

Properties of determinant

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Abbreviations. These are defined below.

\mathcal{MuL} , multi-linear.
 \mathcal{AnS} , anti-symmetric.
 \mathcal{Alt} , alternating.

Intro. Below, all VSes are over a common field,

$$\mathbb{F} = (\mathbb{F}, +, 0, \cdot, 1)$$

In the defn below, I use “5” as a stand-in for N .

A map $g: \mathbf{V}_1 \times \cdots \times \mathbf{V}_5 \rightarrow \mathbf{X}$ is **5-multi-linear** [5- \mathcal{MuL}] if for all tuples $(\mathbf{u}_1, \dots, \mathbf{u}_5) \in \mathbf{V}_1 \times \cdots \times \mathbf{V}_5$, the following 5 maps are each linear:

$$\begin{aligned} g(\cdot, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5): \mathbf{V}_1 &\rightarrow \mathbf{X} \quad , \\ g(\mathbf{u}_1, \cdot, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5): \mathbf{V}_2 &\rightarrow \mathbf{X} \quad , \\ g(\mathbf{u}_1, \mathbf{u}_2, \cdot, \mathbf{u}_4, \mathbf{u}_5): \mathbf{V}_3 &\rightarrow \mathbf{X} \quad , \\ g(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdot, \mathbf{u}_5): \mathbf{V}_4 &\rightarrow \mathbf{X} \quad , \\ g(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \cdot): \mathbf{V}_5 &\rightarrow \mathbf{X} \quad . \end{aligned}$$

Sometimes 5- \mathcal{MuL} is called **5-linear**, or just **multi-linear** if the 5 is understood. [We use *bilinear* rather than “2-linear”, typically.]

When $\mathbf{V}_1 = \mathbf{V}_2 = \cdots = \mathbf{V}_5$. Map $h: \mathbf{V} \times \cdot^5 \times \mathbf{V} \rightarrow \mathbf{X}$ is **anti-symmetric** [\mathcal{AnS}] if

†: For all index-pairs $j < k$, switching the entries at indices j and k **negates** the h -value.

E.g, for $j=2$ and $k=5$:

$$h(\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_2) = -1 \cdot h(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5) .$$

A map $h: \mathbf{V} \times \cdot^5 \times \mathbf{V} \rightarrow \mathbf{X}$ is **alternating** [\mathcal{Alt}] if for each tuple $\widehat{\mathbf{u}} = (\mathbf{u}_1, \dots, \mathbf{u}_5)$:

*: If there exists an index-pair $j < k$ with $\mathbf{u}_j = \mathbf{u}_k$, then $h(\widehat{\mathbf{u}}) = \mathbf{0}_\mathbf{X}$.

1: AC-Alt Thm. Consider a map $h: \mathbf{V} \times \cdot^N \times \mathbf{V} \rightarrow \mathbf{X}$. Then

i: [\mathcal{MuL} & \mathcal{Alt}] \implies [\mathcal{MuL} & \mathcal{AnS}].

ii: When $\text{Char}(\mathbb{F}) \neq 2$ then the converse holds:
 [\mathcal{MuL} & \mathcal{AnS}] \implies [\mathcal{MuL} & \mathcal{Alt}]. \diamond

Proof of (i). WLOG, $N = 2$. For arbitrary $\mathbf{u}, \mathbf{w} \in \mathbf{V}$, note

$$\begin{aligned} \mathbf{0}_\mathbf{X} &\stackrel{\text{by } \mathcal{Alt}}{=} h(\mathbf{u} + \mathbf{w}, \mathbf{u} + \mathbf{w}) \\ &\stackrel{\text{by } \mathcal{MuL}}{=} h(\mathbf{u}, \mathbf{u}) + h(\mathbf{u}, \mathbf{w}) + h(\mathbf{w}, \mathbf{u}) + h(\mathbf{w}, \mathbf{w}) \\ &\stackrel{\mathcal{Alt}}{=} \mathbf{0}_\mathbf{X} + h(\mathbf{u}, \mathbf{w}) + h(\mathbf{w}, \mathbf{u}) + \mathbf{0}_\mathbf{X} . \end{aligned}$$

Hence $h(\mathbf{w}, \mathbf{u}) = -h(\mathbf{u}, \mathbf{w})$. \diamond

Pf of (ii). \mathcal{AnS} implies that $h(\mathbf{u}, \mathbf{u}) = -h(\mathbf{u}, \mathbf{u})$. Hence $[1 + 1] \cdot h(\mathbf{u}, \mathbf{u}) = \mathbf{0}_\mathbf{X}$. Since $\text{Char}(\mathbb{F}) \neq 2$, we may divide by $[1 + 1]$, obtaining the desired $h(\mathbf{u}, \mathbf{u}) = \mathbf{0}_\mathbf{X}$. \diamond

To verify that a map is \mathcal{AnS} , it suffices to check that

‡: Switching adjacent entries, j and $j+1$, negates the value,

as a transposition is a composition of *oddly many* adjacent-transpositions.

Filename: Problems/Algebra/LinearAlg/determinant-prop.latex

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