

Dehn's solution to Hilbert's third problem

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ABSTRACT: The cube and the tetrahedron of equal volume are not equidecomposable by means of planar cuts. Along the way this proves that $\arccos(k/n)$, for $n \geq 3$ with the fraction in lowest terms, is incommensurate with π .

Entrance. Here a *polygon* P is a certain type of compact simply-connected subset of the plane with non-empty interior. It is obtained by taking a polygonal Jordan loop L with finitely many sides, then letting P be the closure of the “inside” of L ; thus $\partial P = L$. The motivation for Hilbert's question, below, is the observation that every two polygons of the same area are “scissor congruent” by straight line cuts. (*Should the Greeks have discovered this?*)

To be specific, suppose we cut P into two polygons Q_0 and Q_1 by a chord connecting two points on ∂P (and so that the “interior” of the chord lies in the interior of P). Thus $Q_0 \cup Q_1 = P$ and $Q_0 \cap Q_1$ is the chord. For two finite collection of polygons

$$\mathcal{C} = \{P_1, Q_2, \dots, Q_N\} \quad \mathcal{F} = \{R_0, R_1, R_2, \dots, R_N\}$$

say that the two collections are *straight-line congruent*, $\mathcal{C} \equiv \mathcal{F}$, if we can cut P_1 into two pieces, Q_0 and Q_1 as above, such that each Q_j is congruent with R_j via an orientation-preserving isometry of the plane. Finally, let ‘ \equiv ’ actually be the *transitive closure* of the above relation. Interpret the statement “ $P \equiv Q$ ” by identifying P with the singleton collection $\{P\}$ and Q with $\{Q\}$.

1: Bolyai (-Gerwien) Theorem. *Two polygons are straight-line congruent IFF they have the same area.* \diamond

Proof. See the pictures on the preceding handwritten page. \diamond

Hilbert's Third Problem

An analogous question can be posed for polyhedra, in 3-space. For simplicity, just consider a convex polyhedron P ; a bounded subset of $\mathbb{R}^{\times 3}$, with non-void interior, obtained by intersecting finitely many closed half-spaces.

2: Question. *If P and Q are convex polyhedra of equal volume, are they “straight-plane congruent”?* \diamond

Dehn answered Hilbert's question in the negative.

3: Dehn's Theorem. *The cube and regular tetrahedron (of the same volume) are **not** straight-plane congruent.* \diamond

Proof. Given two intersecting planes in 3-space, let the *scaled-angle* between them denote the dihedral angle between them divided by π . Thus, the scaled-angle of each edge of the cube is $\frac{1}{2}$. Let α denote the scaled-angle of the tetrahedron. Now suppose, for the sake of contradiction, that the cube and tetrahedron can both be planarly-cut into a collection $\{P_1, \dots, P_N\}$ of polyhedra. Let Θ denote the (finite) set of numbers which appear as the scaled-angles of the $\{P_j\}_{j=1}^N$. Let \mathbf{V} denote the \mathbb{Q} -vectorspace spanned by the “vectors” $1, \alpha$ and the members of Θ ; thus the dimension of \mathbf{V} is at most $2 + |\Theta|$.

The lemma below will show that α is irrational. Thus “vectors” 1 and α are independent and so we can define a \mathbb{Q} -linear functional $\Lambda: \mathbf{V} \rightarrow \mathbb{R}$ which maps 1 –and consequently all rational numbers– to zero, and maps $\alpha \mapsto 1$.

For an edge e of a polyhedron P , let \hat{e} denote the scaled-angle between the two faces meeting at e . If the scaled-angles of P all lie in \mathbf{V} we can define

$$\text{I-a:} \quad \Upsilon\langle P \rangle := \sum_{\text{edges } e} \text{Len}(e) \cdot \Lambda(\hat{e}),$$

where the sum is taken over all the edges of P . Evidently $\Upsilon\langle \text{cube} \rangle = 0$ yet $\Upsilon\langle \text{tetrahedron} \rangle$ is positive since it equals 6 times the edge-length of the tetrahedron. So demonstrating that

$$\text{I-b:} \quad \begin{array}{l} \text{If } P \text{ is cut by a plane into } \{Q_0, Q_1\} \\ \text{then } \Upsilon\langle Q_0 \rangle + \Upsilon\langle Q_1 \rangle = \Upsilon\langle P \rangle \end{array}$$

will yield the contradiction that

$$\Upsilon\langle \text{cube} \rangle = \sum_{j=1}^N \Upsilon\langle P_j \rangle = \Upsilon\langle \text{tetrahedron} \rangle,$$

whence the theorem.

$\Upsilon\langle \rangle$ is an invariant of straight-plane congruence

The first possibility, for an edge e of P , is that the cutting-plane slices across e . Then e is cut into two pieces a_0 and a_1 , with a_j an edge of Q_j , satisfying $\text{Len}(a_0) + \text{Len}(a_1) = \text{Len}(e)$. Since each $\widehat{a}_j = \widehat{e}$ we have that

$$\text{Len}(a_0) \cdot \Lambda(\widehat{a}_0) + \text{Len}(a_1) \cdot \Lambda(\widehat{a}_1) = \text{Len}(e)\Lambda(\widehat{e}).$$

This holds also in the case where e was not touched by the plane and hence one of a_0 or a_1 is the “empty edge”.

The second possibility for an edge, e , of P is that the cutting-plane contains e . Thus e is bifurcated into two edges c_0 and c_1 , with c_j an edge of Q_j . Moreover, $\text{Len}(c_j) = \text{Len}(e)$ and $\widehat{c}_0 + \widehat{c}_1 = \widehat{e}$. Thus and consequently

$$\text{Len}(c_0)\Lambda(\widehat{c}_0) + \text{Len}(c_1)\Lambda(\widehat{c}_1) = \text{Len}(e)\Lambda(\widehat{e}).$$

These two possibilities account for all the edges of P . Moreover, wherever the cutting-plane cuts a face of P into two faces, we can imagine that P had a “degenerate” edge there of scaled-angle 1. And since $\Lambda(1)$ equals zero, the sum in (I-a) is unaffected by addition of a degenerate edge. So by means of these degenerate edges, we have arranged that every edge of the Q_j arises from either the first xor second possibility above. Summing the two foregoing displays over all the edges of Q_0 , Q_1 and P yields the desired $\Upsilon\langle Q_0 \rangle + \Upsilon\langle Q_1 \rangle = \Upsilon\langle P \rangle$ and completes the proof. \blacklozenge

4: Lemma. *The dihedral angle θ of a regular tetrahedron is incommensurate with 2π . (Hence $\alpha \stackrel{\text{def}}{=} \theta/\pi$ is irrational.)* \blacklozenge

Proof. Take the tetrahedron to have sidelength 1. On an equilateral triangle of sidelength 1, an altitude has length $\ell := \sqrt{3}/2$. Thus θ is the angle between the two “ $\sqrt{3}/2$ ” sides of the $\sqrt{3}/2, \sqrt{3}/2, 1$ isosceles triangle. By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(C)$, we get that

$$\begin{aligned} 1^2 &= \ell^2 + \ell^2 - 2\ell\ell\cos(\theta) \\ &= 2\ell^2[1 - \cos(\theta)] = \frac{3}{2}[1 - \cos(\theta)]. \end{aligned}$$

Hence $\cos(\theta) = 1/3$.

Consequently, the complex number $\mathbf{z} := e^{i\theta}$ —which is *not* real—satisfies

$$\mathbf{z} + \frac{1}{\mathbf{z}} = \frac{2}{3}$$

and so $3\mathbf{z}^2 - 2\mathbf{z} + 3 = 0$.

FTSOContradiction, suppose that θ is some rational multiple $\frac{p}{q} \cdot 2\pi$ (with $q \in \mathbb{Z}_+$). Then $\mathbf{z}^q = 1$. So \mathbf{z} is a root of the two polynomials

$$U(x) := x^q - 1 \quad \text{and} \quad D(x) := 3x^2 - 2x + 3.$$

These are members of $\mathbb{Q}[\cdot]$, the ring of polynomials over the rationals. The Euclidean Algorithm gives us quotient and remainder polys \mathbf{m} and r . They satisfy

$$U = \mathbf{m}D + r, \quad \text{where } \mathbf{m}, r \in \mathbb{Q}[\cdot]$$

with $\text{Deg}(r) < \text{Deg}(D) = 2$. So $r(\cdot)$ is linear. But $r(\mathbf{z}) = 0$ and linear polynomials only have real roots. Thus r must be the zero-polynomial. \heartsuit^1

Since $U = \mathbf{m}D$, the Gauss Lemma [S.Lang’s *Algebra* p.127, or T.Hungerford’s *Algebra* p.162, or my notes] says that

$$\text{GC}(U) = \text{GC}(\mathbf{m}) \cdot \text{GC}(D);$$

their algebraic “content”s multiply. [Define the *content* (Gauss-content) of the zero-polynomial as $\text{GC}(\text{Zip}) := 0$. For a non-zip poly $F \in \mathbb{Q}[\cdot]$, its *content* $\text{GC}(F)$ is the unique rational number $\frac{p}{q}$ so that $F() = \frac{p}{q} \cdot H()$, where H is the unique polynomial in $\mathbb{Z}[\cdot]$ with positive lead coefficient and whose coefficients have no common divisor. A polynomial F is *primitive* if $\text{GC}(F) = 1$ and its lead-coeff is positive.] But U and D have content 1 and so \mathbf{m} must have integral coefficients. Thus factorization $U = \mathbf{m}D$ is over $\mathbb{Z}[\cdot]$. But U is monic, so the leading-coeff of every factor of it must be ± 1 . Unlike the lead-coeff of D . \blacklozenge

Remark. Using the Hahn-Banach theorem, we can extend our linear functional Λ to be defined on all of \mathbb{R} , thus making Υ defined for all polyhedra, rather than just those with dihedral angles spanned by a certain finite set. On the other hand, the advantage of using vectorspace \mathbf{V} is that the Axiom of Choice is avoided. \square

\heartsuit^1 Here is an alternate way to see that $D(\cdot)$ divides $U(\cdot)$. Both \mathbf{z} and $1/\mathbf{z}$ are zeros of $D(\cdot)$ and of $U(\cdot)$. And $\mathbf{z} \neq \frac{1}{\mathbf{z}}$, since θ is not real. So the D -zeros are distinct, and are also zeros of U .

By rewording the argument, it generalizes to the following.

5: Corollary. *Suppose $k \perp n$ with $k \in \mathbb{Z}$ and $n \in [3 .. \infty)$. Then $\theta := \arccos(k/n)$ is incommensurate with π . \diamond*

Pf. Set $\mathbf{z} := e^{i\theta}$. Since $\mathbf{z} + \frac{1}{\mathbf{z}} = \frac{2k}{n}$, our \mathbf{z} is a root of

$$D(x) := nx^2 - 2kx + n.$$

If θ is of the form $\frac{p}{q} \cdot 2\pi$ then \mathbf{z} is also a root of $U(x) := x^q - 1$. As before, D divides U in $\mathbb{Q}[[x]]$, and so we can write

$$*: \quad 2U = \mathbf{m} \cdot D$$

for some polynomial \mathbf{m} . Now $\text{GC}(D) = \text{Gcd}(n, 2k)$ and thus equals 1 or 2, since $k \perp n$. Hence $\text{GC}(D)$ divides $2 = \text{GC}(2U)$ and so, by the Gauss Lemma, \mathbf{m} has integral coefficients. Equating the “ x^q ” coefficients in (*) yields

$$2 = \text{integer} \cdot n;$$

an impossibility, since n is at least 3. \diamond

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