ACES-ACT **Prac-D** Prof. JLF King

Hello. Write **DNE** in a blank <u>if</u> the described object does not exist or if the indicated operation cannot be performed.

Below, (X, d) and (Ω, μ) are MSes. Use *Id* for the identity fnc, $x \mapsto x$, on any specified space. Below, let $\mathbf{J}$ be the $\mathbb{R}$-open interval $(3, 5)$. Use $\mathbb{I} := \mathbb{R} \smallsetminus \mathbb{Q}$ for the set of irrationals.

Fill-in *all* blanks on this sheet **<u>including</u>** the blanks for the essay questions!

D0: (*Definitions.*) Show no work.

a A subset $S \subset \Omega^{\mathrm{MS}}$ is a ***neighborhood*** of point $p \in \Omega$ IFF ....

In $\Omega := \mathbb{R}$, the set $\mathbb{Q}_+ \cap [5, \infty)$ is a neighborhood...

... of 6: *T* *F*. ... of 5: *T* *F*.

b MS (X, d) is ***cover-positive*** IFF (Defn.) ....

c MS (Ω, d) has $Y \subset \Omega$. So (Y, d) is ***totally-bounded*** IFF [Put Ω- or Y- before "closed/open/interior" etc.] ....

d That 7 is <u>a</u> ***Lebesgue number*** of open-cover $\mathcal{C}$ of (X, d), means that ....

Patches $\mathcal{C} := \left\{(-\infty, 28], [7, 33), [29, +\infty)\right\}$ cover $\mathbb{R}$. Thus MaxLebesgueNumber$(\mathcal{C})=$ ....

e MS (Ω, d) has $Y \subset \Omega$. So (Y, d) is ***cluster-point compact*** IFF [Put Ω- or Y- before "closed/open/interior" etc.] ....

And $\mathbb{Q} \cap [3, 5]$ is cluster-point compact: *T* *F*

f A map $f: \mathbf{V} \times \mathbf{E} \to \mathbf{W}$ (where $\mathbf{V}, \mathbf{E}, \mathbf{W}$ are $\mathbb{R}$-vectorspaces) is ***bilinear*** if $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$, $\forall \mathbf{e}_1, \mathbf{e}_2 \in \mathbf{E}$ and

$\forall$ .... : ....

and ....

A map $\langle \cdot, \cdot \rangle$ from $\mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is an ***inner product*** if

I1: ....

I2: ....

I3: ....

g On $\mathbb{R}$-VS X, a ***norm*** $\|\cdot\|$ is a map ...... $\to$ ...... satisfying these three axioms. [*Hint:* Quantifiers.]

N1: ....

N2: ....

N3: ....

N4: ....

h On a set Y, a ***metric*** m is a map ...... $\to$ ...... such that $\forall$ ...... :

MS1: ....

MS2: ....

MS3: ....

MS4: ....

i [!] MS (Ω, d) has $Y \subset \Omega$. So Y is Ω-***precompact*** IFF [Put Ω- or Y- before "closed/open/interior" etc.] ....

j [!] An ***isometry*** from MS (X, d) to MS (Ω, μ) is ... .

See next page.

D1: (*Examples.*) Show no work.

a Fnc $h(x) :=$ ____ is a bounded continuous map $\mathbf{J}\to\mathbb{R}$ which is *not* uniformly-cts.

b Fnc $g(x) :=$ ____ is a bounded uniformly-cts map $\mathbf{J}\to\mathbb{R}$ which is *not* Lipschitz cts.

c P.L fncs f_n converge ptwise, but **not** uniformly, to $[1+Id]$, where the cutpoint and height tuples of f_n are $\vec{\mathbf{p}} := (2,$ ____ $,$ ____ $,5)$ and $\vec{\mathbf{h}} := (3,$ ____ $,$ ____ $,6)$.

d Use $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ for the arctan & stereogr. metrics. With $b_n :=$ ____, seq $\vec{\mathbf{b}} \subset \mathbb{R}$ is $\boldsymbol{\alpha}$-Cauchy but not $\boldsymbol{\sigma}$-Cauchy. With $c_n :=$ ____, sequence $\vec{\mathbf{c}} \subset \mathbb{R}$ is $\boldsymbol{\sigma}$-Cauchy but not $\boldsymbol{\alpha}$-Cauchy.

e Define $\Omega :=$ ____ $\subset \mathbb{R}$ st. the Ω-*closed* ball $C := \Omega\text{-CldBal}_5(0) =$ ____ satisfies $C \supsetneqq \text{Itr}_\Omega(C) =$ ____ $\supsetneqq \Omega\text{-Bal}_5(0) =$ ____.

f Define $X :=$ ____ $\subset \mathbb{R}$ st. the X-open ball $B := X\text{-Bal}_3(0) =$ ____ satisfies $B \subsetneqq \text{Cl}_X(B) =$ ____ $\subsetneqq X\text{-CldBal}_3(0) =$ ____

g Sets $A :=$ ____ and $B :=$ ____ have $\partial_{\mathbb{R}}(A) =$ ____ and $\partial_{\mathbb{R}}(B) =$ ____. Moreover,
____ $= \partial_{\mathbb{R}}(A) \cap \partial_{\mathbb{R}}(B) \subsetneqq \partial_{\mathbb{R}}(A\cap B) =$ ____.

h Sets $C :=$ ____ and $D :=$ ____ have $\partial_{\mathbb{R}}(C) =$ ____ and $\partial_{\mathbb{R}}(D) =$ ____. Further,
____ $= \partial_{\mathbb{R}}(C) \cap \partial_{\mathbb{R}}(D) \supsetneqq \partial_{\mathbb{R}}(C\cap D) =$ ____.

i Let $S := \{q \in \mathbb{Q}_+ \mid 5 \leqslant q^2 < 9\}$. Then:
$\text{Cl}_{\mathbb{R}}(S) =$ ____. $\text{Itr}_{\mathbb{R}}(S) =$ ____.
$\text{Cl}_{\mathbb{Q}}(S) =$ ____. $\text{Itr}_{\mathbb{Q}}(S) =$ ____.

j Let $S := \{q \in \mathbb{Q}_+ \mid 5 \leqslant q^2 < 9\}$. Then:
$\partial_{\mathbb{R}}(S) =$ ____. $\partial_{\mathbb{Q}}(S) =$ ____.

D2: (*Computations.*) Show no work.

a Let $\mathbf{v} := (3,-3,2,1,1) \in \mathbb{R}^5$; so $\|\mathbf{v}\|_3 =$ ____.

b Using the stereographic-metric on $\dot{\mathbb{R}}$:
$\boldsymbol{\sigma}$-Diam(Primes)= ____.

c With $\boldsymbol{\alpha}(\cdot,\cdot)$ the arctan metric on $\mathbb{R}$, the $\boldsymbol{\alpha}$-Diam(PRIMES)= ____.
[*Hint:* No $\boldsymbol{\alpha}()$ should appear in your ans. But arctan() can.]

d Fnc $\arctan : \mathbb{R}\to(\frac{-\pi}{2}, \frac{\pi}{2})$ has Lipschitz constant (circle)
$\frac{1}{\pi}$ $\frac{2}{\pi}$ 1 2 π *DNE*

e Let $f:\mathbb{R}\to\mathbb{R}$ by $f(x) := [3x - x^3] - 1$. Define restrictions $g := f\restriction_{[-2,1]}$ and $h := f\restriction_{[-3,3]}$. Then the sup-norm $\|g\|_{\sup} =$ ____ and $\|h\|_{\sup} =$ ____.

f [!] For $B \in \mathbb{R}_+$, the stereographic-distance $\boldsymbol{\sigma}(B, \frac{1}{B}) =$ ____.

g [!] Let $X := (-\infty, 2) \sqcup [5, +\infty)$. As a union of intervals, the punctured-ball $X\text{-PBal}_6(7) =$ ____.

DA: (All of these True/False questions are new.) [!] We have a fixed MS (X, d) and subsets $B, C \subset X$ that share a common point $p \in B \cap C$.

a Suppose each of B and C is compact.
Then $B \cup C$ is compact. *T* *F*
Then $B \cap C$ is compact. *T* *F*

b Suppose each of B and C is path-connected.
Then $B \cup C$ is path-connected. *T* *F*
Then $B \cap C$ is path-connected. *T* *F*

c Suppose each of B and C is connected.
Then $B \cup C$ is connected. *T* *F*
Then $B \cap C$ is connected. *T* *F*

DB: (All of these are new.) [!]

a The map $x \mapsto \boldsymbol{\pi} - x$ from $\mathbb{R}\circlearrowleft$ is an *isometry*, when $\mathbb{R}$ is equipped with the... Usual metric: *T* *F*.
Stereographic-metric: *T* *F*. Arctan-metric: *T* *F*.

b In $\mathbb{R}^3$, letting $p := (x, y, z)$:

$$\lim_{p \to \hat{\mathbf{0}}} \frac{xy - z^2}{x^2 + y^2 + z^2} \quad \text{exists.} \qquad T \quad F$$

Essay questions: **Fill-in all blanks.** *For each question, carefully write a triple–spaced essay solving the problem.*

D3: Let $\mathbf{J}$ be the interval $(2, 6)$. Suppose functions $H_n \xrightarrow{\text{uniformly}} f$, where $f, H_n : \mathbf{J} \to \mathbb{R}$. If each H_n is uniformly-cts, prove that f is **uniformly-cts**.

D4: State and prove the Intermediate-value theorem.

D5: We have sequences $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{R}$ with $\lim(\vec{\mathbf{x}}) = 6$ and $\lim(\vec{\mathbf{y}}) = 2$. Letting $p_n := x_n / y_n$, give a rigorous ε-proof that $\lim(\vec{\mathbf{p}}) = 3$.

(You may quote, without proof, this result: *If* $\vec{\mathbf{b}}$ *convergent, then* $\vec{\mathbf{b}}$ *is Cauchy. A fortiori,* $\text{Diam}\big(\text{Range}(\vec{\mathbf{b}})\big) < \infty$.)

D6: In a normed-VS $(\mathbf{W}, \|\cdot\|)$, suppose we have a sequence $\vec{\mathbf{x}} \in \mathbf{W}$ and a number $r \in [0, 1)$ such that
$\forall n \in \mathbb{Z}_+ : \|x_n - x_{n+1}\| \leqslant r^n$. Prove that sequence $\vec{\mathbf{x}}$ is $\|\cdot\|$-Cauchy.

D7: [!] Prove that the interval $\mathbf{J} := [3, 7]$ is connected.

D8: [!] State the Heine-Borel thm. State the Bolzano-Weierstrass thm.

D9: [ACT] [!] A MS (X, d) is ***countable self-dense*** (***CSD***) if there exists a *countable* subset $D \subset X$ which is X-dense, i.e $\text{Cl}_X(D) = X$. Prove, given a subset $\Omega \subset X$, that (Ω, d) is CSD.

[*Aside:* In general TSes this is false, but is easy in MSes.]