Algorithms for solving some differential equations [v.7]

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Number Sets. An expression such as $k \in \mathbb{N}$ (read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$") means that k is a natural number; a ***natnum***.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive ***ratnums*** and $\mathbb{Q}_-$ for the negative ratnums.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

An ***"interval of integers"*** $[b\,..\,c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e}\,..\,2\pi] = \{3, 4, 5, 6\} = [3\,..\,6] = (2\,..\,6]$. We allow b and c to be $\pm\infty$; so $(-\infty\,..\,-1]$ is $\mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2i| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'.

Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand Side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', but pt: '*point*', as in "a fixed-pt of a map".

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'. CoV: '*Change-of-Variable*'.

Prefix nt- means '*non-trivial*'. E.g "a *nt*-soln to $f' = 5f$ is $f(t) := \mathrm{e}^{5t}$; a *trivial* soln is $f \equiv 0$."

Phrases. WLOG: '*Without loss of generality*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. Use iff: '*if and only if*'.

IST: '*It Suffices to*' as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Some differentiation formulas. Below, italic boldface parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{f}$ represent *numbers*. Here, differentiation is w.r.t variable t.

1.1: $$t \cdot \mathrm{e}^{t/\boldsymbol{c}} = \Big[\mathrm{e}^{t/\boldsymbol{c}} \cdot [\boldsymbol{c}t - \boldsymbol{c}^2]\Big]'.$$

1.2: $$t^2 \cdot \mathrm{e}^{t/\boldsymbol{c}} = \Big[\mathrm{e}^{t/\boldsymbol{c}} \cdot [\boldsymbol{c}t^2 - 2\boldsymbol{c}^2 t + 2\boldsymbol{c}^3]\Big]'.$$

1.3: $$\frac{\boldsymbol{c}}{\boldsymbol{a} + \boldsymbol{b}t} = \Big[\tfrac{\boldsymbol{c}}{\boldsymbol{b}} \cdot \log(\boldsymbol{a} + \boldsymbol{b}t)\Big]'.$$

Use expressions $E(t) := \mathrm{e}^{\boldsymbol{a}t}$, $S(t) := \sin(\boldsymbol{f} \cdot t)$ and $C(t) := \cos(\boldsymbol{f} \cdot t)$, below. The number $\boldsymbol{f}$ can be thought of as "frequency" and, in some contexts, the $\boldsymbol{a}$ can be thought of as "attenuation". We have

1.4: $$[\boldsymbol{a}^2 + \boldsymbol{f}^2] \cdot \int E \cdot S = E \cdot [\boldsymbol{a}S - \boldsymbol{f}C].$$

1.5: $$[\boldsymbol{a}^2 + \boldsymbol{f}^2] \cdot \int E \cdot C = E \cdot [\boldsymbol{f}S + \boldsymbol{a}C].$$

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Introduction

[Use **NSS 6** for the 6th edition of the *Nagle,Saff,Snider* textbook. Use, e.g, #7 P193.NSS6, to refer to problem #7 on page 193 of **NSS 6**.] [Use **ZW 8** for 8th edition of *Zill* & *Wright*, using e.g, #7 P193.ZW8, to refer to problems.]

For the following algorithms, the *unknown function* is $y = y(t)$. For a DE of form

$$\mathrm{Fnc}(y, y', y'', \ldots) \quad = \quad G(),$$

we will call $G()$ the ***tarGet fnc***.

Use $\mathbf{D}$ for the *differentiation operator*, and $\mathbf{I} := \mathbf{D}^0$ for the *identity operator*. So $\mathbf{D}^3(y)$ means $\mathbf{D}\Big(\mathbf{D}(\mathbf{D}(y))\Big)$, i.e y'''. So $\mathbf{I}(y) = \mathbf{D}^0(y) = y$.

Use DE: '*Differential Equation*', LDE: '*Linear DE*', ODE: '*Ordinary DE*' and PDE: '*Partial DE*'. IVP: '*Initial-Value Problem*'.

Use boldface $\mathbf{1} := [t \mapsto 1]$, for the constant-1 fnc. For the ***identity fnc***, use $Id(t) = t$. Differentiating, $Id' = \mathbf{1}$.

The Easy Scan

Below, $\alpha, \beta, A, B, \mathbf{r}$ range over all numbers; $\mathbb{R}$ or $\mathbb{C}$, as appropriate.

Before we work on solving a DE with U.F $y(t)$, let's glean some properties of $\mathcal{S}$, the soln-set of the DE.

What is the name of: *The indep.var? The U.F? What are the parameters in the DE?* And: *What is the order of the DE?*

Types of functions.

a1: *Is the zero-fnc a soln? Are there constant-solns?*

a2: *Are there non-constant polynomial solns?* [This usually involves examining how the DiffOp affects the degree of a polynomial.]

a3: *Could a nt-exponential,* $A{\cdot}\mathrm{e}^{Bt}$ *with* $B{\neq}0$ *and* $A{\neq}0$*, be a soln to the DE?*

Closure properties of $\mathcal{S}$.

b1: *Is* $\mathcal{S}$ *sealed* [*closed*] *under horizontal translation? I.e, for soln* f *and number* $\mathbf{r}$*, must* $\mathbf{T_r}(f)$ *also be a soln? I.e, is the DE autonomous?*

b2: *Is* $\mathcal{S}$ *sealed* [*closed*] *under scaling?, i.e, for* $f \in \mathcal{S}$*, must each* αf *also be a soln?*

For $f,g \in \mathcal{S}$*, must* $f{+}g \in \mathcal{S}$*?*

[This $\mathcal{S}$ is sealed under scaling *and* under addition IFF the DE can be written in form $\mathrm{LinearOp}(y) = 0$.]

b3: *If not* (b2)*, then is* $\mathcal{S}$ *at least sealed under* ***averaging****? I.e,* $\forall f,g \in \mathcal{S}$ *and all scalars* α,β *with* $\alpha + \beta = 1$*, is average* $\Big[[\alpha f] + \beta g\Big]$ *a soln?*

b4: *Special? Is the DOp linear, affine, equidimensional, a CCLDOp? Is the DE autonomous, separable, EXACT(ifiable), FOLDE, Bernoulli-type?*

Easy-Scan Example. Consider U.F $y{=}y(t)$ satisfying

2: $$\frac{dy}{dt} \;=\; 6t^2 \cdot [y - 4]\,.$$

Checking types (a1,a2,a3)**.** For analysis, define operators [Left and Right]

$$\mathsf{L}(y) \;:=\; \frac{dy}{dt}\,; \qquad [\text{so here, } \mathsf{L} = \mathbf{D}]$$
$$\mathsf{R}(y) \;:=\; 6t^2 \cdot [y - 4]\,.$$

Since $\mathsf{L}(y){\equiv}0$ IFF $y{\equiv}$Constant, *the only constant soln to* (2) *is* $y{\equiv}4$. And for a poly y of degree $N{\geqslant}1$, necessarily $\mathrm{Deg}(\mathsf{R}(y)) = N{+}2$, whereas $\mathrm{Deg}(\mathsf{L}(y)) = N{-}1$. *So no non-constant polynomial solns.*

Lastly, $\mathsf{L}(A{\cdot}\mathrm{e}^{Bt})$ equals [with $A,B \neq 0$] another nt-exponential, $AB{\cdot}\mathrm{e}^{Bt}$. But $\mathsf{R}(A{\cdot}\mathrm{e}^{Bt})$ is not a pure exponential, because of the polynomial factor. *So* (2) *has no nt-exponential solns.*

Checking closure properties. hi
More to come. . . □

Separation of variables [SoV]

Consider a fnc f defined on interval $\mathbb{J} := [-3, 7]$ satisfying IVP

3a: $$f'(x) = \boldsymbol{\beta}(x)/\boldsymbol{\mu}(f(x))\,,$$ with $f(5) = 9$.

Let $\mathbb{K} := f(\mathbb{J})$, the interval which is the f-image of $\mathbb{J}$.

Together with functions $\boldsymbol{\beta}, \boldsymbol{\mu}$, suppose we have three other fncs $\mathsf{B}, \mathsf{M}, \mathsf{M}^{\text{InvF}}$ satisfying:

3b:
Fncs $\boldsymbol{\beta}, \mathsf{B}$ are defined on $\mathbb{J}$, with $\mathsf{B}' = \boldsymbol{\beta}$.

Fncs $\boldsymbol{\mu}, \mathsf{M}$ are defined on $\mathbb{K}$, with $\boldsymbol{\mu} \neq 0$ and $\mathsf{M}' = \boldsymbol{\mu}$.

Fnc M is invertible with M^{InvF} its inverse-fnc.

Re-write the top-line of (3a) as

3c: $$\boldsymbol{\mu}(f(t)) \cdot f'(t) = \boldsymbol{\beta}(t)\,.$$

For each $x \in \mathbb{J}$, then, we have

3d: $$\int_5^x \boldsymbol{\mu}(f(t)) \cdot f'(t)\,dt = \int_5^x \boldsymbol{\beta}(t)\,dt\,.$$

Substitution $y = f(t)$ says LhS(3d) equals

$$\int_{f(5)}^{f(x)} \boldsymbol{\mu}(y) \cdot dy \overset{\text{by FTC}}{=\!=} \mathsf{M}(f(x)) - \mathsf{M}(f(5))\,.$$

And RhS(3d) equals $\mathsf{B}(x) - \mathsf{B}(5)$. Hence

$$\mathsf{M}(f(x)) = \mathsf{B}(x) + [\mathsf{M}(f(5)) - \mathsf{B}(5)]\,.$$

Consequently, initial condition (3a) produces

3e: $$f(x) = \mathsf{M}^{\text{InvF}}\Big(\mathsf{B}(x) + \mathsf{M}(9) - \mathsf{B}(5)\Big)\,.$$

Example of SoV

Consider U.F. $f = f(x)$ satisfying

3a†: $$f'(x) = \mathsf{e}^{-2f(x)} \cdot x \overset{\text{note}}{=\!=} 2x/2\mathsf{e}^{2f(x)}\,,$$ with $f(0) = 9$.

Soln. [Do Easy-Scan first.] Define the following fncs:

3b†:
$$\boldsymbol{\beta}(x) := 2x \quad \text{and} \quad \mathsf{B}(x) := x^2.$$
$$\boldsymbol{\mu}(y) := 2\mathsf{e}^{2y} \quad \text{and} \quad \mathsf{M}(y) := \mathsf{e}^{2y}.$$
$$\text{Hence} \quad \mathsf{M}^{\text{InvF}} = \tfrac{1}{2} \cdot \log.$$

Computing, $\mathsf{B}(x) + \mathsf{M}(9) - \mathsf{B}(0) = x^2 + \mathsf{e}^{18} - 0$. Hence

3e†: $$f(x) = \tfrac{1}{2}\log(x^2 + \mathsf{e}^{18})\,.$$

Check. To verify that (3e†) satisfies (3a†), note

∗: $$f'(x) = \frac{2x + 0}{2 \cdot [x^2 + \mathsf{e}^{18}]} \overset{\text{note}}{=\!=} \frac{x}{x^2 + \mathsf{e}^{18}}\,.$$

And $\mathsf{e}^{2f(x)} = \mathsf{e}^{\log(x^2 + \mathsf{e}^{18})} = x^2 + \mathsf{e}^{18}$. Hence $\mathsf{e}^{-2f(x)} \cdot x$ equals $x/[x^2 + \mathsf{e}^{18}]$, which indeed equals RhS(∗).

Finally, to verify the initial condition, note $f(0)$ equals $\frac{1}{2}\log(\mathsf{e}^{18}) = \frac{1}{2} \cdot 18 = 9$.

CoV to SoV

A function $F(x_1, \ldots, x_N)$ is ***scale-invariant*** [or "***homogeneous*** of degree-0"] if

4.1: $\forall s \neq 0: \; F(sx_1, \ldots, sx_N) \;=\; F(x_1, \ldots, x_N)\,.$

[I.e, the fnc is unchanged by scaling.] More generally, for a $\mathbf{d} \in \mathbb{R}$, say that $F()$ is "***homogeneous*** of degree $\mathbf{d}$" if

4.2: $\forall s \neq 0: \; F(sx_1, \ldots, sx_N) \;=\; s^{\mathbf{d}} \cdot F(x_1, \ldots, x_N)\,.$

4.3: Scale-invariant to SoV. Consider a scale-invariant $F(x, y)$, U.F $y = y(x)$, and DE

4.3a: $$\frac{dy}{dx} \;=\; F(x, y)\,.$$

Define CoV $\boxed{v := \frac{y}{x}}$ and fnc $G(v) := F(1, v)$. Solve

4.3b: $$\frac{1}{G(v) - v} \cdot dv \;=\; \frac{1}{x} \cdot dx$$

using SoV. For each number α, then,

4.3c: $$y_\alpha(x) \;:=\; x \cdot v_\alpha(x)$$

solves (4.3a). [**Note:** You might only obtain *implicit* solutions.] □

Why does this work? Substitution $v := \frac{y}{x}$ yields that
$$F(x, y) \;=\; F(1, \tfrac{y}{x}) \;\overset{\text{note}}{=\!=}\; G(v)\,.$$

Rewrite $v := \frac{y}{x}$ as $y = x \cdot v$. The Product Rule gives
$$G(v) \;=\; \frac{dy}{dx} \;\overset{\text{P.R.}}{=\!=}\; 1 \cdot v \;+\; x \cdot \frac{dv}{dx}\,.$$

This separable DE, rewritten, is (4.3b).

Scale-invariant CoV Example. To find U.F $y = y(x)$, divide by x in DE
$$x \cdot \frac{dy}{dx} \;=\; x + 5y\,, \qquad \textit{obtaining}$$

4.3a†: $$\frac{dy}{dx} \;=\; 1 + 5 \cdot \frac{y}{x}\,. \quad \left[\begin{array}{l}\text{Note RhS is}\\ \text{scale-invariant.}\end{array}\right]$$

So define $G(v) := 1 + 5v$. Then $G(v) - v \;=\; 1 + 4v$. So (4.3b) becomes

4.3b†: $$\frac{1}{1 + 4v} \cdot dv \;=\; \frac{1}{x} \cdot dx\,.$$

Integrating each side, using α as CoI, produces
$$\tfrac{1}{4}\log(|1 + 4v|) \;=\; \alpha + \log(|x|)\,.$$

Letting $\beta := 4\alpha$ gives
$$\log(|1 + 4v|) \;=\; \beta + 4\log(|x|)\,.$$

Exponentiating,
$$|1 + 4v| \;=\; e^\beta \cdot |x|^4\,.$$

With $\gamma := \pm e^\beta$, discard the abs.values, obtaining
$$1 + 4v \;=\; \gamma \cdot x^4\,.$$

Recovering y, we now have that
$$\frac{y}{x} \;\overset{\text{def}}{=\!=}\; v \;=\; \tfrac{1}{4}\gamma x^4 - \tfrac{1}{4}\,.$$

With $\sigma := \frac{1}{4}\gamma$, multiplying both sides by x delivers

4.3c†: $$y_\sigma(x) \;=\; \sigma x^5 - \tfrac{1}{4}x\,.$$

Checking. Does (4.3c†) satisfy $x \cdot \frac{dy}{dx} = x + 5y$? Computing its LhS,

***:** $$x \cdot \frac{dy}{dx} \;=\; x \cdot [5\sigma x^4 - \tfrac{1}{4}] \;=\; 5\sigma x^5 - \tfrac{1}{4}x\,.$$

Again using (4.3c†),
$$x + 5y \;=\; x + [5\sigma x^5 - \tfrac{5}{4}x] \;\overset{\text{note}}{=\!=}\; \text{RhS}(*)\,. \quad \square$$

4.4: Linear-CoV to SoV. A function $H()$, and numbers P,Q, define DE

4.4a: $$\frac{dy}{dx} = H(Px + Qy).$$

WLOG, $Q \neq 0$. CoV $\boxed{z := Px + Qy}$ implies that

$$\frac{dz}{dx} = P\cdot 1 + Q\cdot\frac{dy}{dx} \stackrel{\text{note}}{=} P + Q\cdot H(z).$$

Apply SoV to

4.4b: $$\frac{1}{P + Q\cdot H(z)}\cdot dz = 1\cdot dx.$$

Each number α, then, gives a soln

4.4c: $$y_\alpha(x) := [z_\alpha(x) - P\cdot x]/Q$$

to (4.4a). [These solns might only be implicit solns.] □

Now let's put it all together.

Deg1-Ratio-CoV to SoV. Given six numbers A, B, C, D, S, T, consider DE

4.5a: $$\frac{dy}{dx} = \varphi\Big(\frac{Ax + By + S}{Cx + Dy + T}\Big),$$

with not all of C,D,T zero.

If C=D=0, then $T \neq 0$, so define numbers

$$\widehat{A} := A/T, \quad \widehat{B} := B/T, \quad \widehat{S} := S/T,$$

and function $H(t) := \varphi(t + \widehat{S})$. Then (4.5a), rewritten, is

$$\frac{dy}{dx} = H(\widehat{A}x + \widehat{B}y).$$

Consequently, the (4.4)-method applies here. So in our (4.5a), henceforth: *Not both C and D are zero.*

Looking ahead, define matrix and determinant

$$\mathsf{M} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \Delta := \mathrm{Det}(\mathsf{M}) \stackrel{\text{note}}{=} AD - BC.$$

CASE: $\Delta = 0$ Since AD = BC, there is a number, ρ, such that

$$A = \rho C \quad \text{and} \quad B = \rho D;$$

simply let $\rho := \frac{B}{D}$ or $\rho := \frac{A}{C}$, depending on whether D≠0 or C≠0. For convenience, define number

$$\widetilde{S} := S - \rho T.$$

Notice that

$$\begin{aligned} Ax + By + S &= Ax + By + \rho T + \widetilde{S} \\ &= \rho\cdot[Cx + Dy + T] + \widetilde{S}. \end{aligned}$$

Rewriting the ratio from (4.5a),

$$\frac{Ax + By + S}{Cx + Dy + T} = \rho + \frac{\widetilde{S}}{Cx + Dy}.$$

So define fnc $H(t) := \varphi(\rho + \frac{\widetilde{S}}{t})$. We can thus rewrite (4.5a) as

$$\frac{dy}{dx} = H(Cx + Dy).$$

Thus (4.4) method applies.

CASE: $\Delta \neq 0$ We seek numbers $\mathcal{J}, \mathcal{K}$ so that CoV

4.5b: $$\begin{aligned} v &:= x + \mathcal{J}, \quad \text{and} \\ w &:= y + \mathcal{K} \end{aligned}$$

arranges that

$*$: $$\begin{aligned} Av + Bw &= Ax + By + S, \quad \text{and} \\ Cv + Dw &= Cx + Dy + T. \end{aligned}$$

This eqn-pair can be written as a single matrix eqn

$$\mathsf{M}\cdot\begin{bmatrix} \mathcal{J} \\ \mathcal{K} \end{bmatrix} = \begin{bmatrix} S \\ T \end{bmatrix}.$$

Consequently, its soln is

4.5c: $$\begin{bmatrix} \mathcal{J} \\ \mathcal{K} \end{bmatrix} = \mathsf{M}^{-1}\cdot\begin{bmatrix} S \\ T \end{bmatrix} \stackrel{\text{note}}{=} \frac{1}{\Delta}\cdot\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}\begin{bmatrix} S \\ T \end{bmatrix}.$$

CoV (4.5b) has that $dv = dx$ and $dw = dy$. Thus

$$\frac{dw}{dv} = \frac{dy}{dx} \stackrel{\text{by }(*)}{=} \varphi\Big(\frac{Av + Bw}{Cv + Dw}\Big)$$

Easily, $F(v, w) := \varphi\Big(\frac{Av+Bw}{Cv+Dw}\Big)$ is scale-invariant. Solve it via the (4.3)-method.

Complex numbers

[Complex arithmetic done in class. Treasure problem.]

Remark. The ***discriminant*** of quadratic [i.e, $A\neq 0$] polynomial $q(z) := Az^2 + Bz + C$ is

6.1: $$\text{Discr}(q) \;:=\; B^2 - 4AC\,.$$

The zeros ["roots"] of q are

6.2: $$\text{Roots}(q) \;=\; \frac{1}{2A}\Big[-B \pm \sqrt{\text{Discr}(q)}\,\Big]\,.$$

Hence when A,B,C are *real*, then the zeros of q form a complex-conjugate pair. And q has a *repeated root* IFF $\text{Discr}(q)$ is zero.

A monic $\mathbb{R}$-irreducible quadratic has form

6.3: $$q(z) \;=\; z^2 - Sz + P \;=\; [z - \mathbf{r}]\cdot[z - \overline{\mathbf{r}}]\,,$$

where $\mathbf{r} \in \mathbb{C}\smallsetminus\mathbb{R}$. Note $S = \mathbf{r} + \overline{\mathbf{r}} = 2\text{Re}(\mathbf{r})$ is the *Sum* of the roots. And $P = \mathbf{r}\cdot\overline{\mathbf{r}} = |\mathbf{r}|^2$ is the *Product* of the roots. The discriminant of g, $\text{Discr}(g)$, equals

6.4: $$S^2 - 4P \overset{\text{note}}{=\!=} [\mathbf{r} - \overline{\mathbf{r}}]^2 \;=\; -4\cdot[\text{Im}(\mathbf{r})]^2\,.$$

Completing-the-square yields

6.5: $$q(z) \;=\; [z - \tfrac{S}{2}]^2 + F^2,\ \text{where } F := |\text{Im}(\mathbf{r})|\,,$$

which is easily checked. [Exercise] □

7: Fundamental Theorem of Algebra (Gauss and others). *Consider a monic $\mathbb{C}$-polynomial*

$$p(z) \;:=\; z^N + B_{N-1}z^{N-1} + \ldots + B_1 z + B_0\,.$$

Then p factors completely over $\mathbb{C}$ as

$$p(z) \;=\; [z - \mathbf{r}_1]\cdot[z - \mathbf{r}_2]\cdot\ldots\cdot[z - \mathbf{r}_N]\,,$$

for a list $\mathbf{r}_1,\ldots,\mathbf{r}_N \in \mathbb{C}$, possibly with repetitions. This list is unique up to reordering.

*If p is a **real** polynomial, i.e $\overline{p} = p$, then p factors over $\mathbb{R}$ as a product of monic $\mathbb{R}$-irreducible linear and $\mathbb{R}$-irred. quadratic polynomials. The product is unique up to reordering.* ◊

ℂ-exponential

For $z := x\cdot 1 + y\cdot\boldsymbol{i}$ with $x,y \in \mathbb{R}$, its ***complex conjugate*** $\overline{z}$ is $x\cdot 1 - y\cdot\boldsymbol{i}$. Its real and imaginary parts are

$$\text{Re}(z) := x = \frac{z + \overline{z}}{2}\,,\quad \text{Im}(z) := y = \frac{z - \overline{z}}{2\boldsymbol{i}}\,.$$

By the Pythagorean thm, $|z|^2 = x^2 + y^2 = z\overline{z}$.

For $\mu,\nu \in \mathbb{C}$, note, $\overline{\mu + \nu} = \overline{\mu} + \overline{\nu}$ and $\overline{\mu\cdot\nu} = \overline{\mu}\cdot\overline{\nu}$.

Let's extend the exponential fnc to the complex plane.

8a: Defn. *For $z \in \mathbb{C}$, define*

$$\exp(z) := \mathsf{e}^z := \sum_{n=0}^{\infty}\frac{1}{n!}\cdot z^n = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \ldots;$$

$$\cos(z) := \sum_{j=0}^{\infty}\frac{[-1]^j}{[2j]!}\cdot z^{2j} = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 - \ldots;$$

$$\sin(z) := \sum_{k=0}^{\infty}\frac{[-1]^k}{[2k+1]!}\cdot z^{2k+1} = z - \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 - \ldots.$$

Each series has ∞-RoC. ◊

Since we have absolute convergence of each series at each z, we can re-order terms without changing convergence.

8b: Lemma. *Fix $\alpha,\beta \in \mathbb{C}$. Then*

$$\mathsf{e}^\alpha\cdot\mathsf{e}^\beta \;=\; \mathsf{e}^{\alpha+\beta}\,.$$ ◊

Proof. For natnum N, recall the Binomial thm which says that

∗: $$\sum_{j+k=N}\binom{N}{j,k}\cdot\alpha^j\beta^k \;=\; [\alpha+\beta]^N\,,$$

where the sum is over all ordered-pairs (j,k) of natnums. By its defn [and abs.convergence], $\mathsf{e}^\alpha\mathsf{e}^\beta$ equals

$$\Big[\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\alpha^j\Big]\cdot\Big[\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\beta^k\Big] \;=\; \sum_{N=0}^{\infty}\Big[\sum_{j+k=N}\frac{1}{j!}\frac{1}{k!}\cdot\alpha^j\beta^k\Big]\,.$$

But $\frac{1}{j!\cdot k!}$ equals $\frac{1}{N!}\cdot\frac{N!}{j!\cdot k!}$. Hence $\mathsf{e}^\alpha\mathsf{e}^\beta$ equals

$$\sum_{N=0}^{\infty}\frac{1}{N!}\Big[\sum_{j+k=N}\binom{N}{j,k}\cdot\alpha^j\beta^k\Big] \overset{\text{by }(*)}{=\!=} \sum_{N=0}^{\infty}\frac{1}{N!}[\alpha+\beta]^N\,,$$

which is the defn of $\mathsf{e}^{\alpha+\beta}$. ♦

8c: Lemma. *For θ,x,y,z complex numbers:*

8.1: $e^{i\theta} = [\cos(\theta) + i\sin(\theta)] =: \text{cis}(\theta)$. *Hence*

8.2: $\dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta)$, $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta)$. *Also,*

8.3: $e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot [\cos(y) + i\sin(y)]$, *so*

8.4: $e^{x-iy} = e^x \cdot [\cos(y) - i\sin(y)]$,

since $\cos(-y) = \cos(y)$ *and* $\sin(-y) = -\sin(y)$. *When* θ *is real, then,*

8.5: $\text{Re}(e^{i\theta}) = \cos(\theta)$ *and* $\text{Im}(e^{i\theta}) = \sin(\theta)$.

Since the coefficients in their power-series expansions are all real, our exp(),cos(),sin() *fncs each commute with complex-conjugation, i.e*

8.6: $\overline{\exp(z)}=\exp(\overline{z})$, $\overline{\cos(z)}=\cos(\overline{z})$, $\overline{\sin(z)}=\sin(\overline{z})$;

Finally, the familiar translation-identities

8.7: $\cos(z - \frac{\pi}{2}) = \sin(z)$, $\sin(z + \frac{\pi}{2}) = \cos(z)$

extend to the complex plane. ♢

Proof. Exercise, using (8b). ♦

Same-frequency cosines/sines. Consider a sum of same-frequency cosines

$$h(t) := \sum_{j=1}^{N} A_j \cdot \cos(\nu_j + Kt),$$

where $A_j \in \mathbb{R}$ is *amplitude*, $\nu_j \in \mathbb{R}$ is *phase-shift* and $K \in \mathbb{R}$ determines the *frequency*. [Courtesy (8.7), we could include sine fncs in the sum.] We seek a phase-shift θ and amplitude $R \geqslant 0$ so that

$$h(t) = R \cdot \cos(\theta + Kt).$$

From (8.5), we have that $h(t)$ equals

$$\sum_{j=1}^{N} A_j \cdot \text{Re}(e^{i[\nu_j + Kt]}) \overset{\text{note}}{=\!=} \text{Re}\Big(\sum_{j=1}^{N} A_j \cdot e^{i[\nu_j + Kt]}\Big)$$
$$= \text{Re}\Big(\Big[\sum_{j=1}^{N} A_j \cdot e^{i\nu_j}\Big] \cdot e^{iKt}\Big).$$

Thus we are led to define $S \in \mathbb{C}$ and $X,Y \in \mathbb{R}$ by

$$S := \Big[\sum_{j=1}^{N} A_j \cdot e^{i\nu_j}\Big] =: X + iY.$$

Since each A_j and ν_j is real,

$$X = \sum_{j=1}^{N} A_j \cdot \cos(\nu_j) \quad \text{and} \quad Y = \sum_{j=1}^{N} A_j \cdot \sin(\nu_j).$$

8h: Same-freq Lemma. [*With notation from above.*] *Set* $R := |S| \overset{\text{note}}{=\!=} \sqrt{X^2 + Y^2}$.

If $S = 0$, *then* $h()$ *is the zero-fnc; so can set* $\theta := 0$. *Otherwise, if* $X = 0$, *then set* θ *to* $\frac{\pi}{2}$ *or* $-\frac{\pi}{2}$ *as* Y *is positive or negative.*

Otherwise: If $X > 0$ *then set* $\theta := \arctan(Y/X)$; *and if* $X < 0$ *then set* $\theta := \pi + \arctan(Y/X)$. ♢

The CCLDE algorithm [“Constant-Coeff LDE”]

Initially, we handle only when the target is the zero-fnc.

Step S0. Consider numbers $C_0, \ldots, C_N$ and U.F $y{=}y(t)$ satisfying

$$*: \quad C_N y^{(N)} + C_{N-1} y^{(N-1)} + \ldots + C_1 y' + C_0 y = 0,$$

with $C_N \neq 0$. Define the ***auxiliary polynomial***

$$q(z) := C_N z^N + C_{N-1} z^{N-1} + \ldots + C_1 z^1 + C_0 z^0.$$

We can now re-write $(*)$ as

9a: $$[q(\mathbf{D})](y) = 0.$$

Step S1. Let $\mathcal{Z}$ denote the <u>set</u> of **distinct** roots [i.e, zeros] of $q()$. For each root $\mathbf{r} \in \mathcal{Z}$, let $M_\mathbf{r} \in \mathbb{Z}_+$ denote the *multiplicity* of $\mathbf{r}$ in $q()$. Thus $\sum_{\mathbf{r} \in \mathcal{Z}} M_\mathbf{r}$ equals N, i.e, $\text{Deg}(q)$.

The above says that our polynomial factors as

9b: $$q(z) = C_N \cdot \prod_{\mathbf{r} \in \mathcal{Z}} [z - \mathbf{r}]^{M_\mathbf{r}}.$$

Step S2. The general solution to (9a) is

9c: $$y(t) = \sum_{\mathbf{r} \in \mathcal{Z}} \sum_{j \in [0..M_\mathbf{r})} \big[\lambda_{\mathbf{r},j} \cdot t^j \cdot e^{\mathbf{r} \cdot t}\big],$$

freely choosing the N many numbers, $\{\lambda_{\mathbf{r},j}\}_{\mathbf{r},j}$.

Step S3. Now suppose we were given initial conditions, e.g, given specified numbers for values $y(0), y'(0), y''(0), \ldots, y^{(N-1)}(0)$. Or perhaps we are given the value of y'' at N different points.

Differentiate (9c) appropriately and plug in the given points to obtain N equations [“high school” linear equations] which you solve for the values of the N many unknowns $\{\lambda_{\mathbf{r},j}\}_{\mathbf{r},j}$.

CCLDE Example. U.F. $y = y(t)$ satisfies DE

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} + 10y'' - 36y' + 24y = 0\,.$$

Define $p(z) := z^5 - 6z^4 + 9z^3 + 10z^2 - 36z^1 + 24z^0$; the aux-poly of the above DE. We can re-write the DE as

9a†: $$[p(\mathbf{D})](y) = 0\,.$$

Step S1. Factor polynomial p as

9b†: $$\begin{aligned} p(z) &= [z^2-3]\cdot[z-2]^3 \\ &= [z-U]\cdot[z-V]\cdot[z-2]^3\,, \end{aligned}$$

where $U := \sqrt{3}$ and $V := \text{-}U$. I.e, $\mathcal{Z} = \{U, V, 2\}$ and $\mathrm{M}_U = 1$, $\mathrm{M}_V = 1$ and $\mathrm{M}_2 = 3$.

Step S2. For five arbitrary [possibly complex] numbers $\alpha, \beta, \lambda_0, \lambda_1, \lambda_2$, the function

9c†: $$y(t) := \alpha e^{Ut} + \beta e^{-Ut} + \Big[\sum_{j=0}^{2} \lambda_j \cdot t^j e^{2t}\Big]$$

is the general soln to (9a†).

Step S3. Consider IVP (9a†) with

$$y(0) = 2;\quad y'(0) = 0;\quad y''(0) = 4;\quad y^{(3)}(0) = \text{-}12;\quad y^{(4)}(0) = \text{-}30\,.$$

Solving for the coefficients in (9c†) gives

9d: $$\alpha = \beta = 1;\quad \lambda_0 = \lambda_1 = 0;\quad \lambda_2 = \text{-}1\,.$$

Consequently, the soln to this IVP is

9e: $$y(t) = [e^{\sqrt{3}\cdot t}] + [e^{-\sqrt{3}\cdot t}] - [t^2 e^{2t}]\,.$$

***Complex-root* Example.** Your experiments with fluid-flow[♡1] produce U.F. $f = f(t)$ such that

9f: $$f''' - [2+\boldsymbol{i}]f'' + [1+4\boldsymbol{i}]f' + [2-\boldsymbol{i}]f = 0\,.$$

[♡1] Wine...

Defining the auxiliary polynomial, then factoring, gives

9g: $$\begin{aligned} q(z) &:= z^3 - [2+\boldsymbol{i}]z^2 + [1+4\boldsymbol{i}]z + [2-\boldsymbol{i}] \\ &= \big[z - \boldsymbol{i}\big]^2 \cdot \big[z - [2-\boldsymbol{i}]\big]\,. \end{aligned}$$

The solns, $f(t)$, to (9f) are the linear-combinations of

$$e^{it},\ te^{it},\ e^{[2-i]t}\,.$$

If desired, write $e^{[2-\boldsymbol{i}]t}$ as $e^{2t}\cdot[\cos(t) - \boldsymbol{i}\sin(t)]$, since cos() is an even-fnc and sin() an odd-fnc.

Polynomial target

[In text **NSS6** in 4.4, P.175, "Undetermined coeffs.".] For our differential operator $q(\mathbf{D})$, write its auxiliary-polynomial as

$$q(z) = \sum_{n=L}^{N} \mathrm{C}_n z^n\,,$$

for integers $L \leqslant N$ with $\mathrm{C}_L \neq 0$. Generalizing the target to a *polynomial* $G(t) = \sum_{j=0}^{K} B_j t^j$, we seek a soln f to $[q(\mathbf{D})](f) = G$.

Since our CCLDOp $q(\mathbf{D})$ carries polys to polys, solving this DE is easy. Write a candidate soln as

$$f(t) = \sum_{j=0}^{K} u_j \cdot t^{j+L}$$

for undetermined numbers $\vec{\mathbf{u}} = (u_0, u_1, \ldots, u_K)$. Equating coefficients in $[q(\mathbf{D})](f) = G$ gives $K{+}1$ "high school" [e.g, linear] eqns in the $K{+}1$ unknowns $\vec{\mathbf{u}}$. This system will have (exercise!) a unique soln.

***Polynomial-target* Example.** Consider DE

9h: $$[f'' - 3f' + 2f](t) = 4t\,.$$ So we let $$q(z) := z^2 - 3z + 2 = [z-2]\cdot[z-1]\,.$$

Hence $L = 0$ and $N = 2$. Write $f(t) = vt + u$, for undetermined *numbers* v,u. Thus $[q(\mathbf{D})](f)$ equals

$$[\text{-}3\mathbf{D} + 2\mathbf{I}](f) \overset{\text{note}}{=\!=} \Big[t \mapsto [2vt + [2u - 3v]]\Big]\,.$$

[*Why? Did you detect that* $\mathbf{D}^2(f) = 0$ *?*]

Setting $2vt + [2u - 3v]$ equal to our target, $4t + 0$, gives equations $2v = 4$ and $2u - 3v = 0$. High-schooling gives $v = 2$ and $u = 3$. I.e, $f(t) := 2t + 3$ is sent by $q(\mathbf{D})$ to $4t$.

IVP. Mystery function $h = h(t)$ satisfies

†: $[q(\mathbf{D})](h) = 4Id$, together with

‡: $h(0) = 0$ and $h'(0) = 1$.

From (9h), we know that e^{2t} and e^t are each mapped to 0 by $q(\mathbf{D})$. Consequently, the general soln, h, to (†) has form $h(t) = \alpha e^{2t} + \beta e^t + 2t + 3$, for constants α,β. Eqns (‡) yield $\alpha = 2$ and $\beta = -5$. Thus function

$$h(t) = 2e^{2t} - 5e^t + 2t + 3$$

is the unique soln to "Mystery" IVP (†,‡).

Linear maps

A *vector space* is like $\mathbb{R}\times\mathbb{R}$ [or $\mathbb{C}\times\mathbb{C}$] with component-wise addition: For vectors $\mathbf{v}_j := (x_j, y_j)$, their *sum* $\mathbf{v}_1 + \mathbf{v}_2$ is $(x_1 + x_2,\, y_1 + y_2)$. More generally, a ***vector space***[♡2] is a set $\mathbf{V}$ (or it might be called $\mathbf{W}$ or $\mathbf{E}$ or $\mathbf{H}$ or...) together with an addition which is *commutative* and *associative*. Also, we can multiply a vector by a ***scalar*** which is either a real number or, more generally, a complex number.

So a VS is a tuple $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{R})$ when the scalars are reals, or $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{C})$ when we allow complex scalars.

10a: *Defn.* Now consider a map $\mathsf{L}\colon\mathbf{V}\to\mathbf{W}$ between vector spaces $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{C})$ and $(\mathbf{W}, +, \mathbf{0}, \cdot, \mathbb{C})$. This map L is ***linear*** IFF:

$\forall\, \mathbf{v}_1,\mathbf{v}_2,\mathbf{v} \in \mathbf{V}$ *and for all scalars* α, *our* L *satisfies*

£1: $\mathsf{L}(\mathbf{v}_1 + \mathbf{v}_2) = \mathsf{L}(\mathbf{v}_1) + \mathsf{L}(\mathbf{v}_2)$ *and*

£2: $\mathsf{L}(\alpha\cdot\mathbf{v}) = \alpha\cdot\mathsf{L}(\mathbf{v})$.

Equivalently: *For all vectors* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *and for all scalars* $\alpha_1, \dots, \alpha_N$:

$$\mathsf{L}\Big(\sum_{j=1}^N \alpha_j \mathbf{v}_j\Big) = \sum_{j=1}^N \alpha_j \mathsf{L}(\mathbf{v}_j). \quad \Box$$

[♡2] I'll abbreviate 'vector space' as VS, and 'vector spaces' as VSes.

10b: *Defn.* The *set* of all linear-combinations [***lin-combs***] of a collection $\mathcal{S} := \{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ of vectors is called **"the *span* of $\mathcal{S}$"**. I.e, Span($\mathcal{S}$) equals

$$\operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_N) := \Big\{ \sum_{j=1}^N \alpha_j \mathbf{v}_j \;\Big|\; \text{Where } \alpha_1, \dots, \alpha_N \text{ are scalars.} \Big\}.$$

Our $\mathcal{S}$ is a ***linearly-independent set***[an ***L.I-set***] if the **only** list $\beta_1, \dots, \beta_N$ of scalars satisfying

$$\Big[\sum_{j=1}^N \beta_j \mathbf{v}_j\Big] = \mathbf{0} \quad \text{[the zero vector]}$$

is $\beta_1{=}0$, $\beta_2{=}0 \dots, \beta_N{=}0$. $\Box$

Conjugate-root example

A polynomial with all real coeffs [a "real-poly" or "$\mathbb{R}$-poly"] factors into a product of $\mathbb{R}$-irreducible linear and quadratic real-polys.

The ***discriminant*** of quadratic [i.e, $A{\neq}0$] polynomial $q(z) := Az^2 + Bz + C$ is

11.1: $$\operatorname{Discr}(q) := B^2 - 4AC,$$

and its zeros ["roots"] are

11.2: $$\frac{1}{2A}\Big[-B \pm \sqrt{\operatorname{Discr}(q)}\,\Big].$$

When A,B,C are real, then, the non-real zeros of q come in complex-conjugates pairs.

12: Same-span Lemma. *Here,* Span *means* $\mathbb{C}$-Span. *Fix* J,K *complex numbers* [*usually real, in practice*]. *Then*

$$\operatorname{Span}\Big(e^{[J+iK]t}, e^{[J-iK]t}\Big) = \operatorname{Span}\Big(e^{Jt}\cdot\cos(Kt),\; e^{Jt}\cdot\sin(Kt)\Big).$$

Indeed, for numbers α, β, μ, ν, *we have*

12a: $$\alpha\cdot e^{[J+iK]t} + \beta\cdot e^{[J-iK]t} \;\text{ equals }\; e^{Jt}\cdot[\mu\cdot\cos(Kt) + \nu\cdot\sin(Kt)],$$

where the scalars are related by

12b: $\mu = \alpha + \beta$ and $\nu = \boldsymbol{i}\cdot[\alpha - \beta]$;

12c: $\alpha = \dfrac{\mu - \boldsymbol{i}\cdot\nu}{2}$ and $\beta = \dfrac{\mu + \boldsymbol{i}\cdot\nu}{2}$. ◊

Proof. Lemma (8c) and routine algebra. ♦

Eric's requested IVP.

Let's create a CCLDE whose diff-operator polynomial $q()$ has a specified complex-conjugate roots; say $U := 3 + 2\boldsymbol{i}$ and $\overline{U} = 3 - 2\boldsymbol{i}$. Define

$$*: \quad q(z) := [z - U] \cdot [z - \overline{U}] \stackrel{\text{note}}{=\!=} z^2 - 6z + 13\,.$$

Let's go through the steps to solve DE

$$**: \qquad f'' - 6f' + 13f = 0$$

with initial conditions $f(0) = 1$ and $f'(0) = 13$.

By CCLDE, the soln-set to $(**)$ is $\mathbb{C}\text{-Spn}(\mathrm{e}^{Ut}, \mathrm{e}^{\overline{U}t})$. To re-write using cos() and sin(), define expressions

$$E := \mathrm{e}^{3t}, \quad C := \cos(2t), \quad S := \sin(2t)\,.$$

Courtesy (12a), there are numbers μ, ν so that

$$f(t) := E \cdot [\mu C + \nu S]$$

satisfies the initial conditions. This gives

$$1 = f(0) = 1 \cdot [\mu{\cdot}1 + \nu{\cdot}0] \stackrel{\text{note}}{=\!=} \mu\,.$$

Diff'ing gives $f'(t) = 3E{\cdot}[\mu C + \nu S] + E{\cdot}[-2\mu S + 2\nu C]$. So $13 = f'(0)$ which equals

$$\begin{aligned} 3{\cdot}[\mu + 0] + 1{\cdot}[2\mu{\cdot}0 + 2\nu{\cdot}1] &= 3\mu + 2\nu \\ &= 3 + 2\nu\,. \end{aligned}$$

Hence $\nu = 5$. Thus the soln to the IVP is

$$*1: \quad f(t) = \mathrm{e}^{3t} \cdot [\cos(2t) + 5\sin(2t)]$$

$$*2: \quad \stackrel{\text{by (12c)}}{=\!=} \tfrac{1-5\boldsymbol{i}}{2}{\cdot}\mathrm{e}^{[3+2\boldsymbol{i}]t} + \tfrac{1+5\boldsymbol{i}}{2}{\cdot}\mathrm{e}^{[3-2\boldsymbol{i}]t}\,.$$ □

The FOLDE algorithm ["First-Order LDE"]

Step F0. Write the DE in the form

13a: $$\frac{\mathrm{d}y}{\mathrm{d}x} + [C(x)\cdot y] = G(x).$$

Pick [i.e, compute] an antiderivative $B()$ of $C()$, i.e

13b: $$B(x) := \int^x C().$$

For later use, store this *multiplier function*[♡3] M:

13c: $$M(x) := \mathrm{e}^{B(x)} = [\text{simplified}].$$

Observe that $M' = M\cdot C$. Hence

$$\begin{aligned}[M\cdot y]' &= [M\cdot C\cdot y] + [M\cdot y'] \\ ^{**}: \quad &= M\cdot\Big[[C\cdot y] + y'\Big] \\ &\stackrel{\text{by (13a)}}{=\!=} M\cdot G.\end{aligned}$$

Step F1. Define product $P(x) := M(x)\cdot G(x)$. Compute an antiderivative,

13d: $$Q(x) := \int^x P().$$

Step F2. Now, for $\alpha :=$ [an arbitrary constant], the following definition of y will satisfy equation (13a):

13e: $$y(x) = y_\alpha(x) := \frac{\alpha}{M(x)} + \frac{Q(x)}{M(x)}.$$

Step F3. Use (13e) to compute y'. Plug in to (13a) to see if your formula for y satisfies it. [It is at this point that I sometimes find that I have made a computational error.]

Step F4. If the problem asks that y satisfy –in addition to (13a)– an initial condition of the form $y(x_0) = y_0$, then substitute $x = x_0$ and $y = y_0$ into (13e) and solve for α. You will get that

13f: $$\alpha = [y_0\cdot M(x_0)] - Q(x_0).$$

That's all there is to it! It's all copasetic.

FOLDE Example. Given DE

13a†: $$\begin{aligned} x^3y' + x^2y &= 7x^8 - x^5, \quad \text{re-write it as} \\ y' + \tfrac{1}{x}\cdot y &= 7x^5 - x^2, \end{aligned}$$

to fit form (13a). So $G(x) = [7x^5 - x^2]$.

Applying step (**F0**), we have $C(x) = 1/x$, and can define $B := \log$. Hence

13c†: $$M(x) \stackrel{\text{def}}{=\!=} \mathrm{e}^{\log(x)} \stackrel{\text{note}}{=\!=} x.$$

Step F1. Define $P(x) := x\cdot[7x^5 - x^2] = 7x^6 - x^3$. Antidifferentiate to get

13d†: $$Q(x) := x^7 - \tfrac{1}{4}x^4.$$

Step F2. For each constant, α, the function

13e†: $$y_\alpha(x) := \frac{\alpha}{x} + [x^6 - \tfrac{1}{4}x^3]$$

is supposed to satisfy (13a†). **Check that it does!**

Step F4. Imagine we are given initial condition

13g: $$y(2) = 66.5.$$

For the corresponding α, compute

$$y_\alpha(2) = \frac{\alpha}{2} + 64 - 2 = \frac{\alpha}{2} + 62.$$

Hence $\alpha/2 = 66.5 - 62 = 4.5$, so $\boxed{\alpha = 9}$. Alternatively, formula (13f) gives

$$\begin{aligned}\alpha &= [66.5\cdot M(2)] - Q(2) \\ &= [66.5\cdot 2] - [128 - 4] \\ &= 133 - 124 \stackrel{\text{note}}{=\!=} 9.\end{aligned}$$

THE UPSHOT: The <u>unique</u> soln to IVP (13a†, 13g) is

$$y(x) = [9/x] + x^6 - \tfrac{1}{4}x^3.$$

[♡3] Using functional notation, we could write $M := \exp\circ B$.

CoV to FOLDE [“Change-of-Variable”]

Consider a positive-valued fnc $y = y(x)$ satisfying DE

14a: $$y' + [\widetilde{G}(x)\cdot y] = \widetilde{C}(x)\cdot y{\cdot}\log(y)\,.$$

Happily, we can convert this to a FOLDE, by setting $z := \log(y)$. Divide by y and re-order as

$$[y'/y] - \widetilde{C}(x)\log(y) = \text{-}\widetilde{G}(x)\,.$$

Our substitution allows us to re-write this as

14b: $$z' - \widetilde{C}(x){\cdot}z = \text{-}\widetilde{G}(x)\,,$$

which has form (13a).

Example of CoV-to-FOLDE. We seek a positive-valued fnc $y = y(t)$ satisfying

14a†: $$ty' = 2t^2y + [y{\cdot}\log(y)]\,.$$

Dividing by $t{\cdot}y$ and re-ordering gives

$$\frac{y'}{y} - [\tfrac{1}{t}\cdot\log(y)] = 2t\,.$$

Substitution $z := \log(y)$ gives

14b†: $$z' - [\tfrac{1}{t}\cdot z] = 2t\,.$$

Matching to (13a), we define

$$G(t) := 2t,\ C(t) := \tfrac{-1}{t},\ B := -\log\,,$$
$$\text{and}\ \ M(t) := \mathsf{e}^{B(t)} = \tfrac{1}{t}\,.$$

Step (**F1**) gives $P(t) := \frac{1}{t}{\cdot}2t = 2$, hence $Q(t) := 2t$. For an arbitrary constant α, then,

13e†: $$z_\alpha(t) := \alpha{\cdot}t + 2t{\cdot}t\,.$$

“Un-substituting” [returning to y], then, yields

14c: $$y_\alpha(t) = \mathsf{e}^{\alpha t + 2t^2}\,.$$

Have you <u>checked</u> that this really satisfies (14a†)*?*

Bernoulli eqn using FOLDE

Given coefficient and target fncs $\widetilde{C}$ and $\widetilde{G}$, we seek solutions $y() > 0$ to

15a: $$y' + \widetilde{C}{\cdot}y = \widetilde{G}\cdot y^{\text{-}[N-1]}\,,$$

where $N \in \mathbb{R}$ with $N \neq 0$.

To convert such to a LDE, multiply both sides by $N{\cdot}y^{[N-1]}$ to get

$$Ny^{[N-1]}{\cdot}y' + N\widetilde{C}{\cdot}y^N = N{\cdot}\widetilde{G}\,.$$

With CoV $z = y^N$, this becomes

15b: $$z' + N{\cdot}\widetilde{C}(x){\cdot}z = N{\cdot}\widetilde{G}(x)\,.$$

Apply the FOLDE algorithm to obtain a general soln z_α. Finally, take the (positive) N^{th}-root to get

15c: $$y_\alpha := [z_\alpha]^{1/N}\,.$$

Bernoulli eqn Example. Consider

15a‡: $$y' + 2y = x\cdot y^{\text{-}2} \overset{\text{note}}{=\!=} \frac{x}{y^{3-1}}\,.$$

So $N = 3$ and $\widetilde{C}(x) = 2$ and $\widetilde{G}(x) = x$. Change-of-variable $z := y^3$ gives [via DE $3y^2y' + 6y^3 = 3x$]

15b‡: $$z' + 6z = 3x\,.$$

So $B(x) := 6x$ and $M(x) = \mathsf{e}^{6x}$. Thus product

$$P(x) := M(x){\cdot}3x \overset{\text{note}}{=\!=} 3x{\cdot}\mathsf{e}^{6x}\,.$$

Courtesy (1.1), one antiderivative is

$$Q(x) := \mathsf{e}^{6x}\cdot\Big[\frac{x}{2} - \frac{1}{2\cdot 6}\Big]\,.$$

For α an arbitrary number, then,

13e‡: $$z_\alpha(x) = \alpha\mathsf{e}^{-6x} + [\tfrac{x}{2} - \tfrac{1}{12}]\,.$$ Hence

15c‡: $$y_\alpha(x) = \Big[\alpha\mathsf{e}^{-6x} + \tfrac{x}{2} - \tfrac{1}{12}\Big]^{1/3}\,.$$

The EXACT algorithm [§2.4–NSS6. §2.4–ZW8.]

Write your differential equation in the form

16a: $$\left[\mathcal{N}(x,y)\cdot\tfrac{\mathrm{d}y}{\mathrm{d}x}\right] + \mathcal{M}(x,y) \quad = \quad 0\,.$$

Our goal is to describe y as an ***implicit solution***: We seek a non-trivial function $\mathbf{F}(\cdot,\cdot)$ so that each solution y to (16a) satisfies

16b: $$\mathbf{F}(x, y(x)) \;=\; \alpha\,,$$

for some constant α. [If we are interested in complex-valued solutions, then we will allow α to be a complex number.]

Step E1. Does

16c: $$\frac{\partial\mathcal{N}}{\partial x} \;=\; \frac{\partial\mathcal{M}}{\partial y}\ ?$$

If yes, then (16a) is **"an *exact DE*"**; this means [courtesy of our theorem] that there exists a differentiable fnc $\mathbf{F}(x,y)$ such that

16c′: $$\frac{\partial\mathbf{F}}{\partial y} = \mathcal{N} \qquad\text{and}\qquad \frac{\partial\mathbf{F}}{\partial x} = \mathcal{M}\,.$$

In this case, proceed to step (**E2**). Conversely, if (16a) is not exact, go to (**E1.1**) and (**E1.2**).

Step E2. Compute $\mathbf{F}()$ as follows. Compute two antiderivatives, and their difference:

$$\mathcal{B}(x,y) \;:=\; \int^y \mathcal{N}(x,\widetilde{y})\,\mathrm{d}\widetilde{y}\ ;$$
$$\mathcal{A}(x,y) \;:=\; \int^x \mathcal{M}(\widetilde{x},y)\,\mathrm{d}\widetilde{x}\ ;$$
$$\mathrm{Diff}(x,y) \;:=\; \mathcal{B}(x,y) - \mathcal{A}(x,y)\,.$$

Since (16c′) holds, this difference $\mathrm{Diff}(x,y)$ can be written as the difference between a pure function of y and a pure function of x. We do that next.

Step E3. Find functions $g(y)$ and $h(x)$ so that [this can usually be done by inspection]

16d: $$\mathrm{Diff}(x,y) \;=\; g(y) - h(x)\,.$$

[The pair of functions g, h is *almost* unique —adding a constant to g and the same constant to h, gives another a soln-pair.] One can compute a function $\mathbf{F}()$ which satisfies (16c′), by either

16e: $$\mathbf{F}(x,y) \;:=\; \mathcal{A}(x,y) + g(y) \qquad \textit{or}$$
$$\mathbf{F}(x,y) \;:=\; \mathcal{B}(x,y) + h(x)\,.$$

Step E4. Now use (16b) to discern what you need to know about $y(x)$, such as asymptotic behavior as $x\to\pm\infty$. You might do this by solving (16b) explicitly for $y(x)$, or you might use qualitative methods.

Step E1.1. [§2.5–NSS6. §2.4–ZW8.] When (16a) is *not* exact, check to see if we can create an exact-ifying fnc $W(x)$, as follows. Compute

16f: $$C(x,y) \;:=\; \frac{\mathcal{N}_x(x,y) - \mathcal{M}_y(x,y)}{\mathcal{N}(x,y)}\,.$$

Simplify $C(x,y)$ to see if it is a fnc of x only. If "no", then (16a) cannot be made exact by multiplying by a pure fnc of x. Try (**E1.2**), later in these notes.

If "yes", then write $C(x) := C(x,y)$, and note $W()$ satisfies DE

$*$: $$W'(x) + C(x)W(x) \quad = \quad 0\,.$$

Applying FOLDE, define $B() := \int C()$. Then

16g: $$W(x) \;:=\; 1/\mathrm{e}^{B(x)}$$

satisfies $(*)$.

Finally, define two new functions

16h: $$\widehat{\mathcal{N}}(x,y) \;:=\; W(x)\cdot\mathcal{N}(x,y) \qquad\text{and}$$
$$\widehat{\mathcal{M}}(x,y) \;:=\; W(x)\cdot\mathcal{M}(x,y)\,.$$

Automatically, differential eqn

16a.1: $$\left[\widehat{\mathcal{N}}(x,y)\cdot\tfrac{\mathrm{d}y}{\mathrm{d}x}\right] + \widehat{\mathcal{M}}(x,y) \quad = \quad 0\,.$$

is exact. Apply steps (**E2**,**E3**,**E4**) to (16a.1).

EXACT Example of (E1.1)

Consider DE

$$\text{16a†:}\quad \underbrace{[x+1]\cdot 2y}_{\mathcal{N}(x,y)}\cdot y' \;+\; \underbrace{3\cdot[5+y^2]}_{\mathcal{M}(x,y)} \;=\; 0\,.$$

Is this Exact? Applying (**E1**), note

$$\text{16c†:}\quad \mathcal{N}_x - \mathcal{M}_y \;=\; 2y - 6y \overset{\text{note}}{=\!=} \text{-}4y$$

is *not* the zero-fnc, so (16a†) is not an exact DE. To attempt an exact-ifying factor, (**E1.5**), we compute

$$\text{16f†:}\quad C(x,y) \;:=\; \frac{\text{-}4y}{[x+1]\cdot 2y} \;=\; \text{-}2/[x+1]\,.$$

This is a pure fnc of x, so we anti-diff w.r.t x and get $B(x) := \text{-}2\cdot\log(x+1)$. Our exact-ifying factor is thus

$$W(x) \overset{\text{def}}{=\!=} \mathsf{e}^{\text{-}B(x)} \overset{\text{note}}{=\!=} [x+1]^2\,.$$

Good! We now have Exact DE (16*a*.1), where

$$\text{16h†:}\quad \begin{aligned} \widehat{\mathcal{N}}(x,y) &= [x+1]^3\cdot 2y \qquad \text{and} \\ \widehat{\mathcal{M}}(x,y) &:= 3\cdot[x+1]^2\cdot[5+y^2]\,. \end{aligned}$$

Applying* (E2)*, then* (E3)*. Anti-differentiating w.r.t y, respectively, x gives

$$\begin{aligned} \mathcal{B}(x,y) &:= \int^y \widehat{\mathcal{N}} \overset{\text{note}}{=\!=} [x+1]^3\cdot y^2\,; \\ \mathcal{A}(x,y) &:= \int^x \widehat{\mathcal{M}} \overset{\text{note}}{=\!=} [x+1]^3\cdot[5+y^2]\,.\text{ Thus} \\ \mathcal{B} - \mathcal{A} &\overset{\text{note}}{=\!=} \text{-}5\cdot[x+1]^3 \;=\; g(y) - h(x)\,,\text{ where} \end{aligned}$$

we can define $g(y) := 0$ and $h(x) := 5\cdot[x+1]^3$. Finally, (16e) tells us that $\mathbf{F} = \mathcal{A} + g \overset{\text{note}}{=\!=} \mathcal{A}$.

Checking. Consider a fnc $y = y(x)$ satisfying

$$\text{**:}\quad \text{Const} \;=\; [x+1]^3\cdot[5+y(x)^2]\,.$$

Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ hands us

$$0 \;=\; 3[x+1]^2\cdot[5+y(x)^2] \;+\; [x+1]^3\cdot 2y(x)\cdot y'(x)\,.$$

Dividing by $[x+1]^2$ yields (16a†). *Nice. . .*

Step E1.2. When step (**E1.1**) fails, check for an exact-ifying fnc $V(y)$, as follows. Compute

$$\text{16i:}\quad C(x,y) \;:=\; \frac{\mathcal{M}_y(x,y) - \mathcal{N}_x(x,y)}{\mathcal{M}(x,y)}\,.$$

Simplify $C(x,y)$ to see if it is a fnc of y alone. If "yes", write $C(y) := C(x,y)$. Note $V()$ satisfies DE

$$*:\quad V'(y) + C(y)V(y) \;=\; 0\,.$$

Applying FOLDE, define $B() := \int C()$. Then

$$V(y) \;:=\; 1/\mathsf{e}^{B(y)}$$

satisfies $(*)$. Define two new functions

$$\text{16j:}\quad \begin{aligned} \widehat{\mathcal{N}}(x,y) &:= V(y)\cdot\mathcal{N}(x,y) \qquad \text{and} \\ \widehat{\mathcal{M}}(x,y) &:= V(y)\cdot\mathcal{M}(x,y)\,. \end{aligned}$$

Apply steps (**E2**,**E3**,**E4**) to DE

$$\text{16}a\text{.2:}\quad [\widehat{\mathcal{N}}(x,y)\cdot\tfrac{\mathrm{d}y}{\mathrm{d}x}] + \widehat{\mathcal{M}}(x,y) \;=\; 0\,,$$

which is exact.

EXACT Example of (E1.2)

Consider $y = y(x)$ in

$$\text{16a‡:}\quad \underbrace{x^2}_{\mathcal{N}(x,y)}\cdot y' \;-\; \underbrace{[y^2 + 2xy]}_{\mathcal{M}(x,y)} \;=\; 0\,.$$

Firstly,

$$\mathcal{N}_x - \mathcal{M}_y \;=\; 2x - [\text{-}[2y+2x]] \overset{\text{note}}{=\!=} 2[y+2x]$$

is not the zero-fnc, so (16a‡) is not exact. Secondly, ratio

$$\frac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{N}} \;=\; \frac{2\cdot[y+2x]}{x^2}$$

is not a pure fnc of x, so (**E1.1**) is inapplicable.

Applying (**E1.2**), we compute $C(x,y)$ as

$$\text{16i‡:}\quad \frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{M}} \overset{\text{note}}{=\!=} \frac{\text{-}[2\cdot[y+2x]]}{\text{-}[y^2 + 2xy]} \overset{\text{note}}{=\!=} \frac{2}{y}\,.$$

Yes! –this is a pure fnc of y. Applying FOLDE, we anti-diff w.r.t y, obtaining $B(y) := 2\cdot\log(y)$. Our exact-ifying factor is thus

$$V(y) \overset{\text{def}}{=\!=} \mathsf{e}^{\text{-}B(y)} \overset{\text{note}}{=\!=} 1/y^2\,.$$

Multiplying (16a‡) by $\frac{1}{y^2}$ yields exact $\widehat{\mathcal{N}}\cdot y' + \widehat{\mathcal{M}} = 0$, where

16j‡:
$$\widehat{\mathcal{N}}(x,y) := \frac{x^2}{y^2} \quad \text{and}$$
$$\widehat{\mathcal{M}}(x,y) := -\Big[1 + \frac{2x}{y}\Big].$$

***Applying* (E2,E3).** Anti-differentiating w.r.t y and x, yields

$$\mathcal{B}(x,y) := \int^y \widehat{\mathcal{N}} \overset{\text{note}}{=\!=} -\frac{x^2}{y};$$
$$\mathcal{A}(x,y) := \int^x \widehat{\mathcal{M}} \overset{\text{note}}{=\!=} -\Big[x + \frac{x^2}{y}\Big].$$ Thus
$$\mathcal{B} - \mathcal{A} = x = g(y) - h(x),$$ where

we define $g(y) := 0$ and $h(x) := -x$. Finally, (16e) tells us that $\mathbf{F} = \mathcal{A} + g \overset{\text{note}}{=\!=} \mathcal{A}$.

Checking. Consider a fnc $y = y(x)$ satisfying

**:
$$\alpha = -\Big[x + \frac{x^2}{y(x)}\Big],$$

for some number α. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ produces that

$$0 = -\Big[1 + \frac{2xy - x^2y'}{y^2}\Big] \overset{\text{note}}{=\!=} \frac{x^2y' - 2xy - y^2}{y^2}.$$

Multiplying by y^2 yields (16a‡), as desired.

In this instance, we can actually solve (**) for $y()$ as

$$y_\alpha(x) = \frac{-x^2}{\alpha + x}.$$

Nifty. . .

Does an exactifying factor gain/lose solns? Do these DEs

Given: $\mathcal{N}\cdot y' + \mathcal{M} = 0$,

e1: $[W\mathcal{N}]\cdot y' + W\mathcal{M} = 0$ and

e2: $[V\mathcal{N}]\cdot y' + V\mathcal{M} = 0$

have the same solution-set?

Compartmental analysis

Brine with $1.3\frac{\text{lb}}{\text{gal}}$ salt flows at rate $4\frac{\text{gal}}{\text{min}}$ into a tank that initially hold 12gal of $2\frac{\text{lb}}{\text{gal}}$-salt brine. The tank is well-mixed, and brine is flowing out at rate $4\frac{\text{gal}}{\text{min}}$. We seek a formula for $y(t)$, the number of lbs of salt in the tank at time t.

Units:	Symbol:	Description:
lb	$y(t)$	Salt in tank @ t.
gal	$W(t)$	Water in tank @ t.
	$\mathrm{U} := W(0)$	Initial amount of water.
lb/gal	S	Input salinity.
	$\sigma(t)$	Salinity in tank @ t.
	$\mathrm{D} := \sigma(0) - \mathrm{S}$	Initial Difference in Salinities. (salt-concentration.)
gal/min	R	Input-flow of water.
	ρ	Output-flow of water.
	$\mathrm{A} := \mathrm{R} - \rho$	Accumulation flow-rate.
	$\mathrm{L} := \rho - \mathrm{R} = -\mathrm{A}$	Loss flow-rate.
min	$\mathrm{E} := \mathrm{U}/\mathrm{L}$	Time-to-Empty, when L>0
1/min	$\mathbf{\Gamma} := \frac{\mathrm{R}}{\mathrm{U}}$	A useful constant.

Use italic boldface $\boldsymbol{0}$ to mean 0min.

By definition of the quantities involved

17a: $$W(t) = \mathrm{U} + \mathrm{A}t \quad \text{and} \quad \sigma(t) = \frac{y(t)}{W(t)}.$$

Our salt-fnc y satisfies DE

17b: $$y'(t) = \overbrace{\mathrm{R}\cdot\mathrm{S}}^{\text{Input}} - \overbrace{\rho\cdot\sigma(t)}^{\text{Output}} \overset{\text{note}}{=\!=} \mathrm{SR} - \tfrac{\rho}{W(t)}\cdot y(t).$$

To match our FOLDE notation, let

$$\mathrm{G} := \mathrm{SR} \quad \textit{and} \quad C(t) := \frac{\rho}{W(t)}.$$

So we can re-write (17b) as

$$y'(t) + C(t)y(t) = \mathrm{G}.$$

Case: R=ρ, not zero. Hence $C()$ is the constant $\mathbf{\Gamma} := \frac{\mathrm{R}}{\mathrm{U}} \neq 0$. Step (**F0**) of FOLDE has us anti-diff, then exponentiate, to get

17c: $$M(t) := \mathrm{e}^{\mathbf{\Gamma}t}.$$

Step (**F1**): Anti-diff'ing product $\mathrm{G}\cdot e^{\boldsymbol{\Gamma}t}$ gives

$$Q(t) := \frac{\mathrm{G}}{\boldsymbol{\Gamma}} \cdot e^{\boldsymbol{\Gamma}t} \stackrel{\text{note}}{=\!=} \mathrm{SU} \cdot e^{\boldsymbol{\Gamma}t}.$$

For an arb.constant α, then, step (**F2**) gives

17d: $$y(t) = e^{-\boldsymbol{\Gamma}t} \cdot \left[\alpha + \mathrm{SU}\cdot e^{\boldsymbol{\Gamma}t}\right] = \frac{\alpha}{e^{\boldsymbol{\Gamma}t}} + \mathrm{SU}.$$

Divide through by U, and rename $\frac{\alpha}{\mathrm{U}}$ to α [which is, after all, arbitrary] to get

$$\sigma(t) = \frac{\alpha}{e^{\boldsymbol{\Gamma}t}} + \mathrm{S}.$$

Solve for α, and re-order, to obtain that

17e: $$\sigma(t) = \mathrm{S} + \frac{\mathrm{D}}{e^{\frac{\mathrm{R}}{\mathrm{U}}\cdot t}}.$$

Or use SoV. Alternatively, write (17b) as

$$\frac{dy}{dt} = \mathrm{G} - \boldsymbol{\Gamma}\cdot y$$

and separate variables to get

$$\frac{1}{\mathrm{G} - \boldsymbol{\Gamma}y} \cdot dy = 1 \cdot dt.$$

Only considering when $\mathrm{G} - \boldsymbol{\Gamma}y > 0$, we anti'diff to get

$$\frac{1}{-\boldsymbol{\Gamma}} \cdot \log(\mathrm{G} - \boldsymbol{\Gamma}y) = t + \alpha,$$

using arb.constant α. Cross-mult then exponentiate to get $\mathrm{G} - \boldsymbol{\Gamma}y = 1/e^{\boldsymbol{\Gamma}t + \boldsymbol{\Gamma}\alpha}$. Replace $e^{-\boldsymbol{\Gamma}\alpha}$ by $-\alpha$ [skipping some details] to get

$$\mathrm{G} - \boldsymbol{\Gamma}y = \frac{-\alpha}{e^{\boldsymbol{\Gamma}\cdot t}}.$$

Solve for $y{=}y(t)$, giving

$$y(t) = \frac{\alpha}{e^{\boldsymbol{\Gamma}\cdot t}} + \frac{\mathrm{G}}{\boldsymbol{\Gamma}} \stackrel{\text{note}}{=\!=} \frac{\alpha}{e^{\boldsymbol{\Gamma}t}} + \mathrm{SU}.$$

And this is RhS(17*d*).

Case: $\mathrm{R} \neq \rho$.

I.e, $\mathrm{A} \neq 0$, so $W()$ is not constant.

$*$: *In this section, we only consider values of t where $W(t) \stackrel{\text{note}}{=\!=} \mathrm{U} + \mathrm{A}t$ is positive.*

Step (**F0**): Anti-diff $C(t) = \frac{\rho}{\mathrm{U}+\mathrm{A}t}$ to get

$$B(t) := \frac{\rho}{\mathrm{A}} \cdot \log(\mathrm{U} + \mathrm{A}t),$$

using $(*)$. Setting $\theta := \frac{\rho}{\mathrm{A}}$, then, exponentiating gives

$$M(t) = [\mathrm{U} + \mathrm{A}t]^{\theta}.$$

Step (**F1**): Anti-diff'ing product $\mathrm{G}\cdot M(t)$ hands us

$$Q(t) := \frac{\mathrm{G}}{\mathrm{A}\cdot[\theta+1]} \cdot [\mathrm{U} + \mathrm{A}t]^{\theta+1}.$$

Note $[\mathrm{A}\theta] + \mathrm{A} = \mathrm{R}$ and $\frac{\mathrm{G}}{\mathrm{R}} = \mathrm{S}$. Step (**F2**) has us add an arb.constant α, then divide by $M(t)$, giving

$$y(t) = \frac{1}{M(t)} \cdot \left[\mathrm{S}\cdot[\mathrm{U} + \mathrm{A}t]^{\theta+1} + \alpha\right].$$

Dividing by $W(t) \stackrel{\text{note}}{=\!=} [\mathrm{U} + \mathrm{A}t]$ yields

$$\sigma(t) = \mathrm{S} + \frac{\alpha}{[\mathrm{U} + \mathrm{A}t]^{\mathrm{R}/\mathrm{A}}},$$

since $\theta + 1 = \frac{\mathrm{R}}{\mathrm{A}}$. Dividing top and bottom by $[\mathrm{U}]^{\mathrm{R}/\mathrm{A}}$, and solve for α to arrive at this:

17f: $$\sigma(t) = \mathrm{S} + \frac{\mathrm{D}}{[1 + \frac{\mathrm{A}}{\mathrm{U}}\cdot t]^{\mathrm{R}/\mathrm{A}}},$$

The A rate is positive:negative as the tank is filling:draining. When draining, it is convenient to express this formula in terms of the *Loss flow-rate*, L, and *time-to-Empty*, E. Since $\frac{\mathrm{A}}{\mathrm{U}} = \frac{-\mathrm{L}}{\mathrm{U}} = \frac{-1}{\mathrm{E}}$, our (17f) becomes

17g: $$\sigma(t) = \mathrm{S} + \mathrm{D}\cdot\left[1 - \frac{1}{\mathrm{E}}\cdot t\right]^{\mathrm{R}/\mathrm{L}},$$

Plausibility. Soln (17f) handles when $\mathrm{A} \neq 0$. Do we get our A=0 soln, (17e), as a limit when we send A to zero? Let's check, by applying L'Hôpital's rule to the denominator of (17f). Let

$$\mathscr{L} := \lim_{\mathrm{A}\to 0} [1 + \tfrac{\mathrm{A}}{\mathrm{U}}\cdot t]^{\mathrm{R}/\mathrm{A}}.$$

Since log is continuous, $\log(\mathscr{L}) = \widehat{\mathscr{L}}$, where

$$\widehat{\mathscr{L}} := \lim_{\mathrm{A}\to 0} \frac{\mathrm{R}}{\mathrm{A}} \cdot \log(1 + \tfrac{\mathrm{A}}{\mathrm{U}}\cdot t).$$

Applying L'Hôpital's, L'Hôpital's rule

$$\lim_{\mathrm{A}\to 0} \frac{\log\left(1 + \frac{t}{\mathrm{U}}\cdot\mathrm{A}\right)}{\mathrm{A}} \stackrel{\text{l'Hôp}}{=\!=} \lim_{\mathrm{A}\to 0} \frac{\left[\frac{1}{1+\frac{t}{\mathrm{U}}\cdot\mathrm{A}}\right]\cdot\frac{t}{\mathrm{U}}}{1}$$
$$= \lim_{\mathrm{A}\to 0} \left[\frac{t}{\mathrm{U} + t\cdot\mathrm{A}}\right]$$
$$= \frac{t}{\mathrm{U} + [t\cdot 0]} = \frac{t}{\mathrm{U}}.$$

Hence $\widehat{\mathscr{L}} = \mathrm{R}\cdot\frac{t}{\mathrm{U}}$. Consequently

$$\mathscr{L} \;=\; \mathsf{e}^{\frac{\mathrm{R}}{\mathrm{U}}\cdot t},$$

which indeed equals the denominator of (17e).

Cascading tanks

Calling the above tank "tank-1", we generalize to have tank-1 feed into tank-2, which feeds into tank-3 etc. Each tank has constant input and output flow-rate R. The amount of water in each tank is U.

Use $\sigma_N(t)$ for the salt-concentration in tank-N at time t, and use [recall that italic boldface $\boldsymbol{0}$ means 0min.]

$$\mathrm{D}_N \;:=\; \sigma_N(\boldsymbol{0}) - \mathrm{S}. \quad \text{As a convenience,}$$

$$\mathrm{D}_0 \;=\; \mathrm{S} - \mathrm{S} \overset{\text{note}}{=\!=} 0\tfrac{\mathsf{lb}}{\mathsf{gal}} \quad \text{and}$$

$$*: \qquad \sigma_0(\cdot) \;\equiv\; \mathrm{S},$$

by imagining that the source is an ∞-volume tank-0.

We will show, for $N = 0, 1, 2, \ldots$, that[♡4]

17h:

$$\sigma_N(t) \overset{?}{=} \mathrm{S} + \frac{f_N(t)}{\mathsf{e}^{\mathbf{\Gamma}t}}, \quad \text{where}$$

$$f_N(t) := \sum_{k=0}^{N} \tfrac{1}{k!}\cdot \mathrm{D}_{N-k}\cdot[\mathbf{\Gamma}t]^k \overset{\text{note}}{=\!=} \sum_{k=0}^{N-1} \tfrac{1}{k!}\cdot \mathrm{D}_{N-k}\cdot[\mathbf{\Gamma}t]^k,$$

since D_0 is zero.

For future use, verify this recurrence relation:

$$**: \qquad [f_{N+1}]' \;=\; \mathbf{\Gamma}\cdot f_N.$$

Proving (17h). Product $\mathbf{\Gamma}t$ is unitless, so $f_N(t)$ is in lb/gal; hence so is $\mathrm{S}+[f_N(t)/\mathsf{e}^{\mathbf{\Gamma}t}]$, as it should be.

Secondly $f_N(\boldsymbol{0}) = \frac{1}{0!}\cdot \mathrm{D}_{N-0}\cdot 1 \overset{\text{note}}{=\!=} \mathrm{D}_N$. Thus $\mathrm{S}+\frac{f_N(0)}{\mathsf{e}^0}$ equals $\mathrm{S}+\mathrm{D}_N$, which indeed equals $\sigma_N(\boldsymbol{0})$, as it should. What remains, is for us to verify that (17h) satisfies the appropriate DE.

Base case. Note $f_0(\cdot) = \frac{1}{0!}\cdot \mathrm{D}_0 \overset{\text{note}}{=\!=} 0\frac{\mathsf{lb}}{\mathsf{gal}}$. Hence $\sigma_0(\cdot)$ is the constant-fnc S, as ($*$) indeed says.

Induction. Fix a natnum N for which (17h) holds. Here, let y and σ denote y_{N+1} and σ_{N+1}. Our (17b) DE becomes

$$y'(t) \;=\; \overbrace{\mathrm{R}\cdot\sigma_N(t)}^{\text{Input}} - \overbrace{\mathrm{R}\cdot\sigma(t)}^{\text{Output}}.$$

Divide by U, the [constant] amount of water in each tank, to get FOLDE

$$\textbf{17i:} \qquad \sigma'(t) + \mathbf{\Gamma}\cdot\sigma(t) \;=\; \mathbf{\Gamma}\cdot\sigma_N(t).$$

As in (17c), FOLDE give multiplier-fnc $M(t) := \mathsf{e}^{\mathbf{\Gamma}t}$. We wish to anti-diff product

$$P(t) \;:=\; \mathsf{e}^{\mathbf{\Gamma}t}\cdot\mathbf{\Gamma}\cdot\sigma_N(t) \;\overset{\text{by (17h)}}{=\!=}\; \mathrm{S}\mathbf{\Gamma}\cdot\mathsf{e}^{\mathbf{\Gamma}t} + \mathbf{\Gamma}\cdot f_N(t).$$

Courtesy ($**$), we can choose anti-deriv

$$Q(t) \;:=\; \int^t P \;=\; \mathrm{S}\cdot\mathsf{e}^{\mathbf{\Gamma}t} + f_{N+1}(t).$$

Adding the appropriate constant α, then dividing by $M(t)=\mathsf{e}^{\mathbf{\Gamma}t}$, produces

$$\sigma_{N+1}(t) \;=\; \mathrm{S} + \frac{f_{N+1}(t)+\alpha}{\mathsf{e}^{\mathbf{\Gamma}t}}.$$

We've already checked that (17h) gives the correct value at $t=\boldsymbol{0}$, hence α must be 0. The conclusion is that formula (17h) is correct at stage $N+1$. QED

[♡4] Note that $\mathrm{Deg}(f_N)$ is $N-1$, since D_0 is zero.

Falling things

Consider an object falling in a uniform gravitational field, with air resistance.

We are assuming there is a ***PC***, a ***Proportionality Constant*** [depending on the object] which, when multiplied by the speed of the object, give the *drag*, the retarding *force*. Below, I use Φ as the "PC density", the PC per unit-mass. This Φ, times the airspeed of the object, gives the resulting *deceleration*.

A unit of force is the Newton, written N, which equals a $\frac{\mathsf{kg}\cdot\mathsf{m}}{\mathsf{sec}^2}$. Text **NSS(6^th)** expesses the PC as $\frac{\mathsf{N}\cdot\mathsf{sec}}{\mathsf{m}}$, which thus equals a $\frac{\mathsf{kg}}{\mathsf{sec}}$. Hence our Φ is in reciprocal-seconds.

Units:	Symbol:	Description:
ⓓ/ⓣ^{2}	$\mathsf{A} \overset{\text{Earth}}{\approx} 10\frac{\mathsf{m}}{\mathsf{sec}^2}$	Surface-accel from gravity.
1/ⓣ	Φ	PC per unit-mass.
ⓓ/ⓣ	$\Gamma := \mathsf{A}/\Phi$	A useful ratio.
ⓓ	$y(t)$	is height above planet's surface @ t.
ⓓ/ⓣ	$s(t) := y'(t)$	Speed of object @ t.
ⓓ/ⓣ^{2}	$a(t) := s'(t)$	Accel of object @ t.

Use italic boldface $\boldsymbol{0}$ to mean 0min, and set

$$\mathrm{s}_0 := s(\boldsymbol{0}) \quad \text{and} \quad \mathrm{y}_0 := y(\boldsymbol{0}).$$

We place the y-axis with pos-direction *up* from the planet's surface. Hence once the object is descending, its speed $s()$ will be negative.

Speed. Our DE is

18a:
$$\begin{aligned} a(t) &= -\mathsf{A} - [\Phi\cdot s(t)], \quad \text{i.e} \\ s'(t) + [\Phi\cdot s(t)] &= -\mathsf{A}. \end{aligned}$$

Applying FOLDE and solving for inital conditions gives

18b:
$$\begin{aligned} s(t) &= -\Gamma + [\mathrm{s}_0 + \Gamma]\cdot \mathsf{e}^{-\Phi t} \\ &= \mathrm{s}_0\cdot\mathsf{e}^{-\Phi t} + \mathsf{A}\cdot\left[\frac{\mathsf{e}^{-\Phi t} - 1}{\Phi}\right]. \end{aligned}$$

Sending $\Phi \to 0$, do we recover the frictionless case? Well,

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\mathsf{e}^{-\Phi t} - 1\right) = \left[-t\cdot\mathsf{e}^{-\Phi t}\right] - 0 = -t\cdot\mathsf{e}^{-\Phi t}.$$

So L'Hôpital's, L'Hôpital's rule gives

$$\lim_{\Phi\to 0}\frac{\mathsf{e}^{-\Phi t} - 1}{\Phi} = \lim_{\Phi\to 0}\frac{-t\cdot\mathsf{e}^{-\Phi t}}{1} = -t.$$

From (18b), then,

18b∗:
$$\lim_{\Phi\to 0} s(t) = \mathrm{s}_0 - \mathsf{A}t,$$

as expected.

Height. Integrating (18b) w.r.t time, then solving for the initial height, gives

18c:
$$\begin{aligned} y(t) &= \mathrm{y}_0 + -\Gamma t + \tfrac{1}{\Phi}[\mathrm{s}_0 + \Gamma]\cdot[1 - \mathsf{e}^{-\Phi t}] \\ &= \mathrm{y}_0 + \mathrm{s}_0\cdot\tfrac{1 - \mathsf{e}^{-\Phi t}}{\Phi} + \Gamma\cdot\tfrac{[1 - \mathsf{e}^{-\Phi t}] - t\Phi}{\Phi}. \end{aligned}$$

Hence $\lim_{\Phi\to 0} y(t)$ equals $[\mathrm{y}_0 + \mathrm{s}_0\cdot t]$ plus A times

$$\lim_{\Phi\to 0}\frac{[1 - \mathsf{e}^{-\Phi t}] - t\Phi}{\Phi^2}.$$

This, by L'Hôpital's, equals

$$\lim_{\Phi\to 0}\frac{[0 + t\mathsf{e}^{-\Phi t}] - t}{2\Phi} = \frac{t}{2}\cdot\lim_{\Phi\to 0}\frac{\mathsf{e}^{-\Phi t} - 1}{\Phi} = \frac{-1}{2}\cdot t^2.$$

Hence

18c∗:
$$\lim_{\Phi\to 0} y(t) = \mathrm{y}_0 + \mathrm{s}_0\cdot t - \tfrac{1}{2}\mathsf{A}t^2,$$

unsurprisingly.

Falling Example

[From P.112 from **N,S,S(6^th)**.] Parachutist weighs 75kg. Her PC with chute closed:open is $15\frac{\mathsf{kg}}{\mathsf{sec}}$:$105\frac{\mathsf{kg}}{\mathsf{sec}}$. The closed-chute density is:

$$\Phi_{\mathrm{Cl}} := \frac{15}{75}\frac{1}{\mathsf{sec}} = \frac{1}{5}\frac{1}{\mathsf{sec}}; \text{ so } \quad \Gamma_{\mathrm{Cl}} := \frac{10\frac{\mathsf{m}}{\mathsf{sec}^2}}{\Phi_{\mathrm{Cl}}} = 50\frac{\mathsf{m}}{\mathsf{sec}}.$$

Since $\frac{105}{15} = 7$, the open-chute density is

$$\Phi_{\mathrm{Op}} := \frac{7}{5}\frac{1}{\mathsf{sec}}; \quad \text{hence} \quad \Gamma_{\mathrm{Op}} := 350\frac{\mathsf{m}}{\mathsf{sec}}.$$

She free-falls for 1min, then opens her chute. How long after she jumps, from 4000m, did she gently waft to the ground?[♡5]

[♡5] Where's Waldo?

Convolutions

Recall that $Id := [t \mapsto t]$ is the ***identity function***. So $Id^3(x) = x^3$, and Id^0 is the constant-fnc $\mathbf{1}$. Below, let $\mathbb{J} := [0, \infty)$.

Convolution defn. Given (locally-integrable) fncs $\mathbf{f}, \mathbf{g}: \mathbb{J} \to \mathbb{C}$, their one-sided ***convolution*** is the fnc mapping $\mathbb{J} \to \mathbb{C}$ by

19.1: $$[\mathbf{f} \circledast \mathbf{g}](t) := \int_0^t \mathbf{f}(t - v) \cdot \mathbf{g}(v)\, dv\,.$$

Easily, we get these algebraic properties:

Convolution is commutative and associative. Convolution is bilinear[♡1]*, in that*

19.2: $$[\mathbf{f}_1 + \mathbf{f}_2] \circledast \mathbf{g} = [\mathbf{f}_1 \circledast \mathbf{g}] + [\mathbf{f}_2 \circledast \mathbf{g}]\,,$$ $$[5\mathbf{f}] \circledast \mathbf{g} = 5 \cdot [\mathbf{f} \circledast \mathbf{g}]\,,$$

for arb. fncs $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2$ and arbitrary scalar, 5.

Convolution commutes with complex-conjugation: $\overline{f \circledast g} = \overline{f} \circledast \overline{g}$.

We also have this cty property [more is true]:

19.3: If $\mathbf{f}, \mathbf{g}$ continuous, then $[\mathbf{f} \circledast \mathbf{g}]$ is cts.

CAVEAT: We do *not* have a formula for how convolution interacts with multiplication; we have no nice formula for $F \circledast [g \cdot h]$.

Powers. As a shorthand, the "***n^{th} convolution power*** of $\mathbf{f}$",
$$\mathbf{f}^{\circledast n} := \mathbf{f} \circledast \mathbf{f} \circledast \overset{n}{\cdots} \circledast \mathbf{f}\,,$$
is the result of convolving together n copies of $\mathbf{f}$. In particular, $\mathbf{1}^{\circledast[n+1]}$ is the n^{th}-antideriv of $\mathbf{1}$ (i.e, x^0) whose derivatives are zero at the origin. So

19.4a: $$\mathbf{1}^{\circledast[n+1]} = \tfrac{1}{n!} \cdot Id^n \stackrel{\text{i.e}}{=\!=} \Big[x \mapsto \frac{x^n}{n!}\Big]\,.$$

We get this nice corollary.

19.4b: Power-of-x Lemma. *Consider a continuous fnc $\boldsymbol{\beta}: \mathbb{J} \to \mathbb{C}$ and a natnum N. Then*

$\dagger_N$: $$\big[\tfrac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta} = B_N\,,$$

where B_N is the unique function such that

‡: $$0 = B_N(0) = B_N'(0) = B_N''(0) = \ldots = B_N^{(N)}(0)\,.$$

and $B_N^{(N+1)} = \boldsymbol{\beta}$. ◇

[♡1] In the other order, $f \circledast [\mathbf{g}_1 + \mathbf{g}_2] = [f \circledast \mathbf{g}_1] + [f \circledast \mathbf{g}_2]$; in other words: *"Convolution distributes over addition"*. Also, $f \circledast [7g] = 7[f \circledast g]$; i.e: *"Scalars factor-out"*.

Proof. For an arbitrary fnc $\mathbf{g}$, the FTC says that $[\mathbf{1} \circledast \mathbf{g}](t) \stackrel{\text{def}}{=\!=} \int_0^t \mathbf{g}$ is the antideriv G of $\mathbf{g}$ such that $G(0) = 0$. Courtesy (19.4a), our $\big[\frac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta}$ is
$$\mathbf{1} \circledast [\mathbf{1} \circledast \overset{N+1}{\cdots} [\mathbf{1} \circledast \boldsymbol{\beta}] \ldots]\,,$$
using the associativity of convolution. Hence $\big[\frac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta}$ is indeed the B_N defined by (‡). ♦

Alt Pf. Just for fun, here is an alternate proof using a derivative-of-convolution formula, (19.11), that we'll shortly deduce.

Defining $\boldsymbol{\alpha}_k(t) = t^k/k!$, note $[\boldsymbol{\alpha}_{k+1}]' = \boldsymbol{\alpha}_k$. Fix a natnum K satisfying ($\dagger_K$). Differentiating,

$$\big[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}\big]' \stackrel{\text{by (19.11)}}{=\!=\!=} \Big[[\boldsymbol{\alpha}_{K+1}]' \circledast \boldsymbol{\beta}\Big] + \Big[\boldsymbol{\alpha}_{K+1}(0) \cdot \boldsymbol{\beta}\Big]$$
$$= [\boldsymbol{\alpha}_K \circledast \boldsymbol{\beta}]\,,$$

since $\boldsymbol{\alpha}_{K+1}(0)$ is 0, as $K+1$ is positive. So $\big[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}\big]'$ is B_K. Thus

$$\big[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}\big](t) = \int_0^t B_K \stackrel{\text{by FTC}}{=\!=\!=} B_{K+1}(t)\,.$$

Hence ($\dagger_{K+1}$). We've shown that ($\dagger_K$) ⇒ ($\dagger_{K+1}$), as desired. ♦

Ex.C1. Note that $\frac{d}{dv}([t + 1 - v] * e^v) = [t - v] * e^v$. So
$$[Id \circledast \exp](t) = \Big[[t + 1 - v] * e^v\Big]\Big|_{v=0}^{v=t}$$
$$= e^t - [t + 1]\,.$$ □

Ex.C2. Let $f(x) := x^2$ and $\boldsymbol{\beta}(x) := 30[x^4 + x]$. Then
$$[f \circledast \boldsymbol{\beta}](t) = \int_0^t [t - v]^2 \cdot 30[v^4 + v]\, dv\,.$$
The integrand is a poly, which we could multiply-out, then integrate. Alternatively, apply ($\dagger_2$), and antidiff $\boldsymbol{\beta}$ thrice to get
$$\frac{30\, x^7}{5 \cdot 6 \cdot 7} + \frac{30\, x^4}{2 \cdot 3 \cdot 4} = \frac{x^7}{7} + \frac{5x^4}{4}\,.$$
Multiply by 2! to conclude that
$$[f \circledast \boldsymbol{\beta}](t) = \frac{2}{7} \cdot t^7 + \frac{5}{2} \cdot t^4\,.$$ □

Ex.C3. Let's convolve exponentials $f(x) := \mathsf{e}^{Bx}$ and $g(x) := \mathsf{e}^{Cx}$, where $B,C \in \mathbb{C}$.

CASE: $B = C$ The integrand for computing $[f \circledast f](5)$ is $\mathsf{e}^{B[5-v]} \cdot \mathsf{e}^{Bv} \stackrel{\text{note}}{=} \mathsf{e}^{B\cdot 5}$. Its integral is thus $5 \cdot \mathsf{e}^{B\cdot 5}$. Hence

19.5a: $$[x \mapsto \mathsf{e}^{Bx}]^{\circledast 2}(t) = [f \circledast f](t) = t \cdot \mathsf{e}^{Bt}.$$ *In functional notation,* $$f \circledast f = Id \cdot f.$$

[In the $B{=}0$ case, this says $\mathbf{1} \circledast \mathbf{1} = Id$, which is indeed correct.]

CASE: $B \neq C$ Define difference $D := C - B$. The $[f \circledast g](5)$ integrand is $\mathsf{e}^{B[5-v]} \cdot \mathsf{e}^{Cv} \stackrel{\text{note}}{=} \mathsf{e}^{B\cdot 5}\mathsf{e}^{D\cdot v}$. Its integral is $\frac{\mathsf{e}^{B\cdot 5}}{D} \cdot \mathsf{e}^{D\cdot v}\big|_{v=0}^{v=5}$, i.e, $\frac{\mathsf{e}^{B\cdot 5}}{D} \cdot [\mathsf{e}^{D\cdot 5} - 1]$.

This equals $\frac{1}{D}[\mathsf{e}^{C\cdot 5} - \mathsf{e}^{B\cdot 5}]$. Consequently,

19.5b: $$[f \circledast g](t) = \frac{[\mathsf{e}^{Ct} - \mathsf{e}^{Bt}]}{C - B} \stackrel{\text{note}}{=} \frac{[\mathsf{e}^{Bt} - \mathsf{e}^{Ct}]}{B - C}.$$ *I.e,* $$f \circledast g = \frac{g - f}{C - B} = \frac{f - g}{B - C}.$$

This is symmetric in B and C, as it must be. □

A shorthand. I'll write '$[9x] \circledast \mathsf{e}^{3x}$ equals...' to mean:

Let $f(u) := 9u$ *and* $g(z) := \mathsf{e}^{z}$.
Then $[f \circledast g](x)$ *equals...*

I.e, I will sometimes use the same letter for the input-vars, and the output-var. □

Ex.C4.1. We seek to compute $H := [9x] \circledast \mathsf{e}^{3x}$.

Let's solve this just by using properties of convolution. Let $\mathbf{G} := \mathsf{e}^{3x}$. Since $[\int 3\mathbf{G}] = \mathbf{G} + \text{Const}$, and $\mathbf{G}\rfloor_{x=0}$ is 1, it follows that

†: $$1 \circledast [3\mathbf{G}] = \mathbf{G} - 1.$$

Since convolution is bilinear,

$$\begin{aligned} H = 9[x \circledast \mathbf{G}] &= x \circledast [9\mathbf{G}] \\ &= [1 \circledast 1] \circledast [9\mathbf{G}] \\ &= 1 \circledast [1 \circledast [9\mathbf{G}]], \end{aligned}$$

since $\circledast$ is associative. Computing the inside-convolution,

$$1 \circledast [9\mathbf{G}] = 3\cdot[1 \circledast [3\mathbf{G}]] \stackrel{\text{by } (\dagger)}{=} 3\cdot[\mathbf{G} - 1] = 3\mathbf{G} - 3.$$

So, $$\begin{aligned} H &= [1 \circledast [3\mathbf{G}]] - 3\cdot[1 \circledast 1] \\ &= [\mathbf{G} - 1] - 3x = \mathsf{e}^{3x} - 1 - 3x. \end{aligned}$$ □

Ex.C4.2. The preceding example showed that

‡: $$1 \circledast \mathbf{G} = \tfrac{1}{3}[\mathbf{G} - 1], \quad \text{and}$$ $$1^{\circledast 2} \circledast \mathbf{G} = \tfrac{1}{9}[\mathbf{G} - 1 - 3x].$$

Continuing, $[1^{\circledast 3} \circledast \mathbf{G}]$ is one-ninth of

$$\begin{aligned} &[1 \circledast \mathbf{G}] - [1 \circledast 1] - 3[1 \circledast x] \\ &= \tfrac{1}{3}[\mathbf{G} - 1] - x - 3\cdot\frac{x^2}{2} \\ &= \tfrac{1}{3}\Big[\mathbf{G} - 1 - 3x - 3^2\cdot\frac{x^2}{2}\Big] \\ &\stackrel{\text{note}}{=} \tfrac{1}{3}\Big[\mathbf{G} - 3^0\cdot\frac{x^0}{0!} - 3^1\cdot\frac{x^1}{1!} - 3^2\cdot\frac{x^2}{2!}\Big]. \end{aligned}$$

Hence

$$1^{\circledast 3} \circledast \mathbf{G} = \frac{1}{27}\cdot\Big[\mathbf{G} - \frac{3^0}{0!}\cdot x^0 - \frac{3^1}{1!}\cdot x^1 - \frac{3^2}{2!}\cdot x^2\Big].$$

The pattern is clear:

For each natnum N, with $\mathbf{G}$ denoting e^{3x},

19.6a: $$\begin{aligned} \tfrac{1}{N!}\cdot[x^N \circledast \mathbf{G}] &\stackrel{\text{recall}}{=} 1^{\circledast[N+1]} \circledast \mathbf{G} \\ &= \frac{1}{3^{N+1}}\Big[\mathbf{G} - \sum_{k=0}^{N}\Big[\frac{3^k}{k!}\cdot x^k\Big]\Big]. \end{aligned}$$

Rewriting,

19.6b: $$x^N \circledast \mathsf{e}^{3x} = \frac{N!}{3^{N+1}}\Big[\mathsf{e}^{3x} - \sum_{k=0}^{N}\Big[\frac{3^k}{k!}\cdot x^k\Big]\Big].$$

The above sum, $\sum_{k=0}^{N} \frac{3^k}{k!}\cdot x^k$, we recognize as the *$N^{th}$-Maclaurin-polynomial of* e^{3x}; see below. Before generalizing this result, let us compute an example with [*shudder*] *actual numbers*.

Let $R := [6 - 9x + 54x^2] \circledast \mathsf{e}^{3x}$. Then

$$R = 6\cdot[1 \circledast \mathbf{G}] - 9\cdot[x \circledast \mathbf{G}] + 54\cdot[x^2 \circledast \mathbf{G}].$$

From (19.6b), or (‡), note

$$6\cdot[1\circledast \mathbf{G}] = 6\cdot\tfrac{1}{3}\cdot[\mathbf{G}-1] = 2\mathbf{G}-2\,, \quad\text{and}$$
$$\text{-}9\cdot[x\circledast \mathbf{G}] = \text{-}9\cdot\tfrac{1}{9}\cdot[\mathbf{G}-1-3x] = \text{-}\mathbf{G}+1+3x\,, \quad\text{and}$$
$$54\cdot[x^2\circledast \mathbf{G}] = 54\cdot\tfrac{2!}{3^3}\cdot\Big[\text{Terms}\Big] = 4\cdot\Big[\mathbf{G}-1-3x-\tfrac{9}{2}x^2\Big].$$

Adding these together says that

$$R = 5\mathrm{e}^{3x} - 5 - 9x - 18x^2\,. \qquad \square$$

Maclaurin polynomial. For a natnum N, consider a function G which is N-times differentiable. Then the "N^{th}-th ***Maclaurin polynomial*** of G", is the unique polynomial p of $\mathrm{Deg}(p) \leqslant N$, whose first $N{+}1$ derivatives agree with G's at the origin. I.e

$$p(0) = G(0),\ p'(0) = G'(0),\ p''(0) = G''(0),$$
$$\ldots, p^{(N-1)}(0) = G^{(N-1)}(0),\ p^{(N)}(0) = G^{(N)}(0)\,.$$

An explicit formula for p is

$$p(x) \quad := \quad \sum_{k=0}^{N} \frac{G^{(k)}(0)}{k!}\cdot x^k\,.$$

Use $\mathrm{Mac}_{G,N}$ to denote this p, the N^{th}-th Maclaurin polynomial of G. $\square$

19.7: Mac Lemma. *Consider a cts fnc $\boldsymbol{\beta}$ on $[0,\infty)$, and fix a natnum N. Let $g = g_N$ be a fnc whose $[N{+}1]^{st}$-derivative is $\boldsymbol{\beta}$, i.e, $g^{(N+1)} = \boldsymbol{\beta}$. Then*

$$1^{\circledast[N+1]} \circledast \boldsymbol{\beta} \quad = \quad g - \mathrm{Mac}_{g,N}\,. \qquad \lozenge$$

Proof. This follows immediately from Power-of-x Lemma, (19.4b), on page 19. ♦

Derivative notation. Below, for a two-variable function $H(x,y)$, we use $H_1()$ to mean the partial-derivative w.r.t the 1st variable; so $H_1()$ is a synonym for $H_x()$. And $H_2()$ is $H_y()$. $\square$

19.8: Chain-rule Lemma. *Consider equations*

$$x = \boldsymbol{\alpha}(t) \quad\text{and}\quad y = \boldsymbol{\beta}(t) \quad\text{and}\quad z = H(x,y)\,,$$

for differentiable functions $\boldsymbol{\alpha}, \boldsymbol{\beta}, H$. Then composition $\varphi(t) := H(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ is differentiable. Moreover,

19.8a:
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}\,; \quad [\text{Leibniz}]$$
$$\varphi'(t) = H_1(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))\cdot\boldsymbol{\alpha}'(t) + H_2(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))\cdot\boldsymbol{\beta}'(t)\,, \quad [\text{Newton}]$$

where Leibniz names the variables, and Newton names the functions. $\lozenge$

19.9: DUI: Differentiation under Integral. *Consider fnc $G(x,v)$ defined on a rectangle $\mathbf{U} := [x_0,x_1]\times[v_0,v_1]$ in the plane. Suppose partial-deriv $G_1()$ is cts on $\mathbf{U}$. Then for arb. values, say, 3 and 5, in $[v_0\mathinner{..}v_1]$, the fnc*

$$H(x) := \int_3^5 G(x,v)\,\mathrm{d}v$$

is differentiable, and

$$H'(x) = \int_3^5 G_1(x,v)\,\mathrm{d}v\,. \qquad \lozenge$$

Proof. Fix, say, $x{=}7$. From a non-zero ε, form difference quotient

$$\frac{H(7+\varepsilon) - H(7)}{\varepsilon} = \int_3^5 \frac{G(7+\varepsilon,v) - G(7,v)}{\varepsilon}\,\mathrm{d}v\,.$$

Send $\varepsilon \to 0$. In order to pass that limit through the integral sign, note the following. Since G_1 is cts on compact set $\mathbf{U}$, our G is uniformly Lipschitz in the x-direction. Hence we can use the Dominated Convergence thm to commute the limits. ♦

19.10: Leibniz-rule Lemma. *Consider continuous function $G{:}\mathbb{J}{\times}\mathbb{J}{\to}\mathbb{R}$; hence $G_1()$ is cts. Define*

19.10∗:
$$H(x,y) := \int_0^y G(x,v)\,\mathrm{d}v\,.$$

Then $\varphi(t) := H(t,t)$ is diff'able, and

19.10†:
$$\varphi'(t) = \Big[\int_0^t G_1(t,v)\,\mathrm{d}v\Big] + G(t,t)\,. \qquad \lozenge$$

Proof. Notice that $\varphi(t) = H(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$, where fncs $\boldsymbol{\alpha}(t) := t =: \boldsymbol{\beta}(t)$. Applying the Chain rule (19.8a),

$$\begin{aligned} \varphi'(t) &= H_1(t,t)\cdot\tfrac{dt}{dt} + H_2(t,t)\cdot\tfrac{dt}{dt} \\ &= H_1(t,t) + H_2(t,t)\,. \end{aligned}$$

By DUI (19.9), our $H_1(x,y) = \int_0^y G_1(x,v)\,dv$. Hence

$$H_1(t,t) = \int_0^t G_1(t,v)\,dv\,.$$

By FTC, moreover, $H_2(x,t) = G(x,t)$. Thus

$$H_2(t,t) = G(t,t)\,.$$

These three displays, together, yield (19.10†). ♦

19.11: Leibniz corollary. *Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are differentiable fncs on $\mathbb{J}$. Then $[\boldsymbol{\alpha} \circledast \boldsymbol{\beta}]$ is differentiable,*[♡6] *and*

$$\text{19.10‡:} \quad \begin{aligned} [\boldsymbol{\alpha} \circledast \boldsymbol{\beta}]'(t) &= [\boldsymbol{\alpha}' \circledast \boldsymbol{\beta}](t) + \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t) \\ &\underset{\text{by symmetry}}{=} [\boldsymbol{\alpha} \circledast \boldsymbol{\beta}'](t) + \boldsymbol{\alpha}(t)\cdot\boldsymbol{\beta}(0)\,. \end{aligned} \quad \diamondsuit$$

Proof. Define $G(x,v) := \boldsymbol{\alpha}(x-v)\cdot\boldsymbol{\beta}(v)$, then H as in (19.10∗). So $[\boldsymbol{\alpha} \circledast \boldsymbol{\beta}](t) \stackrel{\text{def}}{=} H(t,t)$. Using that $G(t,t) = \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t)$, applying (19.10†) yields (19.10‡). ♦

19.12: Convol-diff Thm. *Fix a natnum N. Consider an $\mathbf{f} \in \mathbf{C}^N$ and $\mathbf{g} \in \mathbf{C}^{N-1}$.* [*When $N=0$, we just need $\mathbf{g}$ locally-integrable.*] *Then $\mathbf{f} \circledast \mathbf{g}$ is in $\mathbf{C}^N$, and*

$$\mathrm{P}_N\text{:} \quad [\mathbf{f} \circledast \mathbf{g}]^{(N)} = [\mathbf{f}^{(N)} \circledast \mathbf{g}] + \sum_{j+k=N-1} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(k)}\,,$$

where the sum[♡7] *is taken over all ordered pairs (j,k) of natnums.* ♢

Proof. For N=0, this says $[\mathbf{f} \circledast \mathbf{g}] = [\mathbf{f} \circledast \mathbf{g}]$, which is true.

Now fix an N for which (P_N) holds. We differentiate RhS(P_N), by setting $\boldsymbol{\alpha} := \mathbf{f}^{(N)}$ and $\boldsymbol{\beta} := \mathbf{g}$, and applying (19.10‡). It yields that $[\mathbf{f} \circledast \mathbf{g}]^{(N+1)}$ equals

$$[\boldsymbol{\alpha}' \circledast \boldsymbol{\beta}](t) + \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t) + \sum_{j+\ell=N-1} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(\ell+1)}(t)\,,$$

summed over ordered-pairs (j,ℓ) of natnums. Setting $k := \ell+1$, we can re-write this as

$$[\boldsymbol{\alpha}' \circledast \boldsymbol{\beta}](t) + \sum_{j+k=N} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(k)}(t)\,.$$

Noting that $\boldsymbol{\alpha}'$ is $\mathbf{f}^{(N+1)}$, gives (P_{N+1}). ♦

[♡6] Wikipedia gives a slightly different formula, but for the derivative, of a 2-sided convolution. Our 1-sided convolution has an edge-effect when differentiated.

[♡7] E.g $[\mathbf{f} \circledast \mathbf{g}]''(7)$ equals $[\mathbf{f}'' \circledast \mathbf{g}](7)$ plus $\mathbf{f}'(0)\mathbf{g}(7) + \mathbf{f}(0)\mathbf{g}'(7)$.

The CCLDE-AnyTar algorithm

[See P.239 of **N,S,S(6^{th})**.] Given a polynomial

$$q(z) \;:=\; \mathrm{C}_N z^N + \ldots + \mathrm{C}_1 z^1 + \mathrm{C}_0 z^0$$

with $\mathrm{C}_N \neq 0$, we seek a fnc h solving DE

9a: $$[q(\mathbf{D})](h) \;=\; G\,,$$

for a given target fnc G.

1st-step. Use CCLDE to produce a function f solving ZeroTar $[q(\mathbf{D})](f) = 0$ with initial conditions

9b: $$f^{(N-1)}(0) \;=\; 1/\mathrm{C}_N\,, \quad \textit{and} \quad 0 = f(0) = f'(0) = \ldots f^{(N-2)}(0)\,.$$

2nd-step. Compute $h := f \circledast G$. To see that this solves (9a) note, because of initial conditions (9b), that we have this:

$$\text{For } j = 0,1,\ldots,N{-}1: \quad h^{(j)} \;=\; [\mathbf{f}^{(j)} \circledast G]\,.$$

$$\text{And} \quad h^{(N)} \;=\; [\mathbf{f}^{(N)} \circledast G] \;+\; [f^{(N-1)}(0)\cdot G] \;=\; [\mathbf{f}^{(N)} \circledast G] \;+\; [\tfrac{1}{\mathrm{C}_N}\cdot G]\,.$$

Using the bilinearity of convolution, (19.2), we have that sum $\sum_{j=0}^{N} \mathrm{C}_j h^{(j)}$ [which is LhS(9a)] equals

$*$: $$\Big[\Big[\sum_{j=0}^{N} \mathrm{C}_j f^{(j)}\Big] \circledast G\Big] \;+\; \mathrm{C}_N\cdot[\tfrac{1}{\mathrm{C}_N}\cdot G]\,.$$

Since $\sum_{j=0}^{N} \mathrm{C}_j f^{(j)}$ is the zero-fnc, the convolution in $(*)$ is 0. Hence $(*)$ equals G, as requested.

Gen soln to the "Targeted" DE. Recall that the general ZeroTar solution F to $[q(\mathbf{D})](F) = 0$ has N free parameters, $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$. Writing

$$\vec{\boldsymbol{\alpha}} \;=\; (\alpha_1, \alpha_2, \ldots, \alpha_N)\,,$$

then, we denote the general *ZeroTar* soln as $F_{\vec{\boldsymbol{\alpha}}}$. It follows that the sum

9c: $$H_{\vec{\boldsymbol{\alpha}}} \;:=\; h + F_{\vec{\boldsymbol{\alpha}}}$$

is the *general AnyTar-Soln* to $[q(\mathbf{D})](H) = G$.

CCLDE-AnyTar Example

We seek the gen-soln to

9a†: $$3h'' \;-\; 4h' \;+\; h \;\;=\;\; \exp\,.$$

So $q(z) := 3z^2 - 4z + 1 = [z-1]\cdot[3z-1]$ is the aux-poly of our DiffOp, $\mathsf{V} := 3\mathbf{D}^2 - 4\mathbf{D} + \mathbf{I}$.

Applying 1st-step. Here, $N = 2$ and $\frac{1}{\mathrm{C}_N} = \frac{1}{3}$.

The gen-soln to the ZeroTar DE $[q(\mathbf{D})](f) = 0$ is $f(x) := \alpha e^x + \beta e^{x/3}$. Solving for α,β so that $f(0) = 0$ and $f'(0) = \frac{1}{3}$ gives $f(x) = \frac{1}{2}\cdot[e^x - e^{x/3}]$. I.e

$$f \;\;=\;\; \tfrac{1}{2}\cdot[\exp - \Phi]\,,$$

where $\Phi(x) := e^{x/3}$.

Applying 2nd-step. The target in (9a†) is exp. The 2nd-step has us compute $h := f \circledast \exp$. Since convolution is bilinear,

$$h \;=\; \tfrac{1}{2}\cdot\Big[[\exp\circledast\exp] \;-\; [\Phi\circledast\exp]\Big]\,.$$

By (19.5a), our $[\exp\circledast\exp](t) = t\cdot e^t$. And courtesy (19.5b),

$$[\Phi\circledast\exp](t) \;=\; \frac{e^t - e^{\frac{1}{3}t}}{1-\frac{1}{3}} \;=\; \frac{3}{2}\cdot\Big[e^t - e^{\frac{1}{3}t}\Big]\,.$$

Consequently, our *General-target Soln* is

9c†: $$H_{\alpha_1,\alpha_2}(t) \;=\; \tfrac{1}{2}te^t \;+\; \alpha_1 e^t \;+\; \alpha_2 e^{\frac{1}{3}t}\,.$$

A subtlety: We never needed to compute $[\Phi\circledast\exp]$, once we noticed from (19.5b) that $[\Phi\circledast\exp]$ is a linear-comb of Φ and exp. For the ZeroTar solns are all such lin-combs, so computing this specific one is irrelevant.

Variation of parameters

[This section assumes knowledge of matrix multiplication, and the determinant of a square matrix.]

10.1: Cramer's "Rule" Thm. *Consider matrices* H *and* T*, and invertible matrix* M*, related by matrix-eqn*

$$\underbrace{\mathsf{M}}_{N\times N}\cdot\underbrace{\mathsf{H}}_{N\times 1} = \underbrace{\mathsf{T}}_{N\times 1}.$$

Here, "Multipler" M *and "Target"* T *are known, but "Huh?"* H *is unknown. Let* $\mathsf{M}_{\mathsf{T},r}$ *be the* $N\times N$ *matrix* M *except that its* r^{th}*-column has been replaced by column-vector* T. *With* h_r *the entry in the* r^{th}*-row of* H*, then*

$$h_r = \text{Det}(\mathsf{M}_{\mathsf{T},r})/\text{Det}(\mathsf{M}). \qquad \diamondsuit$$

Proof. The Determinant fnc is multiplicative, etc. ♦

A list $\vec{\varphi} := (\varphi_0, \varphi_1, \dots, \varphi_{N-1})$ of sufficiently differentiable fncs engenders its ***Wronskian Matrix***

$$\mathsf{WM}(\vec{\varphi}) := \begin{bmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_{N-1} \\ \varphi_0' & \varphi_1' & \cdots & \varphi_{N-1}' \\ \varphi_0'' & \varphi_1'' & \cdots & \varphi_{N-1}'' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0^{(N-1)} & \varphi_1^{(N-1)} & \cdots & \varphi_{N-1}^{(N-1)} \end{bmatrix},$$

also written as $\mathsf{WM}(\varphi_0, \dots, \varphi_{N-1})$. Its determinant,

$$\mathcal{W}(\varphi_0, \dots, \varphi_{N-1}) := \mathcal{W}(\vec{\varphi}) := \text{Det}(\mathsf{WM}(\vec{\varphi})),$$

is called the "***Wronskian*** of $\vec{\varphi}$".

10.2: Wronskian L.I. Thm. *If* $\vec{\varphi} := (\varphi_0, \varphi_1, \dots, \varphi_{N-1})$ *is a linearly-dependent list of functions, then* $\mathcal{W}(\vec{\varphi})$ *is the zero-function.*

Conversely, when each φ_j *is analytic* [is a power-series fnc]*: If* $\mathcal{W}(\vec{\varphi})$ *is the zero-function, then* $\vec{\varphi}$ *is linearly-dependent.* ♢

VoP algorithm ["Variation of Parameters"]

Step VoP0. Consider target fnc $G()$ and monic complex-polynomial

$$q(z) := z^N + C_{N-1}z^{N-1} + \ldots + C_1 z^1 + C_0 z^0.$$

The polynomial determines a differential operator $\boxed{\mathsf{L} := q(\mathbf{D})}$. We seek the general solution, y, to $\mathsf{L}(y) = G$, i.e,

10.3: $$y^{(N)} + C_{N-1}y^{(N-1)} + \ldots + C_1 y' + C_0 y = G.$$

Step VoP1. Use CCLDE to find a linearly-independent list $\vec{\mathbf{Y}} := (Y_0, \dots, Y_{N-1})$ of fncs, with each Y_j satisfying $\mathsf{L}(Y_j) = 0$.

We seek a list $\vec{\mathbf{f}} := (f_0, \dots, f_{N-1})$ of fncs, so that this sum-function

10.4: $$s := \sum_{j=0}^{N-1} f_j \cdot Y_j$$

satisfies (10.3); that is, that $\mathsf{L}(s) = G$.

VoP2. Let $h_j := f_j'$. Define column-vectors

10.5: $$\mathsf{H} := \begin{bmatrix} h_0 \\ \vdots \\ h_{N-2} \\ h_{N-1} \end{bmatrix} \quad\text{and}\quad \mathsf{T} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G \end{bmatrix}.$$

Compute the Wronskian matrix $\mathsf{M} := \mathsf{WM}(\vec{\mathbf{Y}})$. Then H satisfies

†: $$\underbrace{\mathsf{M}}_{N\times N}\cdot\underbrace{\mathsf{H}}_{N\times 1} = \underbrace{\mathsf{T}}_{N\times 1}.$$

Solve for each h_j, either via computing the inverse-matrix of M, or via Cramer's Rule (theorem, actually).

VoP3. Anti-differentiate to compute each function $f_j := \int h_j$. Then, parametrized by a list of numbers $\vec{\boldsymbol{\alpha}} := (\alpha_0, \alpha_1, \dots, \alpha_{N-1})$, the *general soln* to (10.3) is

10.6: $$y_{\vec{\alpha}} := \left[\sum_{j=0}^{N-1} \alpha_j Y_j\right] + \left[\sum_{j=0}^{N-1} f_j \cdot Y_j\right].$$

Why does this nifty VoP algorithm work? Matrix-eqn (†) says, for $k = 0, 1, \dots, N-2$, that

‡(k): $$\sum_{j=0}^{N-1} h_j \cdot Y_j^{(k)} = 0.$$

Differentiating (10.4) says that s' equals

$$\sum_{j=0}^{N-1} [f_j' Y_j + f_j Y_j'] \overset{\text{note}}{=\!=} \left[\sum_{j=0}^{N-1} h_j Y_j\right] + \left[\sum_{j=0}^{N-1} f_j Y_j'\right].$$

By (‡(0)), then,

$$s' = \sum_{j=0}^{N-1} f_j Y_j'.$$

Differentiating again, then using (‡(1)), shows that

$$s'' = \sum_{j=0}^{N-1} f_j Y_j''.$$

Continuing, we conclude, for $k = 1, 2, \ldots, N-1$, that

$$*: \qquad s^{(k)} = \sum_{j=0}^{N-1} f_j Y_j^{(k)}.$$

Differentiating one last time produces

$$**: \qquad s^{(N)} = \underbrace{\Bigl[\sum_{j=0}^{N-1} h_j Y_j^{(N-1)}\Bigr]}_{=:\ \text{Bob}} + \Bigl[\sum_{j=0}^{N-1} f_j Y_j^{(N)}\Bigr].$$

Eqns (10.4), (∗) and (∗∗), together, imply that

$$\mathsf{L}(s) = \text{Bob} + \Bigl[\sum_{j=0}^{N-1} f_j \cdot \mathsf{L}(Y_j)\Bigr].$$

But each $\mathsf{L}(Y_j) = 0$. Our end result is that

$$10.7: \qquad \mathsf{L}(s) = \sum_{j=0}^{N-1} h_j Y_j^{(N-1)}.$$

And $\mathsf{L}(s) \overset{\text{want}}{=\!=} G$. Hence we need to require that H satisfies $\sum_{j=0}^{N-1} h_j Y_j^{(N-1)} = G$. And this is precisely what the bottom row of matrix-eqn (†) says.

The Upshot. This method *indeed* computes an s with $\mathsf{L}(s) = G$ *if* there is a column-vector H fulfilling (†). Happily, our Wronskian L.I. Thm (10.2) guarantees that M is invertible, since we chose $\vec{\mathbf{Y}}$ to be linearly-independent. So define $\boxed{\mathsf{H} := \mathsf{M}^{-1}\mathsf{T}}$. ♦

10.8: *VoP case N=2.* Here, our matrix eqn is

$$\underbrace{\begin{bmatrix} Y_0 & Y_1 \\ Y_0' & Y_1' \end{bmatrix}}_{\mathsf{M}} \cdot \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix}.$$

So $D := \text{Det}(\mathsf{M}) = [Y_0 Y_1'] - [Y_0' Y_1]$. Hence

$$h_0 = -Y_1 \cdot \tfrac{G}{D} \quad \text{and} \quad h_1 = Y_0 \cdot \tfrac{G}{D}. \quad \text{Thus}$$

$$\begin{aligned} y_{\alpha,\beta} &= [\alpha + \textstyle\int h_0]Y_0 + [\beta + \int h_1]Y_1 \\ &= [\alpha Y_0 + \beta Y_1] + [\textstyle\int h_0]Y_0 + [\int h_1]Y_1 \end{aligned}$$

is our general soln (10.3). □

10.9: General VoP Alg. *When the DE is "not monic", i.e*

$$10.3^*: \qquad \mathrm{C}_N y^{(N)} + \mathrm{C}_{N-1} y^{(N-1)} + \ldots + \mathrm{C}_1 y' + \mathrm{C}_0 y = G,$$

then steps VoP1,2,3 remain, except that the target col-vec becomes

$$10.5^*: \qquad \mathsf{T} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G/\mathrm{C}_N \end{bmatrix}.$$

The algorithm persists if the C_j coefficients are allowed to be functions of the independent variable. The only step that get harder is VoP1 [finding fncs sent to zero by the Diff-Op] since CCLDE no longer applies.♢

CC-VoP Example. DE #7[P] *193*.NSS6 is

$$10.10: \qquad y'' + 4y' + 4y = \mathrm{e}^{-2t} \cdot \log(t).$$

Define expressions

$$\mathcal{R} := \mathrm{e}^{-2t} \quad \text{and} \quad \mathcal{L} := \log(t). \quad \text{Note } \mathcal{R}' = -2\mathcal{R}.$$

Our target fnc is $G := \mathcal{R} \cdot \mathcal{L}$.

VoP1. The Op's aux.poly is $z^2 + 4z + 4 = [z - -2]^2$. So

$$Y_0 := \mathcal{R} \quad \text{and} \quad Y_1 := t\mathcal{R}.$$

is an L.I. pair of fncs annihilated by the DiffOp.

VoP2. Differentiating w.r.t t,

$$\begin{aligned} Y_0' = -2\mathcal{R} \quad \text{and} \quad Y_1' &= 1 \cdot \mathcal{R} + t \cdot [-2\mathcal{R}] \\ &= [1 - 2t]\mathcal{R}. \end{aligned}$$

So the Wronskian-determinant $D := \mathcal{W}(Y_0, Y_1)$ is

$$D = \mathcal{R} \cdot [1 - 2t]\mathcal{R} - t\mathcal{R} \cdot [-2\mathcal{R}] \overset{\text{note}}{=\!=} \mathcal{R}^2.$$

Using the convenient (10.8),

$$\begin{aligned} h_0 &= -\tfrac{1}{D} Y_1 G = -\mathcal{R}^{-2} \cdot t\mathcal{R} \cdot \mathcal{R}\mathcal{L} \overset{\text{note}}{=\!=} -t\mathcal{L} \quad \text{and} \\ h_1 &= \tfrac{1}{D} Y_0 G = \mathcal{R}^{-2} \cdot \mathcal{R} \cdot \mathcal{R}\mathcal{L} \overset{\text{note}}{=\!=} \mathcal{L}. \end{aligned}$$

VoP3. Computing anti-derivatives,

$$f_0 = \int [-t \cdot \log(t)]\, dt = \tfrac{1}{4}t^2 \cdot [1 - 2\log(t)] \quad \text{and}$$
$$f_1 = \int \log(t)\, dt = t \cdot [\log(t) - 1]\,.$$

Equidimensional operators

An "***equidimensional operator*** of order 2" [***EquiDim-Op***] has form

$$\mathsf{E}(y) \;:=\; At^2 y'' + Bty' + Cy$$

where $A_{\neq 0}, B, C \in \mathbb{C}$ and $y{=}y(t)$. [See §4.7–ZW8, where such operators are effectively called ***Cauchy-Euler operators***.]

A ***Generalized EquiDim-Op*** [abbrev. ***Gen-EquiDim-Op***] has form

$$\mathsf{E}(y) \;:=\; At^{\Lambda+2} y'' + Bt^{\Lambda+1} y' + Ct^{\Lambda} y$$

for some $\Lambda \in \mathbb{C}$.

For a number $\mathbf{r} \in \mathbb{C}$, observe that

11a:
$$\begin{aligned} \mathsf{E}(t^{\mathbf{r}}) &= At^{\Lambda+2} \cdot \mathbf{r}[\mathbf{r}-1]t^{\mathbf{r}-2} \\ &\quad + Bt^{\Lambda+1} \cdot \mathbf{r}t^{\mathbf{r}-1} + Ct^{\Lambda} \cdot t^{\mathbf{r}} \\ &= t^{\Lambda+\mathbf{r}} \cdot q(\mathbf{r})\,, \end{aligned}$$

$$\textit{where}\quad q(z) := Az^2 + [B - A]z + C$$

is the "***characteristic polynomial*** of E".

The quadratic formula gives the roots, $\mathbf{r}_1$ and $\mathbf{r}_2$, of q. Hence E sends $t^{\mathbf{r}_1}$ and $t^{\mathbf{r}_2}$ to the zero-fnc. If $\text{Discr}(q) = 0$, i.e $\mathbf{r}_1 = \mathbf{r}_2$, then we can use the below Reduction-of-order method. This will give us a fnc $s()$ which is L.I of $t^{\mathbf{r}_1}$ s.t $\mathsf{E}(s) = 0$.

Roo algorithm ["Reduction of order."]

Consider coefficient-fncs $C_j = C_j(t)$, defining linear Diff-Op

$$\mathsf{L}(\varphi) := \varphi'' + C_1\varphi' + C_0\varphi\,.$$

Suppose we have a fnc Y, which is not identically-zero, satisfying $\mathsf{L}(Y) = 0$.

Given a target fnc G, we seek a fnc s which is linearly-indep of Y, s.t $\mathsf{L}(s) = G$.

Step Roo1. Compute anti-deriv $B_1 := \int C_1$, then let

$$M := Y^2 \cdot \mathrm{e}^{B_1}\,.$$

Roo2. *If* G is identically-zero, then set

$$h := \frac{1}{M} \overset{\text{note}}{=\!=} \frac{1}{Y^2} \cdot \mathrm{e}^{-B_1}\,.$$

Otherwise, define

$$h := \frac{1}{M} \cdot \int \frac{MG}{Y}\,.$$

Roo3. Compute anti-derivative

$$f := \int h\,. \quad \text{Finally, define} \quad s := Y \cdot f\,.$$

Why does the Roo algorithm work? We solve for a fnc f such that $s := Y \cdot f$ satisfies $\mathsf{L}(s) = 0$.

Let $h := f'$. Then

$$s' = Y'f + Yf' \overset{\text{note}}{=\!=} Y'f + Yh\,.$$

So

$$\begin{aligned} s'' &= Y''f + Y'f' + Y'h + Yh' \\ &= [Y''f] + [Yh' + 2Y'h]\,. \end{aligned}$$

Thus $\mathsf{L}(s) \overset{\text{def}}{=\!=} s'' + C_1 s' + C_0 s$ equals

$$\begin{aligned} &\mathsf{L}(Y) \cdot f + [Yh' + 2Y'h] + C_1 Yh \\ &\overset{\text{since } \mathsf{L}(Y)=0}{=\!=\!=\!=} Yh' + [2Y' + C_1 Y]h\,. \end{aligned}$$

Consequently, h satisfies $Yh' + [2Y' + C_1 Y]h = G$. Dividing by Y yields FOLDE

11b:
$$h' + [2\tfrac{Y'}{Y} + C_1]h = \tfrac{G}{Y}\,.$$

Note $\frac{Y'}{Y} = [\log(Y)]'$, so $2\frac{Y'}{Y} = [2\log(Y)]' = [\log(Y^2)]'$. Thus the FOLDE anti-deriv of the coeff-fnc is

$$B := \int [2\tfrac{Y'}{Y} + C_1] \overset{\text{note}}{=\!=} \log(Y^2) + B_1\,.$$

Hence the FOLDE multiplier-fnc is

$$M := Y^2 \cdot \mathrm{e}^{B_1}\,.$$

The last FOLDE-step gives the two formulas in **Roo2**. ♦

Equidim + Roo Example. For t positive, let's find the gen.soln $\varphi=\varphi(t)$ of DE

11c: $$t^2\varphi'' - 5t\varphi' + 9\varphi = 0\,.$$

Operator $\mathsf{E}(y) := t^2y'' - 5ty' + 9y$ is equidimensional. Its char-poly is, from (11a),

$$z^2 + [-5-1]z + 9 = z^2 - 6z + 9 = [z-3]^2.$$

Hence $Y(t) := t^3$ is sent to 0 by $\mathsf{E}()$. Checking:

$$\mathsf{E}(t^3) = t^2\cdot[3\cdot 2t] - 5t\cdot[3t^2] + 9\cdot[t^3]$$
$$= 3t^3\cdot[2-5+3] \overset{\text{note}}{=\!=} 0\,.$$

Roo1. We make a monic version of the operator by defining $\mathsf{L} := [1/t^2]\cdot\mathsf{E}$, i.e

$$\mathsf{L}(y) := y'' - \tfrac{5}{t}y' + \tfrac{9}{t^2}y\,.$$

With $C_1(t) := -\frac{5}{t}$, then

$$B_1(t) := \int^t C_1 = -5\cdot\log(t)\,.$$

So $\exp(B_1(t))$ equals t^{-5}. Thus

$$M(t) := [t^3]^2\cdot t^{-5} = t\,.$$

Roo2. Our target fnc is the zero-fnc, so we simply compute

$$h(t) := 1/M(t) = 1/t\,.$$

Roo3. Antidifferentiating gives $f := \int h = \log$. Consequently, the theory tells us that

11d: $$s(t) := f(t)\cdot[Y(t)] \overset{\text{note}}{=\!=} \log(t)\cdot t^3$$

is sent to the zero-fnc by L [hence also by E], *and* is L.I of $Y(t){=}t^3$. *Did you check?*

Checking: Let $\mathcal{G} := \log(t)$. Then

$$\begin{aligned}
s &= \mathcal{G}t^3\,. && \text{Thus}\\
s' &= \tfrac{1}{t}t^3 + \mathcal{G}\cdot 3t^2 = [1+3\mathcal{G}]t^2\,, && \text{so}\\
s'' &= \tfrac{3}{t}\cdot t^2 + [1+3\mathcal{G}]\cdot 2t = [5+6\mathcal{G}]t\,. && \textit{Summing}\\
9s &= [\,0 + 9\mathcal{G}]\,t^3 && \text{with}\\
-5ts' &= [-5 - 15\mathcal{G}]\,t^3 && \text{and with}\\
t^2s'' &= [\,5 + 6\mathcal{G}]\,t^3
\end{aligned}$$

is the defn of $\mathsf{E}(s)$. The sum is indeed zero.

Operators

We already know operators $\mathbf{D}$ and $\mathbf{I}{=}\mathbf{D}^0$. Use $\mathbf{0}$ for the ***zero-operator***. I.e, $\mathbf{0}(y) = 0$ [the zero-fnc] for *every* fnc y.

Translation. Use $\mathbf{T}$ for the family of ***translation operators***. For a *number* $\alpha\in\mathbb{C}$, operator $\mathbf{T}_\alpha$ acts on an arbitrary fnc φ to produce a new function, which is φ but translated [to the right] by α. E.g,

$$\mathbf{T}_5(\varphi) = [t\mapsto\varphi(t-5)]\,.$$

[So $\mathbf{T}_0 = \mathbf{I}$.] For instance, we know that cos() and sin() are translates of each other. Specifically

$$\mathbf{T}_{\pi/2}(\cos) = \sin \quad\text{and}\quad \mathbf{T}_{-\pi/2}(\sin) = \cos\,.$$

Recall that a [complex] number β is j "a ***period*** of f" if $\mathbf{T}_\beta(f) = f$. E.g, $\mathbf{T}_{2\pi}(\cos) = \cos$.

Multiply-operator. Use $\mathbf{M}$ for the family of ***multiply operators***. So $\mathbf{M}_5$ multiplies its argument by [the constant fnc] 5, e.g $\mathbf{M}_5(y) = 5y$, i.e $\mathbf{M}_5 = 5\mathbf{I}$. More generally, for a *function* f, let $\mathbf{M}_f(y) := f{\cdot}y$. That is

$$[\mathbf{M}_f(y)](t) = f(t){\cdot}y(t) \overset{\text{abbrev.}}{=\!=} f(t){\cdot}y\,.$$

By slight abuse of notation, we can also use an *expression* as a subscript, e.g, $\mathbf{M}_{t^2}(y)$ means t^2y; well, actually, the *function* $[t\mapsto t^2y(t)]$.

12.1: Lemma. *Easily, $\mathbf{M}_0 = \mathbf{0}$ and $\mathbf{M}_1 = \mathbf{I} = \mathbf{T}_0$. Also:*

i: Each $\mathbf{T}_\alpha$ is invertible, and $[\mathbf{T}_\alpha]^{-1} = \mathbf{T}_{-\alpha}$.

ii: When f is no-where zero, then $\mathbf{M}_f$ is invertible, with inverse $\mathbf{M}_{1/f}$. ◊

Commutation relations. Boldface symbols

$$\mathbf{D},\ \mathbf{I},\ \mathbf{0},\ \mathbf{T}_?\ \text{and}\ \mathbf{M}_?$$

denote operators with fixed meanings. We'll use san-serif letters $\mathsf{L}, \mathsf{P}, \mathsf{Q}, \mathsf{U}, \mathsf{V}$ for *operator-variables*; variables that we can assign operators to. Make the convention that, e.g, VP means $\mathsf{V}\circ\mathsf{P}$, and V^3 means $\mathsf{V}\circ\mathsf{V}\circ\mathsf{V}$. Hence $\mathsf{V}^0 = \mathbf{I}$.

Use "$\leftrightarrows$" to mean 'commutes with'. So $\mathsf{U} \leftrightarrows \mathsf{V}$ means that $\mathsf{UV} = \mathsf{VU}$.

12.2: Op-commutation lemma. *Here $\alpha,\beta \in \mathbb{C}$, and f,g are functions.*

a: *Translation-ops are linear and commute with each other. Indeed, $\mathbf{T}_\beta\mathbf{T}_\alpha = \mathbf{T}_{\beta+\alpha} = \mathbf{T}_\alpha\mathbf{T}_\beta$.*

b: *Multiply-ops are linear and commute with each other. Specifically, $\mathbf{M}_f\mathbf{M}_g = \mathbf{M}_{f\cdot g} = \mathbf{M}_g\mathbf{M}_f$.*

c: *Each translation-op commutes with $\mathbf{D}$.*

d: *Operator $\mathbf{M}_g$ commutes with $\mathbf{D}$ IFF g is constant. The general commutation relation is*

$$\mathbf{D}\,\mathbf{M}_g = \mathbf{M}_{g'} + [\mathbf{M}_g\,\mathbf{D}], \quad \text{E.g,}$$
$$\mathbf{D}\,\mathbf{M}_t = \mathbf{I} + [\mathbf{M}_t\,\mathbf{D}].$$

e: *Operator $\mathbf{M}_f$ commutes with $\mathbf{T}_\beta$ IFF β is a period of f. The commutation relation [written with composition symbol $\circ$, for clarity] is*

$$\mathbf{T}_\beta \circ \mathbf{M}_f = \mathbf{M}_{\mathbf{T}_\beta(f)} \circ \mathbf{T}_\beta. \qquad \diamond$$

Proof of (c). Exercise. Use the Chain rule. ♦

Pf of (d). Well, $\mathbf{D}\mathbf{M}_g(y) = \mathbf{D}(g\cdot y) = g'y + gy'$, which equals $\mathbf{M}_{g'}(y) + [\mathbf{M}_g\mathbf{D}](y)$, i.e, $\big[\mathbf{M}_{g'} + [\mathbf{M}_g\,\mathbf{D}]\big](y)$. ♦

Pf of (e). Let $f_\beta := \mathbf{T}_\beta(f)$ and, for y an arbitrary fnc, let $y_\beta := \mathbf{T}_\beta(y)$. So $\mathbf{M}_{\mathbf{T}_\beta(f)} = \mathbf{M}_{f_\beta}$. Thus

$$*: \quad \mathbf{M}_{f_\beta}\mathbf{T}_\beta(y) = f_\beta\cdot y_\beta = \mathbf{T}_\beta(f\cdot y) = \mathbf{T}_\beta\mathbf{M}_f(y),$$

yielding the stated commutation relation.

Now, if $\mathbf{M}_\beta \leftrightarrows \mathbf{T}_f$ then $\mathbf{M}_f\mathbf{T}_\beta = \mathbf{T}_\beta\mathbf{M}_f = \mathbf{M}_{f_\beta}\mathbf{T}_\beta$, by (∗). Evaluating at the constant function 1 shows that $\mathbf{M}_{f_\beta}(1) = \mathbf{M}_f(1)$. Consequently $f_\beta = f$. ♦

Example. *Numerical* expressions can be simplified [e.g 7+1 equals 8], as can *fnc* expresions [e.g $\cos^2 + \sin^2$ equals the constant-fnc 1], and so too can *operator* expressions. For example, the above lemma allows this

$$\mathbf{M}_5\mathbf{D}\mathbf{M}_{\sin}\mathbf{D} \overset{\text{by (12.2d)}}{=\!=} \mathbf{M}_5[\mathbf{M}_{\cos} + \mathbf{M}_{\sin}\mathbf{D}]\mathbf{D}$$
$$\overset{\text{by (12.2b)}}{=\!=} \mathbf{M}_{5\cos}\mathbf{D} + \mathbf{M}_{5\sin}\mathbf{D}^2.$$

Another: Note that

$$\mathbf{T}_{\pi/2}\,\mathbf{M}_{\cos} \overset{\text{by (e)}}{=\!=} \mathbf{M}_{\mathbf{T}_{\pi/2}(\cos)}\,\mathbf{T}_{\pi/2} = \mathbf{M}_{\sin}\,\mathbf{T}_{\pi/2}.$$

Hence
$$\mathbf{T}_{\pi/2}\,\mathbf{M}_{\cos}\,\mathbf{T}_{3\pi/2} = \mathbf{M}_{\sin}\,\mathbf{T}_{2\pi}. \qquad \square$$

DiffOps and the Elimination Method

With U.Fs $x{=}x(t)$ and $y{=}y(t)$, and targets $H_j{=}H_j(t)$, consider system

$$\textbf{13a:}\quad \begin{aligned} \mathsf{P}(x) + \mathsf{U}(y) &= H_1 \quad \text{and} \\ \mathsf{Q}(x) + \mathsf{V}(y) &= H_2, \end{aligned}$$

where $\mathsf{P},\mathsf{Q},\mathsf{U},\mathsf{V}$ are DiffOps.

13b: Elimination Thm. [*Notation from* (13a)] *If* U *and* V *are linear,* ***and*** $\mathsf{U} \leftrightarrows \mathsf{V}$ [i.e $\mathsf{VU} = \mathsf{UV}$], *then x satisfies*

$$\mathsf{L}(x) = G, \quad \text{where}$$
$$\mathsf{L} := \mathsf{VP} - \mathsf{UQ} \quad \text{and} \quad G := \mathsf{V}(H_1) - \mathsf{U}(H_2).$$

Proof. Exercise. ◊

Matrix exponential

Fix posint N and let MAT denote the set of $N{\times}N$ matrices. Use $\mathbf{0},\mathbf{I} \in$ MAT for the zero-matrix and identity-matrix. For $\mathsf{M} \in$ MAT, define

$$*: \quad \exp(\mathsf{M}) := e^{\mathsf{M}} := \sum_{k=0}^{\infty}\Big[\tfrac{1}{k!}\cdot \mathsf{M}^k\Big].$$

14.1: *MiniChallenge: MatrixExp by hand.* Fix an $\boldsymbol{\alpha} \in \mathbb{C}$ and set $\mathsf{S} := \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\alpha} \\ 0 & 0 \end{bmatrix}$. Compute e^{S} and $e^{t\mathsf{S}}$. □

Soln. Let's do this for $\boldsymbol{\alpha} := 5$; we'll see the pattern.

Always, S^0 is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. And for $k \in \mathbb{Z}_+$, easily
$$\mathsf{S}^k = \begin{bmatrix} 5^k & 5^k \\ 0 & 0 \end{bmatrix}.$$

Writing S^0 in the same pattern, then,

$$\mathsf{S}^0 = \begin{bmatrix} 5^0 & 5^0 \\ 0 & 0 \end{bmatrix} + \mathsf{C}, \quad \text{where } \mathsf{C} := \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

Applying defn (∗), our $\mathsf{e}^{t\mathsf{S}}$ equals

$$\frac{1}{0!}\cdot t^0\cdot \mathsf{C} + \sum_{k=0}^{\infty}\frac{1}{k!}\cdot t^k\cdot\begin{bmatrix}5^k & 5^k\\ 0 & 0\end{bmatrix}$$
$$= \mathsf{C} + \begin{bmatrix}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot t^k5^k & \sum_{k=0}^{\infty}\frac{1}{k!}\cdot t^k5^k\\ 0 & 0\end{bmatrix}.$$

This $\sum_{k=0}^{\infty}\frac{1}{k!}t^k5^k$ is just the Taylor series of e^{5t}, so

$$\mathsf{e}^{t\mathsf{S}} = \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\mathsf{e}^{5t} & \mathsf{e}^{5t}\\ 0 & 0\end{bmatrix} \overset{\text{note}}{=\!=} \begin{bmatrix}\mathsf{e}^{5t} & \mathsf{e}^{5t}-1\\ 0 & 1\end{bmatrix}.$$

Nothing was special about the complex number 5, so for our original S we conclude that

14.2: $$\mathsf{e}^{t\mathsf{S}} = \exp\left(t\cdot\begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\alpha}\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}\mathsf{e}^{\boldsymbol{\alpha} t} & \mathsf{e}^{\boldsymbol{\alpha} t}-1\\ 0 & 1\end{bmatrix}.$$

Plugging in t=1 gives

14.3: $$\mathsf{e}^{\mathsf{S}} = \exp\left(\begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\alpha}\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}\mathsf{e}^{\boldsymbol{\alpha}} & \mathsf{e}^{\boldsymbol{\alpha}}-1\\ 0 & 1\end{bmatrix}.$$

By the way, at t=0, note that (14.2) is the identity matrix. *Coincidence? Space aliens? I think not!* ♦

Defn. An $N\times N$ matrix M is ***nilpotent*** if $\exists k\in\mathbb{Z}_+$ such that $\boxed{\mathsf{M}^k = \mathbf{0}_{N\times N}}$. The smallest such k is the **"*nilpotency degree* of M"**. Always:

The nilpotency degree of a nilpotent $N\times N$ matrix is $\leqslant N$.

Matrices A,B ∈ MAT are ***similar***[♡8] [to each other] if *there exists*[♡8] an invertible U ∈ MAT such that

$$\mathsf{B} = \mathsf{U}\mathsf{A}\mathsf{U}^{-1}. \quad \text{Write this relation as} \quad \mathsf{A}\sim\mathsf{B}.$$

Easily, relation ∼ is an equivalence relation.

This A is ***diagonalizable*** if A is similar to *some* diagonal matrix.

Read A ⇆ B as "A commutes with B" i.e, AB = BA.□

15: MatExp theorem. *Series (∗) always converges. Moreover, for scalars α,β and A,B,R,D ∈ MAT:*

♡8 We also say "A *and* B *are conjugate to each other*", or "*matrix* U *conjugates* A *to* B." In general, U is *not* unique; there could be an invertible W≠U s.t $\mathsf{W}\mathsf{A}\mathsf{W}^{-1} = \mathsf{B} = \mathsf{U}\mathsf{A}\mathsf{U}^{-1}$.

a: *Exp() of a diagonal matrix* $\mathsf{D} := \begin{bmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_N\end{bmatrix}$ *yields diagonal matrix*

$$\mathsf{e}^{\mathsf{D}} = \begin{bmatrix}\mathsf{e}^{\alpha_1} & & \\ & \ddots & \\ & & \mathsf{e}^{\alpha_N}\end{bmatrix}, \text{ so } \mathsf{e}^{t\mathsf{D}} = \begin{bmatrix}\mathsf{e}^{\alpha_1 t} & & \\ & \ddots & \\ & & \mathsf{e}^{\alpha_N t}\end{bmatrix}.$$

Thus $\mathsf{e}^{\mathbf{0}} = \mathbf{I}$.

b: *If matrices* A ⇆ B, *then* $\mathsf{e}^{\mathsf{A}+\mathsf{B}} = \mathsf{e}^{\mathsf{A}}\cdot\mathsf{e}^{\mathsf{B}}$.

Hence, every e^{R} *is invertible, and* $[\mathsf{e}^{\mathsf{R}}]^{-1} = \mathsf{e}^{-\mathsf{R}}$. *Also,* $\mathsf{e}^{[\alpha+\beta]\mathsf{R}} = \mathsf{e}^{\alpha\mathsf{R}}\cdot\mathsf{e}^{\beta\mathsf{R}}$.

c: *For* R *arbitrary and* U *invertible, let* $\mathsf{D} := \mathsf{U}^{-1}\mathsf{R}\mathsf{U}$; *so* $\mathsf{R} := \mathsf{U}\mathsf{D}\mathsf{U}^{-1}$. *Then* $\boxed{\mathsf{e}^{\mathsf{U}\mathsf{D}\mathsf{U}^{-1}} = \mathsf{U}\,\mathsf{e}^{\mathsf{D}}\,\mathsf{U}^{-1}}$. *I.e,*

[Conjugation by U] *commutes-with* exp().

From above, $t\mathsf{R} = \mathsf{U}\cdot t\mathsf{D}\cdot\mathsf{U}^{-1}$, *since scalars commute with matrices, and thus*

$$\mathsf{e}^{t\mathsf{R}} = \mathsf{U}\cdot\mathsf{e}^{t\mathsf{D}}\cdot\mathsf{U}^{-1}.$$

d: *Function* $[t\mapsto\mathsf{e}^{t\mathsf{R}}]$ *is differentiable, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{e}^{t\mathsf{R}} = \mathsf{R}\cdot\mathsf{e}^{t\mathsf{R}} = \mathsf{e}^{t\mathsf{R}}\cdot\mathsf{R}.$$ ◇

16.1: *MiniChallenge: CEX to* $\mathsf{e}^{\mathsf{A}+\mathsf{B}} = \mathsf{e}^{\mathsf{A}}\mathsf{e}^{\mathsf{B}}$.

Find 2×2 matrices A and B which form a counterexample (abbrev. CEX) to assertion $\mathsf{e}^{\mathsf{A}+\mathsf{B}} = \mathsf{e}^{\mathsf{A}}\mathsf{e}^{\mathsf{B}}$. □

Soln. MatExp (15b) tells us to search among non-commuting pairs; that is, AB ≠ BA. About the simplest non-commuting pair there is, is

16.2: $$\mathsf{A} := \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \text{ and } \mathsf{B} := \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

Is this pair is a CEX ?! (*This is so exciting!*)

Since A is a diagonal matrix, our (15a) says

$$\mathsf{e}^{\mathsf{A}} = \begin{bmatrix}\mathsf{e}^1 & 0\\ 0 & \mathsf{e}^0\end{bmatrix} = \begin{bmatrix}\mathsf{e} & 0\\ 0 & 1\end{bmatrix}.$$

Our B has nilpotency-degree 2 [i.e $\mathsf{B}^2 = \mathbf{0}_{2\times 2}$], so

$$\mathsf{e}^{\mathsf{B}} = \frac{1}{0!}\mathbf{I} + \frac{1}{1!}\mathsf{B} = \mathbf{I} + \mathsf{B} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

Before even computing $\mathsf{e}^{\mathsf{A}+\mathsf{B}}$, note that

16.3: $\mathsf{e}^{\mathsf{A}} \cdot \mathsf{e}^{\mathsf{B}} = \begin{bmatrix} \mathsf{e} & \mathsf{e} \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} \mathsf{e} & 1 \\ 0 & 1 \end{bmatrix} = \mathsf{e}^{\mathsf{B}} \cdot \mathsf{e}^{\mathsf{A}}$.

Since A+B *does* equal B+A, this implies that –in one order or the other– we indeed have a CEX.

To find out which, we compute e^{S}, where the sum

$$\mathsf{S} := \mathsf{A} + \mathsf{B} \stackrel{\text{note}}{=\!=} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Our previous work, (14.3), says that exponential

16.4: $\mathsf{e}^{\mathsf{S}} = \begin{bmatrix} \mathsf{e}^1 & \mathsf{e}^1 - 1 \\ 0 & 1 \end{bmatrix} \stackrel{\text{note}}{=\!=} \begin{bmatrix} \mathsf{e} & \mathsf{e} - 1 \\ 0 & 1 \end{bmatrix}$.

So: *No two of* $\mathsf{e}^{\mathsf{A}}\mathsf{e}^{\mathsf{B}}$, $\mathsf{e}^{\mathsf{B}}\mathsf{e}^{\mathsf{A}}$, $\mathsf{e}^{\mathsf{A}+\mathsf{B}}$ *are equal.* ♦

17: Lemma. *Consider a mystery vector-valued function*

$$\mathsf{Z}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}.$$

Suppose Z *satisfies* $\mathsf{Z}' = \mathsf{R} \cdot \mathsf{Z}$, *where* R *is an* $N \times N$ *matrix of numbers. Then each column,* Y, *of* $\mathsf{e}^{t\mathsf{R}}$ *satisfies* $\mathsf{Y}' = \mathsf{R} \cdot \mathsf{Y}$. *Hence the soln to* $\mathsf{Z}' = \mathsf{RZ}$ *is*

17a: $\mathsf{Z}(t) = \mathsf{e}^{t\mathsf{R}} \cdot \mathsf{Z}(0)$. ◊

18.1: *Diagonalizable Example.* Unknown fncs $x=x(t)$ and $y=y(t)$ satisfy

18.2:
$$x' = \text{-}5x + 9y \quad \text{and}$$
$$y' = \text{-}6x + 10y.$$

So the coeff-matrix is $\mathsf{R} := \begin{bmatrix} \text{-}5 & 9 \\ \text{-}6 & 10 \end{bmatrix}$. Magic [or a nice guy] produces a conjugating matrix $\mathsf{U} := \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ s.t

$$\mathsf{D} := \mathsf{U}^{-1}\mathsf{R}\mathsf{U} \stackrel{\text{note}}{=\!=} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

is a diagonal matrix.[♡9] Hence $\mathsf{e}^{t\mathsf{R}} = \mathsf{U}\,\mathsf{e}^{t\mathsf{D}}\,\mathsf{U}^{-1}$. I.e,

18.3:
$$\mathsf{e}^{t\mathsf{R}} = \mathsf{U}\begin{bmatrix} \mathsf{e}^t & 0 \\ 0 & \mathsf{e}^{4t} \end{bmatrix}\mathsf{U}^{-1}$$
$$\stackrel{\text{note}}{=\!=} \left[\begin{array}{c|c} 3\mathsf{e}^t - 2\mathsf{e}^{4t} & \text{-}3\mathsf{e}^t + 3\mathsf{e}^{4t} \\ 2\mathsf{e}^t - 2\mathsf{e}^{4t} & \text{-}2\mathsf{e}^t + 3\mathsf{e}^{4t} \end{array}\right].$$

Our general soln, parameterized by numbers α and β, is

‡:
$$x_{\alpha,\beta}(t) = [3\mathsf{e}^t - 2\mathsf{e}^{4t}]\cdot\alpha + [\text{-}3\mathsf{e}^t + 3\mathsf{e}^{4t}]\cdot\beta,$$
$$y_{\alpha,\beta}(t) = [2\mathsf{e}^t - 2\mathsf{e}^{4t}]\cdot\alpha + [\text{-}2\mathsf{e}^t + 3\mathsf{e}^{4t}]\cdot\beta.$$

As they must, $\alpha = x(0)$ and $\beta = y(0)$. □

19.1: *Nilpotent Example.* UFs $x = x(t)$ and $y = y(t)$ satisfy

19.2:
$$x' = 2x - y \quad \text{and}$$
$$y' = 4x - 2y.$$

Hence the coeff-matrix is $\mathsf{R} := \begin{bmatrix} 2 & \text{-}1 \\ 4 & \text{-}2 \end{bmatrix}$. Note $\mathsf{R}^2 = \mathbf{0}$. [I.e, R has nilpotency-degree 2.] Thus

19.3: $\mathsf{e}^{t\mathsf{R}} = \mathbf{I} + t\mathsf{R} \stackrel{\text{note}}{=\!=} \left[\begin{array}{c|c} 1 + 2t & -t \\ 4t & 1 - 2t \end{array}\right]$.

Therefore, the soln to (19.2) is

‡: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c|c} 1 + 2t & -t \\ 4t & 1 - 2t \end{array}\right] \cdot \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$. □

Defn. The ***characteristic polynomial*** of an $N \times N$ matrix M is

20.1: $\wp_{\mathsf{M}}(z) := \operatorname{Det}(\mathsf{M} - z\mathbf{I})$

And the ***trace*** of M is

20.2: $\operatorname{Trace}(\mathsf{M}) := \begin{bmatrix} \text{Sum of elements on} \\ \text{main diagonal of M} \end{bmatrix}$.

Consider $\mathsf{Q} := \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$. Then $\operatorname{Trace}(\mathsf{Q}) = [\mathbf{a} + \mathbf{d}]$. And $\mathsf{Q} - z\mathbf{I} = \left[\begin{array}{c|c} \mathbf{a} - z & \mathbf{b} \\ \mathbf{c} & \mathbf{d} - z \end{array}\right]$. Hence

20.3:
$$\wp_{\mathsf{Q}}(z) = z^2 - [\mathbf{a} + \mathbf{d}]z + [\mathbf{ad} - \mathbf{bc}]$$
$$\stackrel{\text{note}}{=\!=} z^2 - \operatorname{Trace}(\mathsf{Q})\cdot z + \operatorname{Det}(\mathsf{Q}).$$

For a general $N \times N$ matrix M: If we write

$$\wp_{\mathsf{M}}(z) = [\text{-}1]^N z^N + \mathbf{\Omega}_{N-1} z^{N-1} + \ldots + \mathbf{\Omega}_0,$$

[♡9] Note that $\mathsf{U}^{-1} = \begin{bmatrix} 1 & \text{-}1 \\ \text{-}2 & 3 \end{bmatrix}$

then $\boldsymbol{\Omega}_0 = \mathrm{Det}(\mathsf{M})$ and $\boldsymbol{\Omega}_{N-1} = [-1]^{N-1}\cdot\mathrm{Trace}(\mathsf{M})$. I.e,

20.4:
$$\wp_{\mathsf{M}}(z) = [-1]^N z^N + [-1]^{N-1}\mathrm{Trace}(\mathsf{M})z^{N-1} + \boldsymbol{\Omega}_{N-2}\, z^{N-2} + \ldots + \boldsymbol{\Omega}_1\, z + \mathrm{Det}(\mathsf{M})\,.$$

Over $\mathbb{C}$, our char-poly factors as

$$\wp_{\mathsf{M}}(z) = [-1]^N\cdot[z-\alpha_1]\cdot[z-\alpha_2]\cdots[z-\alpha_N]\,.$$

This list $\alpha_1, \alpha_2, \ldots, \alpha_N$ of (possibly complex) numbers is the list of ***eigenvalues*** of M. If M is diagonalizable, then

$$\mathsf{M} \sim \begin{bmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_N\end{bmatrix}.$$

Moreover, the *only* diagonal matrices to which M is similar are those whose main diagonal is some permutation of $\alpha_1, \ldots, \alpha_N$. □

20.5: Distinct-roots Thm. *Suppose that the char-poly*

$$\wp_{\mathsf{R}}(z) = [z-\beta_1]\cdot[z-\beta_2]\cdots[z-\beta_N]\cdot[-1]^N$$

of $N\times N$ matrix R has N distinct (possibly complex) eigenvalues. $\beta_1, \ldots, \beta_N$. Then R is indeed similar[♡10] *to diagonal matrix* $\begin{bmatrix}\beta_1 & & \\ & \ddots & \\ & & \beta_N\end{bmatrix}$.

In particular, for column-vector $\mathsf{Z}(t) := \begin{bmatrix}x_1(t)\\ \vdots \\ x_N(t)\end{bmatrix}$ *satisfying DE $\mathsf{Z}' = \mathsf{R}\mathsf{Z}$, each $x_j(t)$ is simply a linear-combination of exponentials* $\mathsf{e}^{\beta_1 t}, \ldots, \mathsf{e}^{\beta_N t}$.

Letting $\mathbf{m}$ denote the maximum of the real-parts of the eigenvalues, it follows that no $x_j(t)$ can grow faster than [constant times $\mathsf{e}^{\mathbf{m}\cdot t}$], as $t\nearrow\infty$. ◊

20.6: *Example.* Consider $\mathsf{X}'(t) = \mathsf{B}\cdot\mathsf{X}(t)$, where,

$$\mathsf{B} := \begin{bmatrix}115 & 207 & -54\\ -72 & -130 & 34\\ -24 & -45 & 13\end{bmatrix}.$$

The char-poly of B is

$$\wp_{\mathsf{B}}(z) = -[z+5]\cdot[z^2-3z+8]\,.$$

The discriminant of quadratic $q(z) := z^2 - 3z + 8$ is $\mathrm{Discr}(g) = [-3]^2 - 4\cdot1\cdot8 = -23$. The roots of q are thus $S := [3+\sqrt{23}\,\boldsymbol{i}]/2$ and $\overline{S} \stackrel{\text{note}}{=\!=} [3-\sqrt{23}\,\boldsymbol{i}]/2$. So $\wp_{\mathsf{B}}(z) = -[z - -5][z-S][z-\overline{S}]$ in std form.

Since the three $\wp_{\mathsf{B}}$-roots are distinct, the Distinct-roots thm tell us that B is similar to diagonal matrix

$$\begin{bmatrix}-5 & & \\ & S & \\ & & \overline{S}\end{bmatrix}.$$

So every entry in $\mathsf{X}(t)$ is a lin-comb of $\mathsf{e}^{-5t}, \mathsf{e}^{St}, \mathsf{e}^{\overline{S}t}$. The maximum of the real-parts of $-5, S, \overline{S}$ is $\frac{3}{2}$. So no soln grows faster than $\mathrm{Const}\cdot\exp(\frac{3}{2}t)$. □

Recoding: Exchanging dimension for DE-order

For numbers $\boldsymbol{\Omega}_k \in \mathbb{C}$ and U.F $x{=}x(t)$,

21a: $$x^{(N)} = \sum_{k=0}^{N-1}\left[\boldsymbol{\Omega}_k \cdot x^{(k)}\right].$$

is an N^{th}-order DE in 1-dim'al space. Define col-vec

$$\mathsf{Z}(t) := \begin{bmatrix}x(t)\\ x'(t)\\ \vdots\\ x^{(N-2)}(t)\\ x^{(N-1)}(t)\end{bmatrix},$$

which is $N\times1$. We can restate (21a) as

21b: $\mathsf{Z}' = \mathsf{R}\cdot\mathsf{Z}$, where R is $N\times N$ matrix[♡11]

21c: $$\mathsf{R} := \begin{bmatrix}0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1\\ \boldsymbol{\Omega}_0 & \boldsymbol{\Omega}_1 & \boldsymbol{\Omega}_2 & \ldots & \boldsymbol{\Omega}_{N-2} & \boldsymbol{\Omega}_{N-1}\end{bmatrix}.$$

[The unshown entries are zero.] The solution to (21a,21b) is $\mathsf{Z}(t) = \mathsf{e}^{t\cdot\mathsf{R}}\cdot\mathsf{Z}(0) = \exp(t\cdot\mathsf{R})\cdot\mathsf{Z}(0)$. But of course, we can solve (21a) with CCLDE, and do not need the matrix-exp. Here is a more interesting example:

21d: *Recoding Example.* Imagine U.Fs $x{=}x(t)$ and $y{=}y(t)$ related by DEs

21a†: $$x''' - 2x'' - 3x + 4y = 0\,, \quad \textit{and} \qquad y' + 5x'' + 6x' + 7x - 8y = 0\,.$$

[♡10] Alas, it may be difficult to compute a conjugating matrix.

[♡11] See "Companion matrix" in Wikipedia.

We can cheerfully recode this system as a 1^{st}-order DE in 3+1 = 4 dim'al space, with U.F $\mathsf{Z}{=}\mathsf{Z}(t)$, as follows.

21b†: *Note* $\mathsf{Z}' = \mathsf{R}{\cdot}\mathsf{Z}$, *where* $\mathsf{Z} := \begin{bmatrix} y \\ x \\ x' \\ x'' \end{bmatrix}$ *and*

21c†: $$\mathsf{R} := \begin{bmatrix} 8 & -7 & -6 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & -0 & 2 \end{bmatrix}.$$

Hence the soln to (21a†, 21b†) is $\mathsf{Z}(t) = \mathsf{e}^{t\cdot\mathsf{R}} \cdot \mathsf{Z}(0)$.

In this instance, $\mathsf{e}^{t\cdot\mathsf{R}}$ is not so easy to compute, but it can be polynomially approximated by, say,

$$\exp(t{\cdot}\mathsf{R}) \approx \sum_{k=0}^{50} \left[t^k \mathsf{R}^k \big/ k! \right],$$

with easily computable error-bounds. □

MacFOLDE

Let's generalize.

22: Product-rule Lemma. *Suppose $\mathsf{A}(t)$ is a $J{\times}K$ matrix, and $\mathsf{B}(t)$ is a $K{\times}N$ matrix, each differentiable fncs. Then $J{\times}N$ matrix $\mathsf{P}(t) := \mathsf{A}(t)\cdot\mathsf{B}(t)$ is differentiable, and*

$$\mathsf{P}'(t) = [\mathsf{A}'(t){\cdot}\mathsf{B}(t)] + [\mathsf{A}(t){\cdot}\mathsf{B}'(t)]. \quad \lozenge$$

N.B. I.e, $\mathsf{P} = [\mathsf{A}'\,\mathsf{B}] + [\mathsf{A}\,\mathsf{B}']$. Matrix-mult is not commutative, so it is possible that P *fails* to equal, e.g, $[\mathsf{B}\mathsf{A}'] + [\mathsf{B}'\mathsf{A}]$. □

23.1: *Warning!* Consider the matrix-valued fnc,

$$\mathsf{B}(t) := \begin{bmatrix} 3 & 2t \\ 0 & 0 \end{bmatrix}, \quad \text{so} \quad \mathsf{B}'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Observe that

$$\mathsf{B}(t){\cdot}\mathsf{B}'(t) = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}, \quad \text{yet} \quad \mathsf{B}'(t){\cdot}\mathsf{B}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, $\mathsf{B}'(t)$ does *not* commute with $\mathsf{B}(t)$. In symbols, $\mathsf{B}' \not\leftrightarrows \mathsf{B}$. □

23.2: Lemma. *Consider a differentiable matrix-valued function $\mathsf{B}(t)$ where, for each t, our $\mathsf{B}(t)$ is an $N{\times}N$ matrix. At each time t, suppose $\mathsf{B}'(t) \leftrightarrows \mathsf{B}(t)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{e}^{\mathsf{B}(t)} = \mathsf{B}'(t)\cdot\mathsf{e}^{\mathsf{B}(t)} = \mathsf{e}^{\mathsf{B}(t)}\cdot\mathsf{B}'(t). \quad \lozenge$$

With C a matrix of numbers, and $\mathsf{B}(t) := \mathsf{C}\cdot t$, note that $\mathsf{B}'(t) = \mathsf{C}$. Hence $\mathsf{B}'(t)$ *does* commute with $\mathsf{B}(t)$.

This "constant coefficient" case is the case that interests us, so I call the following the ***Matrix-CC-FOLDE*** algorithm, abbreviated ***MacFOLDE***, even though the algorithm *does* apply whenever, for all t, matrix $\mathsf{B}'(t)$ commutes with $\mathsf{B}(t)$.

Step MFOL 0. We have U.F $\mathsf{Z}=\mathsf{Z}(t)$ which is a time-varying $N\times 1$ matrix. Write the DE in the form

$$\text{24a:} \qquad \frac{\mathrm{dZ}}{\mathrm{d}t} + [\mathsf{C}\cdot\mathsf{Z}] = \mathsf{G}(t),$$

where C is an $N\times N$ matrix of numbers, and $\mathsf{G}(t)$ is an $N\times 1$ time-varying fnc. An antiderivative of C is $\mathsf{B}(t) := \mathsf{C}\cdot t$.

Define *multiplier function*

$$\text{24b:} \qquad \mathsf{M}(t) := \mathrm{e}^{\mathsf{B}(t)} \stackrel{\text{note}}{=\!=} \mathrm{e}^{t\mathsf{C}}.$$

Observe that $\mathsf{M}'(t) = \mathsf{M}(t)\cdot\mathsf{C}$. By (22), then,

$$\begin{aligned} [\mathsf{M}(t)\cdot\mathsf{Z}]' &= [\mathsf{M}(t)\cdot\mathsf{C}\cdot\mathsf{Z}] + [\mathsf{M}(t)\cdot\mathsf{Z}'] \\ \text{**:} \qquad &= \mathsf{M}(t)\cdot\Big[[\mathsf{C}\cdot\mathsf{Z}] + \mathsf{Z}'\Big] \\ &= \mathsf{M}(t)\cdot\mathsf{G}(t). \end{aligned}$$

Step MFOL 1. Define the column-vector *function* $\mathsf{P}(t) := \mathsf{M}(t)\cdot\mathsf{G}(t)$, then compute

$$\mathsf{Q}(t) := \int^t \mathsf{P}().$$

For an arbitrary column-vec $\vec{\boldsymbol{\alpha}} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{bmatrix}$ of numbers, then,

$$\mathsf{M}\cdot\mathsf{Z} = \vec{\boldsymbol{\alpha}} + \mathsf{Q}.$$

Multiplying by $\mathsf{M}^{-1} \stackrel{\text{note}}{=\!=} \mathrm{e}^{-t\mathsf{C}}$, and putting the t back in the notation, we have that

$$\text{24c:} \qquad \underbrace{\mathsf{Z}_{\vec{\boldsymbol{\alpha}}}(t)}_{N\times 1} = \underbrace{\mathrm{e}^{-t\mathsf{C}}}_{N\times N}\cdot\Big[\underbrace{\vec{\boldsymbol{\alpha}}}_{N\times 1} + \underbrace{\mathsf{Q}(t)}_{N\times 1}\Big].$$

And if we arrange that $\mathsf{Q}(0) = \vec{\mathbf{0}}$, by defining

$$\mathsf{Q}(t) := \int_0^t \mathsf{P}(), \qquad \textit{then}$$

$$\text{24d:} \qquad \underbrace{\mathsf{Z}(t)}_{N\times 1} = \underbrace{\mathrm{e}^{-t\mathsf{C}}}_{N\times N}\cdot\Big[\underbrace{\mathsf{Z}(0)}_{N\times 1} + \underbrace{\mathsf{Q}(t)}_{N\times 1}\Big].$$

Aside: Since C is constant, our $\mathrm{e}^{-t\mathsf{C}}$ is simply $\mathsf{M}(-t)$.

25.1: *Revisiting* (19.1), *from* P.30. Imagine unknown fncs $x = x(t)$ and $y = y(t)$ satisfying system

$$\text{25.2:} \qquad \begin{aligned} x' &= 2x - y \qquad \text{and} \\ y' &= 4x - 2y + 2. \end{aligned}$$

Setting $\mathsf{Z}(t) := \begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$ and $\mathsf{C} := \begin{bmatrix}-2 & 1\\ -4 & 2\end{bmatrix}$ and $\mathsf{G}(t) := \begin{bmatrix}0\\ 2\end{bmatrix}$, we can rewrite (25.2) as

$$\text{*:} \qquad \mathsf{Z}' + \mathsf{C}\cdot\mathsf{Z} = \mathsf{G}.$$

With this Z and C, our (19.2) example from page 30, was $\mathsf{Z}' + \mathsf{C}\cdot\mathsf{Z} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

We get for free that

$$\text{24b†:} \qquad \mathsf{M}(t) := \mathrm{e}^{t\mathsf{C}} \stackrel{\text{note}}{=\!=} \left[\begin{array}{c|c} 1-2t & t \\ -4t & 1+2t \end{array}\right],$$

since C is negative the R from (19.1). Computing,

$$\mathsf{P} := \mathsf{M}\cdot\mathsf{G} = \left[\begin{array}{c|c} 1-2t & t \\ -4t & 1+2t \end{array}\right]\cdot\begin{bmatrix}0\\ 2\end{bmatrix} = \begin{bmatrix}2t\\ 2+4t\end{bmatrix}.$$

Integrating

$$\mathsf{Q} := \int_0^t \mathsf{P} = \begin{bmatrix}t^2\\ 2t+2t^2\end{bmatrix} \stackrel{\text{note}}{=\!=} t\cdot\begin{bmatrix}t\\ 2+2t\end{bmatrix}.$$

Applying (24d), multiplying $\mathrm{e}^{-t\mathsf{C}}\cdot\frac{1}{t}\mathsf{Q}$ gives

$$\left[\begin{array}{c|c} 1+2t & -t \\ 4t & 1-2t \end{array}\right]\cdot\begin{bmatrix}t\\ 2+2t\end{bmatrix} \stackrel{\text{note}}{=\!=} \begin{bmatrix}-t\\ 2-2t\end{bmatrix}.$$

So

$$\mathrm{e}^{-t\mathsf{C}}\cdot\mathsf{Q} = \begin{bmatrix}-t^2\\ 2t-2t^2\end{bmatrix}.$$

With initial condition $x(0) = 0 = y(0)$, then,

$$\begin{aligned} x(t) &= -t^2, \qquad \textit{and} \\ y(t) &= 2t - 2t^2. \end{aligned}$$

Hence the general soln to (25.2) is

$$\text{‡:} \qquad \begin{aligned} x(t) &= [1+2t]\cdot x(0) - t\cdot y(0) - t^2, \qquad \textit{and} \\ y(t) &= 4t\cdot x(0) - [1-2t]\cdot y(0) + [2t-2t^2]. \end{aligned}$$

Compare this with (19.1 ‡), on P.30. □

§A Appendix: Misc examples

These may be cited from anywhere.

26: *Poly-coeffs yet ∃soln not* $\mathbf{C}^2$. Find a non-$\mathbf{C}^2$ function $y = y(t)$ that, for $t \in \mathbb{R}$, satisfies

26a:
$$y'y + y^2 = G, \quad \text{where} \quad G(t) := t^4 - 2t^3 + 2t - 1.$$

ASIDE: This DE has form $P{\cdot}y'y + Q{\cdot}y^2 = G$. The coeff-fncs P,Q and target-fnc G are $\mathbf{C}^\infty$; indeed, *polynomials*; and P,Q are *constant*. Nonetheless, this DE admits a soln that is not even twice-differentiable. □

Soln. EASY SCAN: The DiffOp is invariant under negation; if f is a soln, then so is $-f$.

Could a degree-N poly satisfy (26a)? Well, the y^2 term forces $N \geqslant 2$. Thus $\text{Deg}(y'{\cdot}y) = 2N-1$ and $\text{Deg}(y^2) = 2N$, so N must *be* 2. The method of Undetermined Coeffs applies and we find that

26b:
$$f(t) := [t-1]^2$$

satisfies (26a). Thus $-[t-1]^2$ is also a soln.

IDEA: The 0^{th} and 1^{st} derivatives of these solns *agree* at $t{=}1$, which are the only derivatives used by the DiffOp. So: *At* $t{=}1$, *we can stitch these solns together.* This gives this *new* soln:

†:
$$y(t) := \begin{Bmatrix} +[t-1]^2 & \text{if } t \geqslant 1 \\ -[t-1]^2 & \text{if } t < 1 \end{Bmatrix} \overset{\text{note}}{=\!=} \left|t-1\right| \cdot [t-1].$$

Its derivative,

$$y'(t) = 2 \cdot \left|t-1\right|,$$

fails to be differentiable at $t{=}1$. So (†) is not twice-differentiable, hence not $\mathbf{C}^2$.

Let's check that (†) satisfies (26a). Computing,

$$y'{\cdot}y = 2{\cdot}[t-1]^3 = 2t^3 - 6t^2 + 6t - 2,$$
$$y^2 = [t-1]^4 = t^4 - 4t^3 + 6t^2 - 4t + 1.$$

Adding these together produces (26a). ♦

26c: *N.B.*. Our three fncs, (†) and $\pm[t-1]^2$, each solve first-order DE (26a), *and*: Their 0^{th} *and* 1^{st} derivatives agree at $t{=}1$. So even possession of *two* initial conditions to a first-order DE, need not be sufficient to uniquely specify a soln. □

§Index for DiffyQ

Stated theorems are in the ToC. ***Applications*** *of theorems may appear in this index.*

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