deBruijn's Harmonic Brick Condition is computable

Jonathan L. King

University of Florida, Gainesville
32611-2082, USA squash@math.ufl.edu
Webpage: http://www.math.ufl.edu/~squash/

ABSTRACT. For particular collections $\mathbf{P}$ of integer-sided $\mathfrak{D}$-dimensional bricks, deBruijn gave an IFF-condition for when the only $\mathbf{P}$-packable boxes are those which admit a parallel packing.

We show, for general $\mathbf{P}$, that deBruijn's condition is computable, and provide an algorithm to compute it in time proportional to the product $\mathfrak{D}\mathfrak{N}^3$, where $\mathfrak{N}$ is the number of bricks in $\mathbf{P}$. The method first shows that deBruijn's condition has an equivalent formulation where tilings using *negative*, as well as positive, copies of bricks are permitted. The equivalent formulation is then characterized in terms of the minimal bricks in an associated finite distributive lattice of bricks.

§1 ENTRANCE

In a delightful paper entitled "Filling Boxes with bricks", N.G. deBruijn proposed and solved a problem inspired by his 7 year old son's playing with toy wooden bricks of size 1×2×4 and being unable to pack the 6×6×6 box with them. Subsequently, deBruijn (the father) went on to discover that box $a{\times}b{\times}c$ can be packed using all $6 = 3!$ orientations of 1×2×4 iff $a{\times}b{\times}c$ supports a "parallel packing" –that is, iff $a{\times}b{\times}c$ can be filled using only *one* orientation of 1×2×4. This latter means that there is some re-ordering a', b', c' of a, b, c, so that

$$1 \lhd a' \quad\text{and}\quad 2 \lhd b' \quad\text{and}\quad 4 \lhd c'$$

(I use $\lhd$ to mean "divides", and $\rhd$ to mean "is a multiple of".)

Generalizing 1×2×4 to $\mathfrak{D}$-dimensional bricks, [dB] defines a brick $\mathsf{B} = b_1{\times}\cdots{\times}b_{\mathfrak{D}}$ (all sides positive integers) to be ***harmonic*** if there exists a re-arrangement $b'_1, \ldots, b'_{\mathfrak{D}}$ such that

$$b'_1 \lhd b'_2 \lhd b'_3 \lhd \cdots \lhd b'_{\mathfrak{D}-1} \lhd b'_{\mathfrak{D}},$$

that is, each sidelength divides the next sidelength. To state the primary theorem of [dB], we establish terminology.

Say that brick B ***divides*** (or *parallel-packs*) brick $\mathsf{T} = t_1{\times}\cdots{\times}t_{\mathfrak{D}}$ iff copies of B can pack T. Equivalently,

$$b_1 \lhd t_1 \text{ and } b_2 \lhd t_2 \text{ and } \ldots \text{ and } b_{\mathfrak{D}} \lhd t_{\mathfrak{D}}.$$

We write this as $\mathsf{B} \preccurlyeq \mathsf{T}$. (Equivalently, let $\mathsf{T} \succcurlyeq \mathsf{B}$ mean that T is a multiple of B.) Thus 5×3×2 divides itself, and divides 10×6×6, but does not divide 6×10×6.

Informally, say that a family $\mathbf{P}$ of $\mathfrak{D}$-dimensional bricks ***packs*** box[‡] T if and only if T can be filled by disjoint copies of *translates* –no rotations allowed– of bricks in $\mathbf{P}$. We call the members of brick-set $\mathbf{P}$ the "protobricks".

Here is the property of a proto-set, $\mathbf{P}$, studied in [dB]:

> *Whenever a box is $\mathbf{P}$-packable, then there is a protobrick which parallel-packs it.*

This property I call ***deBruijn's Brick Condition***, dBBC, and call such a $\mathbf{P}$ a ***deBruijn family*** of bricks. Here are theorems 2 and 3 from [dB].

DEBRUIJN'S THEOREM. *Given brick $\mathsf{B} = b_1{\times}\cdots{\times}b_{\mathfrak{D}}$, let $\mathbf{P}$ be the proto-set comprising the bricks*

$$b_{\pi(1)} \times b_{\pi(2)} \times \cdots \times b_{\pi(\mathfrak{D})},$$

as π ranges over all the permutations of $\{1, \ldots, \mathfrak{D}\}$. Then: $\mathbf{P}$ *is deBruijn IFF* B *is harmonic.*

When $\mathbf{P}$ comprises all permutations of a single brick B, the deBruijn Condition is computable in time "big O" of $\mathfrak{D} \cdot \log(\mathfrak{D})$, simply by attempting to sort the sidelengths of B.

The goal of this article is to show that dBBC, with no assumptions on $\mathbf{P}$, is computable. Further, it can be computably verified in time $\mathrm{O}(\mathfrak{D}\mathfrak{N}^3)$, where $\mathfrak{N} := \#\mathbf{P}$ is the number of protobricks, each of dimension $\mathfrak{D}$. I will characterize dBBC and show it identical to a lattice-theoretic property which I call "uncombinability".

1991 *Mathematics Subject Classification.* 05B45.
Key words and phrases. Harmonic bricks, Tiling algorithms, Distributive lattices.

[‡]Boxes and bricks are the same mathematical objects, but I use "bricks" to pack "boxes". A formal definition of packing appears at the end of §1.

Conventions. Use $[k..n]$ for the "interval of integers" $[k, n] \cap \mathbb{Z}$. Use the prefix ***nv-*** to mean *non-void*, e.g, "a nv-collection".

Symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{T}$ name bricks. A lowercase letter denotes the corresponding sidelengths, e.g,

$$\mathsf{A} = a_1 \times \cdots \times a_{\mathfrak{D}} \quad \text{and} \quad \mathsf{T} = t_1 \times \cdots \times t_{\mathfrak{D}} \,.$$

Indices d and e denote ***directions*** in $[1..\mathfrak{D}]$. For a brick-set $\mathbf{S}$ and direction d, the symbol $\mathbf{S}_{\square \to d}$ denotes[†] the set $\{b_d \mid \mathsf{B} \in \mathbf{S}\}$ of d^{th} sidelengths.

When $\mathbf{P}$ is not deBruijn then there is a packable box T which is divisible by no protobrick. Such a box is a ***certificate*** that $\mathbf{P}$ fails dBBC.

Questions. Of examples E1–E4 below, which are deBruijn families?

E1: $\mathbf{P}^{\langle 1 \rangle} := \{3, 7\}$.
E2: $\mathbf{P}^{\langle 2 \rangle} := \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, where $\mathsf{A} = 25{\times}3$, $\mathsf{B} = 9{\times}8$ and $\mathsf{C} = 16{\times}5$.
E3: $\mathbf{P}^{\langle 3 \rangle}$ has $3{\times}20{\times}14$, $33{\times}5{\times}98$ and $99{\times}10{\times}7$.

Shows deBruijns, father and son.

FIGURE E4. (No caption.)

Answers. The $\mathbf{P}^{\langle 1 \rangle}$ "bricks" can be thought of as half-open intervals of length 3 and 7. These pack the interval 10, yet neither 1 nor 5, its proper divisors, are packable. Thus $\mathbf{P}^{\langle 1 \rangle}$ is not deBruijn, and 10 is a certificate of this.

Proto-set $\mathbf{P}^{\langle 2 \rangle}$ packs box $\mathsf{T} := 34{\times}11$.

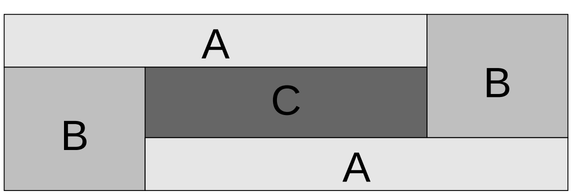

FIGURE 1. Two copies of rectangle A, two of B and a single C pack the $34{\times}11$ rectangle T.

Since T is not a multiple of A, nor of B nor C, this T is a certificate that $\mathbf{P}^{\langle 2 \rangle}$ fails. Of course, $\mathbf{P}^{\langle 2 \rangle}$ can be *augmented* to be a deBruijn family –but, it turns out, only in a trivial way. It will follow from §2 that any finite brick-set $\mathbf{F}$ making $\mathbf{P}^{\langle 2 \rangle} \cup \mathbf{F}$ deBruijn must necessarily own the $1{\times}1$ square.

Collection $\mathbf{P}^{\langle 3 \rangle}$ is deBruijn, courtesy of this mild strengthening of deBruijn's theorem: *Suppose for each $d \in [1..\mathfrak{D}]$ that the poset $(\mathbf{P}_{\square \to d}\,, \triangleleft)$ is totally ordered. Then* $\mathbf{P}$ *has* dBBC.

This will be a direct corollary of the Equivalence Theorem in §2. Here is an application of the corollary. Suppose π and ν are arbitrary maps of $[1..\mathfrak{N}]$ into itself. Then this list of $\mathfrak{N}$ bricks,

$$3 \times 5^{\pi(1)} \times 7^{\nu(1)}, \; \ldots, \; 3^{\mathfrak{N}} \times 5^{\pi(\mathfrak{N})} \times 7^{\nu(\mathfrak{N})} \,,$$

is a deBruijn family.

Example E4 is (part of) a deBruijn family *par excellence*: Father and son together.

The Brickspace Lattice

When equipped with the divisibility relation $\triangleleft$, the set $\mathcal{L}$ of positive integers forms a distributive lattice $(\mathcal{L}\,, \triangleleft)$. The corresponding "inf" operation, $a \wedge b$, is *greatest common divisor*; $\gcd\{a, b\}$. And "sup", $a \vee b$, is $\operatorname{lcm}\{a, b\}$.

The product poset

$$\Lambda := \mathcal{L}^{\times \mathfrak{D}} = \mathcal{L} \times \mathcal{L} \times \overset{\mathfrak{D}}{\ldots} \times \mathcal{L}$$

is our space of bricks. And the *parallel-packs* relation $\preccurlyeq$ is the product order. Consequently $(\Lambda, \preccurlyeq)$ is a distributive lattice. For the "inf" of bricks A and B write $\mathsf{Glb}\{\mathsf{A}, \mathsf{B}\}$, **"**greatest lower brick**"**. Let $\mathsf{Lub}\{\mathsf{A}, \mathsf{B}\}$, **"**least upper brick**"**, denote their "sup".

Multiples and Scalings. In brickspace Λ one can think of the multiples of a fixed brick B as forming a "cone" of boxes, with B being the cone's vertex. Let $\triangleleft(\mathsf{B})$, the ***cone*** over B, denote this set $\{\mathsf{T} \in \Lambda \mid \mathsf{T} \succcurlyeq \mathsf{B}\}$ of multiples. A cone is a particular example of a set $\mathbf{U}$ of bricks satisfying

$$\mathsf{B} \in \mathbf{U} \text{ and } \mathsf{B} \preccurlyeq \mathsf{T} \quad \Longrightarrow \quad \mathsf{T} \in \mathbf{U}\,.$$

Any such set $\mathbf{U}$, closed under going up, I will call an ***up-set***.

The ***cone*** $\triangleleft(\mathbf{S})$ of a *brick-set*, by which I mean

$$\{\mathsf{T} \mid \exists \mathsf{B} \in \mathbf{S} \text{ with } \mathsf{B} \preccurlyeq \mathsf{T}\}\,,$$

is also an up-set. Every up-set $\mathbf{U}$ has this form, since

$$\mathbf{U} \;=\; \triangleleft\big(\mathsf{Mml}(\mathbf{U})\big)\,,$$

where $\mathsf{Mml}(\mathbf{U})$ is the set of $\preccurlyeq$-minimal elements of $\mathbf{U}$.

[†]This notation is meant to suggest *projecting* $\mathbf{S}$ from Brickspace –represented by a square– to the space of dth sidelengths of bricks.

SCALING LEMMA, 2. *Consider a finite brick-set* $\mathbf{F}$. *Suppose that* T *is a box such that for each large integer* k, *the* ***scaled brick***

$$(2') \qquad k\mathsf{T} \coloneqq kt_1 \times kt_2 \times \cdots \times kt_{\mathfrak{D}}$$

is in the cone $\triangleleft(\mathbf{F})$. *Then* T *is, itself, a multiple of some* $\mathbf{F}$*-brick.*

PROOF. For each large k there is an $\mathbf{F}$-brick B dividing $k\mathsf{T}$. Since $\mathbf{F}$ is finite, the Pigeon-hole Principle asserts two large primes $k < p$ and a common brick B in $\mathbf{F}$ which divides *both* $k\mathsf{T}$ and $p\mathsf{T}$. Consequently,

$$\mathsf{B} \preccurlyeq \mathsf{Glb}\{k\mathsf{T}, p\mathsf{T}\} \overset{\text{note}}{=\!=} \gcd\{k,p\}\mathsf{T}\,.$$

And this latter equals T, since $\gcd\{k,p\}$ equals 1.♦

Packings and Tilings. Figure 1 displayed the box $\mathsf{T} = 34{\times}11$ packed by 2 copies of A, 2 of B and one of C. Conversely, we can think of the picture as showing how to "signed pack" C by using positive *and negative* tiles: One copy of T and -2 copies of A and of B. We say that collection $\{\mathsf{T},\mathsf{A},\mathsf{B}\}$ *tiles* C.

Formally, identify each brick B with the corresponding Cartesian product of half-open intervals,

$$\mathsf{B} = [0,b_1) \times [0,b_2) \times \cdots \times [0,b_{\mathfrak{D}})\,.$$

A translate, H, of B, has the form

$$[h_1, h_1{+}b_1) \times [h_2, h_2{+}b_2) \times \cdots \times [h_{\mathfrak{D}}, h_{\mathfrak{D}}{+}b_{\mathfrak{D}})\,.$$

The indicator function $\mathbf{1}_{\mathsf{H}}$, when evaluated at a point $\vec{x} \in \mathbb{R}^{\mathfrak{D}}$, is 1 or 0 as $\vec{x}$ is/is-not in the above product. Box T is ***tilable*** by proto-set $\mathbf{P}$ if

$$\mathbf{1}_{\mathsf{T}} = \sum_{\mathsf{H}\in\mathbf{F}} \gamma_{\mathsf{H}} \mathbf{1}_{\mathsf{H}}\,, \qquad \text{with each coefficient } \gamma_{\mathsf{H}} \in \mathbb{Z},$$

for some finite family $\mathbf{F}$ of protobrick translates and corresponding coefficients γ_{H}. Finally, T is ***packable*** if an $\mathbf{F}$ can be chosen with all the coefficients equaling 1.

Let $\mathsf{Pac}(\mathbf{P})$ and $\mathsf{Til}(\mathbf{P})$ denote the set of packable, respectively, tilable, boxes. Each of these is an up-set and so is determined by its collection of minimal members. When $\mathbf{P}$ is deBruijn then $\mathsf{Mml}(\mathsf{Pac}(\mathbf{P}))$ is a subset of $\mathbf{P}$, hence is finite. But, typically, there are infinitely many minimal packable-boxes.*

What turns out to be decisive for the deBruijn Condition is that the set of minimal *tilable*-boxes is always finite –and computably so.

*With proto-set $\mathbf{P}^{\langle 1\rangle}$, for example, every prime exceeding 11 is a minimal $\mathbf{P}^{\langle 1\rangle}$-packable box.

§2 THE COMBINE OPERATOR

Given bricks $\mathsf{A},\mathsf{B},\ldots,\mathsf{C}$, observe that A parallel-packs the "slab"

$$\mathsf{A}' \coloneqq a_1 \times \ell_2 \times \ell_3 \times \cdots \times \ell_{\mathfrak{D}}\,,$$

where each $\ell_e \coloneqq \operatorname{lcm}\{a_e, b_e, \ldots, c_e\}$. Letting g_1 be the greatest common divisor of $\{a_1,\ldots,c_1\}$, take integers $k_{\mathsf{A}},\ldots,k_{\mathsf{C}}$ such that

$$g_1 = (k_{\mathsf{A}} \cdot a_1) + (k_{\mathsf{B}} \cdot b_1) + \cdots + (k_{\mathsf{C}} \cdot c_1)\,.$$

Then the box

$$g_1 \times \ell_2 \times \ell_3 \times \cdots \times \ell_{\mathfrak{D}}$$

is tiled by k_{A} copies of slab A' and k_{B} copies of B' ... and finally k_{C} copies of slab C'.

The above is an example of the ***combine*** operation; we will write $\mathsf{T} = \mathsf{Comb}_1(\{\mathsf{A},\ldots,\mathsf{C}\})$. Formally, given a finite brick-set $\mathbf{F}$ and direction d, let $\mathsf{Comb}_d(\mathbf{F})$ denote the brick $t_1{\times}\cdots{\times}t_{\mathfrak{D}}$, where

$$t_d \coloneqq \gcd\big(\mathbf{F}_{\Box\to d}\big) \quad\text{and}\quad t_e \coloneqq \operatorname{lcm}\big(\mathbf{F}_{\Box\to e}\big)\,,$$

for each direction $e \neq d$.

Lastly, brick-set $\mathbf{P}$ is ***uncombinable*** if: *For every* $\mathsf{A},\mathsf{B} \in \mathbf{P}$ *and each direction* d, *the box* $\mathsf{Comb}_d\{\mathsf{A},\mathsf{B}\}$ *is divisible by some brick in* $\mathbf{P}$.

Here is the principal result of this note.

EQUIVALENCE THEOREM, 3. *For each finite brick-set* $\mathbf{P}$, *conditions* (i,ii,iii) *are equivalent.*

- i: $\mathbf{P}$ *is a deBruijn set, i.e,* $\mathsf{Pac}(\mathbf{P}) = \triangleleft(\mathbf{P})$.
- ii: $\mathsf{Til}(\mathbf{P}) = \triangleleft(\mathbf{P})$.
- iii: $\mathbf{P}$ *is uncombinable.*

Tools. The theorem will follow from the foregoing Scaling Lemma, as well as two theorems which we cite from [Kin1] but which have antecedents going back at least to 1971.

Let $\mathsf{Ext}_d(\mathbf{P})$, the "$d^{th}$extension of $\mathbf{P}$", denote this set of bricks:

$$\big\{\mathsf{Comb}_d(\mathbf{F}) \bigm| \mathbf{F} \text{ is a finite nv-subset of } \mathbf{P}\big\}$$

It is not difficult to see that the extension operators commute, $\mathsf{Ext}_e \circ \mathsf{Ext}_d = \mathsf{Ext}_d \circ \mathsf{Ext}_e$, and each operator is idempotent. In consequence, the iteration

$$\mathsf{Ext}_{\mathfrak{D}}\Big(\mathsf{Ext}_{\mathfrak{D}-1}\big(\ldots\mathsf{Ext}_2\big(\mathsf{Ext}_1(\mathbf{P})\big)\ldots\big)\Big)$$

is the set of *all* boxes that can be built from $\mathbf{P}$ by means of Combine operators. Abbreviate this set by $\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})$.

Let's apply this operator to the bricks $\{\mathsf{A},\mathsf{B},\mathsf{C}\}$ of figure 1. Since $\mathsf{Ext}_{2..1}(\mathbf{P}^{\langle 2\rangle})$ owns the 1×1 square, this square is tilable. Consequently, for a finite augmentation $\mathbf{F}\cup\mathbf{P}^{\langle 2\rangle}$ to be deBruijn, $\mathbf{F}$ must own 1×1.

History. Versions of the next two results appear in several papers.

Under the assumption that $\mathbf{P}$ is permutation-invariant (permuting the sides of any protobrick yields another), Katona & Szász (1971) prove lemma 5 and give an intricate criterion for when a box is $\mathbf{P}$-tilable.

In two seminal papers, Barnes (1982) develops commutative algebra machinery to determine when a polyomino is tilable by others. He establishes (4) –that $\mathsf{Mml}(\mathsf{Til}\mathbf{P})$ is finite– but with a different characterization of the minimal set, and proves (5) with a common value for $\mathcal{K}_1,\ldots,\mathcal{K}_{\mathfrak{D}}$.

LEMMA 4 [Kin1, Equality Thm]. $\mathsf{Til}(\mathbf{P})$ *equals the cone over* $\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})$.

This says that iterating all possible Combines is sufficient to generate all of the minimal bricks in $\mathsf{Til}(\mathbf{P})$. Here is a sample computation.

Calculating rank. Let A be the 5×6×7 brick and let $\mathbf{P} := \{\mathsf{A},\mathsf{A}',\mathsf{A}''\}$, where each stroke means to rotate the sides by one position; $\mathsf{A}' = 6{\times}7{\times}5$ and $\mathsf{A}'' = 7{\times}5{\times}6$. Necessarily, the set $\mathsf{Mml}(\mathsf{Ext}_{3..1}(\mathbf{P}))$ is rotation invariant. It comprises these five bricks

$$\begin{array}{lcl}
\mathsf{A} & : & 5\times\ 6\ \times\ 7\\
\mathsf{B} := \mathsf{Comb}_1\{\mathsf{A},\mathsf{A}'\} & : & 1\times(6\cdot7)\times(7\cdot5)\\
\dot{\mathsf{B}} := \mathsf{Comb}_1\{\mathsf{A}',\mathsf{A}''\} & : & 1\times(7\cdot5)\times(5\cdot6)\\
\ddot{\mathsf{B}} := \mathsf{Comb}_1\{\mathsf{A}'',\mathsf{A}\} & : & 1\times(5\cdot6)\times(6\cdot7)\\
\mathsf{C} := \mathsf{Comb}_5\{\mathsf{B},\dot{\mathsf{B}},\ddot{\mathsf{B}}\} & : & 1\times\ 1\ \times(5\cdot6\cdot7)
\end{array}$$

and their rotates. Thus $\mathsf{Mml}(\mathsf{Til}(\mathbf{P}))$, which equals $\mathsf{Mml}(\mathsf{Ext}_{3..1}(\mathbf{P}))$, has 15 bricks. Call the cardinality of $\mathsf{Mml}(\mathsf{Til}(\mathbf{P}))$ the ***rank*** of $\mathbf{P}$.

LEMMA 5 [Kin1, Computability Thm]. *There are computable integers* $\mathcal{K}_1,\ldots,\mathcal{K}_{\mathfrak{D}}$ *(depending on* $\mathbf{P}$*) so that whenever* T *is a box with each sidelength* $t_d > \mathcal{K}_d$*, then:*

$$\mathsf{T}\ \textit{tilable} \implies \mathsf{T}\ \textit{packable}.$$

Remark. An algorithm in §3 will use these numbers and so we give a formula here. It suffices‡ to let $\mathcal{K}_d := (J-1)L$, where $L = \mathrm{lcm}(\mathbf{P}_{\square\to d})$ and J is the number of distinct primes in the factorization of L.

We can now verify (3), the Equivalence Theorem.

PROOF OF i⟺ii. Certainly (ii)⇒(i) since, by definition, $\mathsf{Til}(\mathbf{P}) \supseteq \mathsf{Pac}(\mathbf{P}) \supseteq \lhd(\mathbf{P})$.

To establish the converse implication, fix a tilable box T. For all large k, courtesy lemma 5, the scaled box $k\mathsf{T}$ is packable. Hence $k\mathsf{T} \in \lhd(\mathbf{P})$, since $\mathbf{P}$ is deBruijn. By the Scaling Lemma, then, some protobrick parallel-packs T, as desired. ♦

PROOF OF ii⟺iii. Equality (ii) is, thanks to lemma 4, equivalent to $\lhd(\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})) = \lhd(\mathbf{P})$, which is equivalent to

$$\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P}) \subseteq \lhd(\mathbf{P}).$$

This latter inclusion certainly implies that $\mathbf{P}$ is uncombinable. And the converse follows from the observation that Combine is a non-decreasing function of its operands: If $\mathsf{A}' \succcurlyeq \mathsf{A}$ and $\mathsf{B}' \succcurlyeq \mathsf{B}$ then $\mathsf{Comb}\{\mathsf{A}',\mathsf{B}'\} \succcurlyeq \mathsf{Comb}\{\mathsf{A},\mathsf{B}\}$. ♦

COROLLARY 3′. *For each direction* $d \in [1..\mathfrak{D}]$*, suppose that the poset* $(\mathbf{P}_{\square\to d}, \lhd)$ *is totally ordered. Then* $\mathbf{P}$ *is deBruijn.*

PROOF. It is enough to show that the hypothesis on $\mathbf{P}_{\square\to d}$ implies that $\mathsf{Ext}_d(\mathbf{P}) \subseteq \lhd(\mathbf{P})$.

Consider a nv-subset $\mathbf{F} \subseteq \mathbf{P}$ and choose a protobrick $\mathsf{T} \in \mathbf{F}$ so that, with respect to divisibility, each sidelength t_d is the minimum of the integers in $\mathbf{F}_{\square\to d}$. Automatically, t_d divides the d-th sidelength of $\mathsf{Comb}_d(\mathbf{F})$. Thus T divides $\mathsf{Comb}_d(\mathbf{F})$. ♦

§3 QUESTIONS & ALGORITHMS

Let $\beta = \beta(\mathsf{T})$ denote the number of bits needed to describe a box T; so $\beta(\mathsf{T})$ is roughly $\log_2(t_1)+\cdots+\log_2(t_{\mathfrak{D}})$. The foregoing gives an algorithm, linear in β, for whether a candidate box is $\mathbf{P}$-tilable: *Test whether there exists a divisor* $\mathsf{B} \in \mathsf{Mml}(\mathsf{Ext}_{\mathfrak{D}..1}\mathbf{P})$ *of* T. This takes time† proportional to $\mathrm{rank}(\mathbf{P})\cdot\beta$. Several interrelated queries are suggested, two of which are open.

Question Q1. Is there a linear-time algorithm for $\mathbf{P}$*-packability?*

‡A justification, as well as a better (lower) value for $\mathcal{K}_d$ in terms of Frobenius numbers, appears in [Kin1].

†As a function of β, linear is the best we can get. Alas, the algorithm is often impractical, since $\mathrm{rank}(\mathbf{P})$ can be huge. A maliciously chosen proto-set, among those comprising six rectangles, can have rank 7,828,352 (but no larger). Behavior of the rank function is studied in [Kin1,2] and [Ha&To].

Question Q2. Let $\mathbf{M}(\mathbf{P}) \coloneqq \mathsf{Mml}(\mathsf{Pac}(\mathbf{P}))$, the minimal *packable* boxes. *Is "Is* $\mathbf{M}(\mathbf{P})$ *finite?" computable? When finite, is the set* $\mathbf{M}(\mathbf{P})$ *computable?*

Question Q3. Courtesy (3ii), when the righthand inclusion of

$$\mathsf{Til}(\mathbf{P}) \supseteq \mathsf{Pac}(\mathbf{P}) \supseteq \lhd(\mathbf{P})$$

is equality, then so is the lefthand inclusion. *Does the converse hold?*

Discussion. Courtesy lemma 5, were we only to consider boxes whose sidelengths are sufficiently great, the answer to Q1 would be "yes". But without such restriction, Q1 is open –even in the $\mathfrak{D} = 2$ case. However, there is one situation with an unrestricted "yes". This is when $\mathbf{M}(\mathbf{P})$ is *finite*, suggesting Q2.

The proposition, below, answers Q2, as follows: *Compute* $\mathsf{Mml}(\mathsf{Ext}_{\mathfrak{D}..1}\mathbf{P})$. *If all these bricks are packable, then this set is* $\mathbf{M}(\mathbf{P})$; otherwise $\mathbf{M}(\mathbf{P})$ is infinite.

Proposition 6. *Given a finite brick-set* $\mathbf{P}$, *let* $\mathbf{M}$ *be* $\mathsf{Mml}(\mathsf{Pac}(\mathbf{P}))$. *Then the following are equivalent, and are implied by* (3i,ii,iii)

j: $\mathbf{M}$ *is finite.*
jj: $\mathsf{Til}(\mathbf{M}) = \lhd(\mathbf{M})$, *i.e,* $\mathbf{M}$ *is uncombinable.*
jjj: $\mathsf{Til}(\mathbf{P}) = \mathsf{Pac}(\mathbf{P})$.

Proof. Automatically

$$(*) \qquad \mathsf{Pac}(\mathbf{M}) = \mathsf{Pac}(\mathbf{P}) = \lhd(\mathbf{M}),$$

so $\mathbf{M}$ is deBruijn. When $\mathbf{M}$ is finite, (j), then implication (i)⇒(ii) applies to $\mathbf{M}$, yielding (jj).

Evidently $\mathbf{P} \subseteq \lhd(\mathbf{M})$, so $\mathsf{Til}(\mathbf{P}) = \mathsf{Til}(\mathbf{M})$. By (jj) and $(*)$, then, $\mathsf{Til}(\mathbf{P}) = \lhd(\mathbf{M}) = \mathsf{Pac}(\mathbf{P})$, giving (jjj).

From (jjj) and Lemma 4,

$$\mathbf{M} = \mathsf{Mml}(\mathsf{Til}(\mathbf{P})) = \mathsf{Mml}(\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})).$$

The latter is finite, since $\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})$ is. ♦

Gift certificates. With no extra work, the proof of (3) gives an algorithm –when $\mathbf{P}$ fails to be deBruijn– to produce a certificate of failure. This simple algorithm is in terms of the numbers $\mathcal{K}_1, \ldots, \mathcal{K}_{\mathfrak{D}}$ from lemma 5.

a: *Find a brick* T *which is minimal in* $\mathsf{Ext}_{\mathfrak{D}..1}(\mathbf{P})$, *yet is divisible by no protobrick. Let* $\mathcal{K}'$ *be the minimum of* $\mathcal{K}_d/t_d$, *taken over* $d = 1, \ldots \mathfrak{D}$.

b: *Compute integers* $k_0 < k_1 < \cdots < k_{\mathfrak{N}}$, *with* $k_0 > \mathcal{K}'$, *so that each two are relative prime.**

This has arranged that at least one of the scaled bricks, below, is divisible by no protobrick.

c: *Scan the list of scaled bricks,* $k_0\mathsf{T}$, $k_1\mathsf{T}$, ..., $k_{\mathfrak{N}}\mathsf{T}$, *for one which has no protobrick divisor.*

Each such scaled brick is packable, hence is a certificate.

Uncombinability algorithm. Step a:, up above, asked us to find what could be called a minimal "certificate of combinability". This can be accomplished by the algorithm, below, which runs in $\mathrm{O}(\mathfrak{D}\mathfrak{N}^3)$ "ticks". Here, we presume that each of these sidelength calculations,

$$a \lhd b\,, \quad \gcd\{a,b\}\,, \quad \mathrm{lcm}\{a,b\}\,,$$

costs one tick.

LOOP over each pair $\mathsf{A},\mathsf{B} \in \mathbf{P}$, *with* $\mathsf{A} \neq \mathsf{B}$, *DOing steps* 1: 2: 3:, *below.*

IF the EXIT statement of 3: *is never executed, PRINT* "$\mathbf{P}$ is uncombinable".

1: *INITIALIZE* $\mathfrak{D}$*-tuples* L, G *and* Dir, *by*
$\mathsf{L}_d \coloneqq \mathrm{lcm}\{a_d, b_d\}$, $\mathsf{G}_d \coloneqq \gcd\{a_d, b_d\}$
and $\mathrm{Dir}_d \coloneqq$ *FALSE, for* $d = 1, \ldots, \mathfrak{D}$.

2: *FOR each brick* $\mathsf{C} \in \mathbf{P}$:
IF $(\mathsf{C} \preccurlyeq \mathsf{L})$ *THEN*
LET $\mathrm{Dir}_e \coloneqq$ *TRUE for all those directions* e *such that* $c_e \lhd \mathsf{G}_e$.

3: *IF (exists* e *with* $\mathrm{Dir}_e =$ *FALSE) THEN*
PRINT "No protobrick divides $\mathsf{Comb}_e\{\mathsf{A},\mathsf{B}\}$"
and then EXIT.

References

Mathematical Review numbers, where available, are listed at the end of each reference. When *MR:* numbers are not available, a call number may be listed.

[Bar1] F.W. Barnes, *Algebraic theory of brick packing, I*, Discrete Math. **42** (1982), 7–26, *MR:* **84e:05044a**.

[Bar2] F.W. Barnes, *Algebraic theory of brick packing, II*, Discrete Math. **42** (1982), 129–144, *MR:* **84e:05044b**.

*One method is to let $q_0 := 1$ and q_i be the ith prime. Then $q_0 < q_1 < \cdots < q_{\mathfrak{N}}$ is a pairwise co-prime list. To translate this past $\mathcal{K}'$, let P be the product of all the primes which are less-equal the difference $q_{\mathfrak{N}} - q_0$. If M is any multiple of P, then the numbers $k_i := M + q_i$, for $i = 0, \ldots, \mathfrak{N}$, are necessarily pairwise-prime. So we pick M to be the smallest multiple of P such that $M + q_0$ exceeds $\mathcal{K}'$.
An in-depth discussion of related problems appears in [EPP].

[dB] N.G. de Bruijn, *Filling Boxes with bricks*, Amer. Math. Monthly **76** (1969), 37–40, *MR:* **38 #3155**.

[EPP] P. Erd˝os, D.E. Penney & C. Pomerance, *On a class of relatively prime sequences*, J. Number Theory **10** (1978), 451–474.

[Ha&To] T. Hamachi & Y. Tomita, *On the King maximal tiling-rank function*, Preprint.

[Ka&Sz] G. Katona & D. Szász, *Matching Problems*, Jour. Comb. Theoryp **10** (1971), 60-92, *MR:* **46 #798**.

[Kin1] J.L. King, *Brick Tiling and Monotone Boolean Functions*, Preprint available at: `http://www.math.ufl.edu/~squash/tilingstuff.html`.