Let \( \Omega_w \) be the space of continuous functions

\[ z: J_w \to B \text{ such that } z(5) = 0. \]

Our goal is to find a differentiable fnc \( x \in \Omega_w \) fulfilling (1) for all \( t \in J_w \). Henceforth write these two sets as \( J \) and \( \Omega \); the dependency on the width was made explicit, above, since we will later shrink \( w \). Use \([\cdot]\) to indicate the sup-norm,

\[ |z| := \sup_{t \in J} [z(t)], \]

for functions mapping \( J \to \mathbb{H} \). For each \( z \in \Omega \), since \( J \) is compact, \([z] < \infty \).

Convergence in \([\cdot]\) is uniform-convergence. The usual “three-\( \varepsilon \) argument” shows that the uniform-limit of continuous fnecs (exists and) is cts. Thus

2b: Metric space \((\Omega, [\cdot])\) is complete.

(This used that \( B \) is closed.) Each contraction mapping \( \Omega \to \Omega \) has a (unique) fixed-point. Our goal:

2c: Produce a contraction mapping \( z \mapsto \hat{z} \) whose unique fixed-point is a soln to (1).

Conditions on \( K \)

Our kernel \( K \) is a function

A0: \( K: J \times B \to \mathbb{H} \)
satisfying these three conditions:

A1: \( K() \) is continuous.

A2: \( \|K\|_{J \times B} \) is finite. [This boundedness is not needed if \( B \) is all of \( \mathbb{H} \).]

A3: \( K() \) is \( \mathbb{H} \)-wise Lipschitz.

This last means we have a real number \( U < \infty \), where \( U \) is the supremum of

\[ \frac{[K(t; z) - K(t; y)]}{|z - y|} \]

Use w.r.t: ‘with respect to’ and s.t: ‘such that’.
Latin: e.g. exempli gratia, ‘for example’. i.e: id est, ‘that is’. N.B: Nota bene, ‘Note well’. QED: quod erat demonstrandum, meaning “end of proof”.

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taken over all \( t \in J \) and all distinct-point pairs \( z, y \in B \). In particular

\[
\forall t \in J, \forall z, y \in B: \quad [K(t; z) - K(t; y)] \leq U \cdot [z - y].
\]

Evidently

1. **If** \( B \) **is compact then** \( J \times B \) **is cpt, hence** (A2).

2. **And automatically** (A3), when \( B \) **is compact and** the partial derivative \( \frac{dK}{dy}(t; y) \) **exists, and is a continuous function of** \((t, y) \in J \times B\).

Each function \( z \in \Omega \) has a **"K-image"**, which is a fnc \( f_z: J \rightarrow \mathbb{H} \). It is

1a: \( f_z(t) := K(t; z(t)) \).

Our Lipschitz constant \( U \) yields this:

2. **For all functions** \( z, y \in \Omega \):

1b: \( \|f_z - f_y\| \leq U \cdot \|z - y\| \).

**The operator**

Each \( f_z \) is cts, courtesy the cty of \( K \). Thus (1), with DE \( \frac{\text{d}x}{\text{d}t} = f_z \), is equivalent to assertion

1': \( x(t) = \int_0^t f_x(\tau) \, d\tau \), for every time \( t \in J \).

This, courtesy the **Fund.thm of Calculus**, since \( f_x \) is cts so RhS(???) is defined and differentiable.

Each fnc \( z \in \Omega \) yields a new fnc \( \tilde{z}: J \rightarrow \mathbb{H} \),

5: \( \tilde{z} := [t \mapsto \int_0^t f_z(\tau) \, d\tau] \).

A solution \( x \) to (???) is a **fixed-point** of the mapping \( z \mapsto \tilde{z} \). Each \( \tilde{z} \) maps into \( \mathbb{H} \); as a first step, we need to guarantee that \( \tilde{z} \) maps into \( B \). So we need to arrange that the norm of \( \tilde{z} \) is \( \leq \text{Radius}(B) \).

From (5) we have, since \( w \) is the radius of \( J \), that

\[
\|\tilde{z}\| \leq w \cdot \|f_z\| \leq w \cdot \|K\|_{J \times B}.
\]

Courtesy (A2), our \( K() \) is bounded. So we can simply shrink \(^{\circlearrowright} 1 \) width \( w \) until the product

6: \( w \cdot \|K\| \) **is dominated by** \( \text{Radius}(B) \).

This arranges that the \( z \mapsto \tilde{z} \) mapping indeed maps \( \Omega \) into \( \Omega \).

\(^{\circlearrowright} 1 \) Shrinking \( w \) shrinks interval \( J \) and so might decrease norm \( \|K\|_{J \times B} \). Lipschitz constant \( U \) might also decrease.

**The Contraction**

For each two functions \( z, y \in \Omega \) observe that

7: \( [\tilde{z} - \tilde{y}] \leq w \cdot [f_z - f_y] \)

Again \(^{\circlearrowright} 1 \) shrink \( w \), this time so that

6': \( w \cdot U < 1 \).

Now, finally, \( z \mapsto \tilde{z} \) is a contraction-map on \( \Omega \) and thus has a unique fixed-point.

\[ \diamond \]

**A general initial-condition.** If we instead require that \( y(5) = P \) for some particular pt \( P \in \mathbb{H} \), then just center ball \( B \) at \( P \) and replace (5) by

5': \( \tilde{z} := [t \mapsto P + \int_5^t f_z(\tau) \, d\tau] \).

**Higher-order DEs**

We first give an example of "re-coding" a higher-order DE to a 1st-order DE.

**2nd-order to 1st-order.** Over an interval \( J \subset \mathbb{R} \), suppose we seek a soln \( h: J \rightarrow \mathbb{R} \) to DE

8a: \( h''(t) = \sin(t \cdot h'(t)) + h(t) \).

A fnc \( x: J \rightarrow \mathbb{R}^2 \) can be written in components as

\[
x(t) = \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix}.
\]

If each \( h_j \) is differentiable then so is \( x \), and \( x' \) equals \( \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix} \) \( \overset{\text{note}}{\xrightarrow{\text{note}}} \begin{bmatrix} h_1'(t) \\ h_0'(t) \end{bmatrix} \). Consider DE (*)

8b: \( x'(t) \overset{\text{def}}{=} \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix} \overset{\text{note}}{\xrightarrow{\text{note}}} \begin{bmatrix} \sin(t \cdot h_1(t)) + h_0(t) \\ h_1(t) \end{bmatrix} \).

The 0th-component, \( h_0 \), of a soln \( x \) to (8a) is automatically a soln to (8a). Conversely, a soln \( h \) to (8a) yields a soln \( \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix} \) to (8b), by setting \( h_0 \leftarrow h \) and \( h_1 \leftarrow h' \).

Lastly, defining the kernel \( K: J \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by

8c: \( K(t; \begin{bmatrix} h_1(t) \\ h_0(t) \end{bmatrix}) := \begin{bmatrix} \sin(t \cdot h_1(t)) + h_0(t) \end{bmatrix} \),

makes the vector-valued DE \( x'(t) = K(t; x(t)) \) a restatement of (8b), hence of (8a).
Now consider a third-order DE OTForm

9: \[ h''' = G(t; h, h', h''), \]
with \( h^{(k)}(5) = P_k \), for \( k = 0, 1, 2 \).

Here \( h: J\rightarrow \mathbb{H} \) and \( G: J \times \mathbb{H}^3 \rightarrow \mathbb{H} \) and each \( P_k \) is a point in \( \mathbb{H} \). By letting \( h_k \) denote the \( k\)th derivative \( h^{(k)} \), we can restate the DE part of (9) as a vector equation

\[
\begin{bmatrix}
  h_2' \\
  h_1' \\
  h_0'
\end{bmatrix} = \begin{bmatrix} G(t; h_0, h_1, h_2) \\
  h_2 \\
  h_1
\end{bmatrix}.
\]

**Reduction.** In order to write (9) as a first-order eqn, consider a fnc \( x: J \rightarrow \mathbb{H}^3 \) written as a column vector \( x(t) = \begin{bmatrix} h_2(t) \\
  h_1(t) \\
  h_0(t)
\end{bmatrix} \), where each \( h_k \) maps \( J \rightarrow \mathbb{H} \).

This notation hands us a function

10: \( K(t; \begin{bmatrix} h_2 \\
  h_1 \\
  h_0
\end{bmatrix}) := \begin{bmatrix} G(t; h_0, h_1, h_2) \\
  h_2 \\
  h_1
\end{bmatrix}. \)

This allows us to rewrite (9) as

\[ \forall t \in J: \quad x'(t) = K(t; x(t)), \] with \( x(5) = P \),

where \( P \) denotes the point \( \begin{bmatrix} P_2 \\
  P_1 \\
  P_0
\end{bmatrix} \) in \( \mathbb{H}^3 \).

**Commentary.** Suppose \( K \) is, say, 7-times continuously-differentiable. Then a fixed-pt of \( z \mapsto \hat{z} \), where

\[ \hat{z}(t) := P + \int_5^t K(\tau; z(\tau)) \, d\tau, \]
is (at least) 8-times continuously-diff’able.

When \( K() \) comes from (10) then the level of differentiability of \( K() \) is that of \( G() \).

**When our hypotheses fail.** Consider the following \( \mathbb{H} := \mathbb{R} \) case. For \( C \) a real constant, define

11a: \( x(t) = x_C(t) := \frac{1}{1 + Ct}. \)

Note \( x(t) - 1 = \frac{-Ct}{1 + Ct} \), so

\[ x(t) \cdot [x(t) - 1] = \frac{-Ct}{[1 + Ct]^2}. \]

But this latter equals \( t \cdot x'(t) \). hence \( x_C() \) satisfies initial-value problem

11b: \[ t \cdot x'(t) = x(t) \cdot [x(t) - 1], \] with \( x(0) = 1 \).

For each \( C \in [0, 3] \) our \( x_C() \) is well-defined on interval \( J := \left(-\frac{1}{3}, \infty \right) \). Since \( J \ni 0 \) [our initial-cond is at 0], we thus have an infinite family of solns to IVP (11b) on \( J \); one soln for each \( C \in [0, 3] \).

Since the conclusion to FTODE is false, it must be that some FTODE-hypothesis failed.

Writing (11b) in form (1), our kernel is

11c: \( K(t; x) := \frac{1}{t} \cdot [x^2 - x]. \)

But this is not well-defined at \( t=0 \), which is where our initial-condition takes place. Also, no matter how small a \( t_0 > 0 \) is take, the RhS(11c) is not \( \mathbb{H} \)-wise Lipschitz as \( t \) ranges over interval \( (0, t_0) \).