Dehn's solution to Hilbert's third problem

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://www.math.ufl.edu/~squash/
9 July, 2012 (at *00:27*)

ABSTRACT: The cube and the tetrahedron of equal volume are not equidecomposable by means of planar cuts. Along the way this proves that $\arccos(k/n)$, for $n \geqslant 3$ with the fraction in lowest terms, is incommensurate with $\boldsymbol{\pi}$.

Entrance. Here a ***polygon*** P is a certain type of compact simply-connected subset of the plane with non-empty interior. It is obtained by taking a polygonal Jordan loop L with finitely many sides, then letting P be the closure of the "inside" of L; thus $\partial\mathsf{P} = L$. The motivation for Hilbert's question, below, is the observation that every two polygons of the same area are "scissor congruent" by straight line cuts. (*Should the Greeks have discovered this?*)

To be specific, suppose we cut P into two polygons Q_0 and Q_1 by a chord connecting two points on $\partial\mathsf{P}$ (and so that the "interior" of the chord lies in the interior of P). Thus $\mathsf{Q}_0 \cup \mathsf{Q}_1 = \mathsf{P}$ and $\mathsf{Q}_0 \cap \mathsf{Q}_1$ is the chord. For two finite collection of polygons

$$\mathscr{C} = \{\mathsf{P}_1, \mathsf{Q}_2, \ldots, \mathsf{Q}_N\} \qquad \mathscr{F} = \{\mathsf{R}_0, \mathsf{R}_1, \mathsf{R}_2, \ldots, \mathsf{R}_N\}$$

say that the two collections are ***straight-line congruent***, $\mathscr{C} \equiv \mathscr{F}$, if we can cut P_1 into two pieces, Q_0 and Q_1 as above, such that each Q_j is congruent with R_j via an orientation-preserving isometry of the plane. Finally, let '$\equiv$' actually be the *transitive closure* of the above relation. Interpret the statement "$\mathsf{P} \equiv \mathsf{Q}$" by identifying P with the singleton collection $\{\mathsf{P}\}$ and Q with $\{\mathsf{Q}\}$.

1: Bolyai (-Gerwien) Theorem. *Two polygons are straight-line congruent IFF they have the same area.* $\diamondsuit$

Proof. See the pictures on the preceding handwritten page. $\blacklozenge$

Hilbert's Third Problem

An analogous question can be posed for polyhedra, in 3-space. For simplicity, just consider a convex polyhedron P; a bounded subset of $\mathbb{R}^{\times 3}$, with non-void interior, obtained by intersecting finitely many closed half-spaces.

2: Question. *If P and Q are convex polyhedra of equal volume, are they "straight-plane congruent"?* $\diamondsuit$

Dehn answered Hilbert's question in the negative.

3: Dehn's Theorem. *The cube and regular tetrahedron (of the same volume) are **not** straight-plane congruent.* $\diamondsuit$

Proof. Given two intersecting planes in 3-space, let the ***scaled-angle*** between them denote the dihedral angle between them divided by $\boldsymbol{\pi}$. Thus, the scaled-angle of each edge of the cube is $\frac{1}{2}$. Let α denote the scaled-angle of the tetrahedron. Now suppose, for the sake of contradiction, that the cube and tetrahedron can both be planarly-cut into a collection $\{\mathsf{P}_1, \ldots, \mathsf{P}_N\}$ of polyhedra. Let Θ denote the (finite) set of numbers which appear as the scaled-angles of the $\{\mathsf{P}_j\}_{j=1}^N$. Let $\mathbf{V}$ denote the $\mathbb{Q}$-vectorspace spanned by the "vectors" 1, α and the members of Θ; thus the dimension of $\mathbf{V}$ is at most $2 + |\Theta|$.

The lemma below will show that α is irrational. Thus "vectors" 1 and α are independent and so we can define a $\mathbb{Q}$-linear functional $\Lambda\colon \mathbf{V} \to \mathbb{R}$ which maps 1 –and consequently all rational numbers– to zero, and maps $\alpha \mapsto 1$.

For an edge e of a polyhedron P, let $\widehat{\mathsf{e}}$ denote the scaled-angle between the two faces meeting at e. If the scaled-angles of P all lie in $\mathbf{V}$ we can define

I-a: $$\Upsilon\langle\mathsf{P}\rangle := \sum_{\text{edges } \mathsf{e}} \operatorname{Len}(\mathsf{e}) \cdot \Lambda(\widehat{\mathsf{e}}),$$

where the sum is taken over all the edges of P. Evidently $\Upsilon\langle\text{cube}\rangle = 0$ yet $\Upsilon\langle\text{tetrahedron}\rangle$ is positive since it equals 6 times the edge-length of the tetrahedron. So demonstrating that

I-b: *If P is cut by a plane into $\{\mathsf{Q}_0, \mathsf{Q}_1\}$ then $\Upsilon\langle\mathsf{Q}_0\rangle + \Upsilon\langle\mathsf{Q}_1\rangle = \Upsilon\langle\mathsf{P}\rangle$*

will yield the contradition that

$$\Upsilon\langle\text{cube}\rangle = \sum_{j=1}^N \Upsilon\langle\mathsf{P}_j\rangle = \Upsilon\langle\text{tetrahedron}\rangle,$$

whence the theorem.

$\Upsilon\langle\rangle$ is an invariant of straight-plane congruence

The first possibility, for an edge e of P, is that the cutting-plane slices across e. Then e is cut into two pieces a_0 and a_1, with a_j an edge of Q_j, satisfying $\mathrm{Len}(\mathsf{a}_0) + \mathrm{Len}(\mathsf{a}_1) = \mathrm{Len}(\mathsf{e})$. Since each $\widehat{\mathsf{a}_j} = \widehat{\mathsf{e}}$ we have that

$$\mathrm{Len}(\mathsf{a}_0)\cdot\Lambda(\widehat{\mathsf{a}_0}) + \mathrm{Len}(\mathsf{a}_1)\cdot\Lambda(\widehat{\mathsf{a}_1}) \;=\; \mathrm{Len}(\mathsf{e})\Lambda(\widehat{\mathsf{e}})\,.$$

This holds also in the case where e was not touched by the plane and hence one of a_0 or a_1 is the "empty edge".

The second possibility for an edge, e, of P is that the cutting-plane contains e. Thus e is bifurcated into two edges c_0 and c_1, with c_j an edge of Q_j. Moreover, $\mathrm{Len}(\mathsf{c}_j) = \mathrm{Len}(\mathsf{e})$ and $\widehat{\mathsf{c}_0} + \widehat{\mathsf{c}_1} = \widehat{\mathsf{e}}$. Thus and consequently

$$\mathrm{Len}(\mathsf{c}_0)\Lambda(\widehat{\mathsf{c}_0}) + \mathrm{Len}(\mathsf{c}_1)\Lambda(\widehat{\mathsf{c}_1}) \;=\; \mathrm{Len}(\mathsf{e})\Lambda(\widehat{\mathsf{e}})\,.$$

These two possibilities account for all the edges of P. Moreover, wherever the cutting-plane cuts a face of P into two faces, we can imagine that P had a "degenerate" edge there of scaled-angle 1. And since $\Lambda(1)$ equals zero, the sum in (I-a) is unaffected by addition of a degenerate edge. So by means of these degenerate edges, we have arranged that every edge of the Q_j arises from either the first xor second possibility above. Summing the two foregoing displays over all the edges of Q_0, Q_1 and P yields the desired $\Upsilon\langle\mathsf{Q}_0\rangle + \Upsilon\langle\mathsf{Q}_1\rangle = \Upsilon\langle\mathsf{P}\rangle$ and completes the proof. ♦

4: Lemma. *The dihedral angle θ of a regular tetrahedron is incommensurate with $2\boldsymbol{\pi}$. (Hence $\alpha \stackrel{\text{def}}{=} \theta/\boldsymbol{\pi}$ is irrational.)* ◊

Proof. Take the tetrahedron to have sidelength 1. On an equilateral triangle of sidelength 1, an altitude has length $\ell := \sqrt{3}/2$. Thus θ is the angle between the two "$\sqrt{3}/2$" sides of the $\sqrt{3}/2, \sqrt{3}/2, 1$ isosceles triangle. By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(C)$, we get that

$$\begin{aligned} 1^2 \;&=\; \ell^2 + \ell^2 - 2\ell\ell\cos(\theta) \\ &=\; 2\ell^2\big[1-\cos(\theta)\big] \;=\; \frac{3}{2}\big[1-\cos(\theta)\big]\,. \end{aligned}$$

Hence $\cos(\theta) = 1/3$.

Consequently, the complex number $\mathbf{z} := \mathsf{e}^{i\theta}$ –which is *not* real– satisfies

$$\mathbf{z} + \frac{1}{\mathbf{z}} \;=\; \frac{2}{3}$$

and so $3\mathbf{z}^2 - 2\mathbf{z} + 3 = 0$.

FTSOContradiction, suppose that θ is some rational multiple $\frac{p}{q}\cdot 2\boldsymbol{\pi}$ (with $q \in \mathbb{Z}_+$). Then $\mathbf{z}^q = 1$. So $\mathbf{z}$ is a root of the two polynomials

$$U(x) \;:=\; x^q - 1 \quad\text{and}\quad D(x) \;:=\; 3x^2 - 2x + 3\,.$$

These are members of $\mathbb{Q}[\![\cdot]\!]$, the ring of polynomials over the rationals. The Euclidean Algorithm qives us quotient and remainder polys $\mathbf{m}$ and r. They satisfy

$$U \;=\; \mathbf{m}D + r\,, \qquad \text{where } \mathbf{m}, r \in \mathbb{Q}[\![\cdot]\!]$$

with $\mathrm{Deg}(r) < \mathrm{Deg}(D) = 2$. So $r()$ is linear. But $r(\mathbf{z}) = 0$ and linear polynomials only have real roots. Thus r must be the zero-polynomial.[♡1]

Since $U = \mathbf{m}D$, the Gauss Lemma [S. Lang's *Algebra* p. 127, or T. Hungerford's *Algebra* p. 162, or my notes] says that

$$\mathrm{GC}(U) \;=\; \mathrm{GC}(\mathbf{m})\cdot\mathrm{GC}(D)\,;$$

their algebraic "content"s multiply. [Define the ***content*** (Gauss-content) of the zero-polnomial as $\mathrm{GC}(\mathrm{Zip}) := 0$. For a non-zip poly $F \in \mathbb{Q}[\![\cdot]\!]$, its ***content*** $\mathrm{GC}(F)$ is the unique rational number $\frac{p}{q}$ so that $F() = \frac{p}{q}\cdot H()$, where H is the unique polynomial in $\mathbb{Z}[\![\cdot]\!]$ with positive lead coefficient and whose coefficients have no common divisor. A polynomial F is ***primitive*** if $\mathrm{GC}(F) = 1$ and its lead-coeff is positive.] But U and D have content 1 and so $\mathbf{m}$ must have integral coefficients. Thus factorization $U{=}\mathbf{m}D$ is over $\mathbb{Z}[\![x]\!]$. But U is monic, so the leading-coeff of every factor of it must be ± 1. Unlike the lead-coeff of D. ♦

Remark. Using the Hahn-Banach theorem, we can extend our linear functional Λ to be defined on all of $\mathbb{R}$, thus making Υ defined for all polyhedra, rather than just those with dihedral angles spanned by a certain finite set. On the other hand, the advantage of using vectorspace $\mathbf{V}$ is that the Axiom of Choice is avoided. □

[♡1] Here is an alternate way to see that $D()$ divides $U()$. Both $\mathbf{z}$ and $1/\mathbf{z}$ are zeros of $D()$ *and* of $U()$. And $\mathbf{z} \neq \frac{1}{\mathbf{z}}$, since θ is not real. So the D-zeros are distinct, and are also zeros of U.

By rewording the argument, it generalizes to the following.

5: Corollary. *Suppose $k \perp n$ with $k \in \mathbb{Z}$ and $n \in [3 .. \infty)$. Then $\theta := \arccos(k/n)$ is incommensurate with $\boldsymbol{\pi}$.* ♢

Pf. Set $\mathbf{z} := \mathrm{e}^{i\theta}$. Since $\mathbf{z} + \frac{1}{\mathbf{z}} = \frac{2k}{n}$, our $\mathbf{z}$ is a root of

$$D(x) := nx^2 - 2kx + n\,.$$

If θ is of the form $\frac{p}{q} \cdot 2\boldsymbol{\pi}$ then $\mathbf{z}$ is also a root of $U(x) := x^q - 1$. As before, D divides U in $\mathbb{Q}[\![x]\!]$, and so we can write

$$*: \qquad 2U = \mathbf{m} \cdot D$$

for some polynomial $\mathbf{m}$. Now $\mathrm{GC}(D) = \mathrm{Gcd}(n, 2k)$ and thus equals 1 or 2, since $k \perp n$. Hence $\mathrm{GC}(D)$ divides $2 = \mathrm{GC}(2U)$ and so, by the Gauss Lemma, $\mathbf{m}$ has integral coefficients. Equating the "x^q" coefficients in $(*)$ yields

$$2 = \text{integer} \cdot n\,;$$

an impossibility, since n is at least 3. ♦

Filename: `Problems/Geometry/dehn-invariant.latex`
As of: *Monday 17Jan2011.* Typeset: *9Jul2012 at 00:27.*