A countably-valued sleeping stockbroker process

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Abstract. We exhibit a stationary countably-valued process $\{V_n\}_{-\infty}^{\infty}$ which is deterministic, but which is non-deterministic in the sense that whenever $\ldots n_{-2} < n_{-1} < n_0 < n_1 < \ldots$ are indices with no two consecutive, then $\{V_{n_i} \mid i \in \mathbb{Z}\}$ is an independent process. This answers a question of [1].

In addition, although $n \mapsto V_n$ is deterministic, its time reversal $n \mapsto V_{-n}$ is not deterministic.

§B Beginning

The following question was asked by Furstenberg, in the context of ergodic theory. Suppose

$$\mathbf{X} = \ldots, X_{-2}, X_{-1}, X_0, X_1, X_2 \ldots$$

is a stationary process which is ***deterministic***; that is, X_0 is measurable with respect to the σ-field generated by $\ldots, X_{-2}, X_{-1}$. Must

$$\ldots X_{-4}, X_{-2}, X_0, X_2, X_4, \ldots$$

be deterministic?

In "Dilemma of the Sleeping Stockbroker" [1], this was phrased in the fanciful setting of a stockbroker who slept late every second day, and thus missed learning the Dow Jones Average for that particular day. The Dow Jones Average was (realistically?) assumed to be the output of a stationary real-valued process, and was (unrealistically!) assumed to be deterministic. Knowing the Average from yesterday, three days ago, five days past and so on, the article asked whether our hypothetical sleepy stockbroker could predict tomorrow's value.

Using Haar measure on the circle, [1] constructed a real-valued stationary process $\mathbf{V}$ which was counterexample with a twist. It satisfies

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(*a*) The process is deterministic. Indeed, the process is strongly deterministic in that the entire process is a function of any two consecutive terms; each V_n is predictable from knowledge of the pair $V_0 V_1$.

(*b*) The process is ***non-consecutively independent***. This means that whenever $\{n_i\}_{i=-\infty}^{\infty}$ is a sequence of indices with no consecutive pair, $1 + n_i < n_{i+1}$ for all i, then

$$\ldots V_{n_{-2}} V_{n_{-1}} V_{n_0} V_{n_1} \ldots$$

is an independent process.

Generalization. Henceforth, "process" means a stationary process. Say that a process $\mathbf{V}$ is a ***sleeping stockbroker process*** if it is deterministic and non-consecutively independent. Can this be realized with a countably-valued process, that is, where Range(V_0) is countable? Certainly the strong determinism of(a) is impossible, since the pair V_0V_1 takes on only countably many values, whereas $\mathbf{V}$ takes on uncountably many values.

The purpose of the current note is to construct a countably-valued sleeping stockbroker process $\mathbf{V}$. It combines the nesting idea of the original article together with random look-ahead. In order to obtain a process over an alphabet which is only countable, here we use Haar measure on finite groups rather than Haar measure on the circle. The article closes with an open question about the distribution V_0 can have, subject to a necessary entropy constraint.

A side effect of the construction is that although $\mathbf{V}$ is deterministic, the time-reversal process $n \mapsto V_{-n}$ is not deterministic.

Nomenclature. Let $x \coloneqq y$ mean that y is the definition of the symbol x. Let $[a \,..\, b)$ denote the "interval of integers" $[a, b) \cap \mathbb{Z}$, with similar meaning for $[a \,..\, b]$.

§C Construction

Let $\mathbf{L}$ be an iid process taking values in the integers bigger than 2; the distribution of L_0 will be specified later. The random values of $\mathbf{L}$ will describe how far the sleeping stockbroker process $\mathbf{V}$ looks into its own future.

A "word" is to mean a string over the alphabet $\{\mathtt{0}, \mathtt{1}\}$. Let $\mathbf{X}$ be iid so that X_n is a word of length L_n and the word is uniformly distributed over all 2^{L_n} words of length L_n.

The Future Process. Let $\mathbf{F}$ be the following encoding of the future of $\mathbf{X}$ into half-infinite bit-strings. $\mathbf{F}$ is the stationary process defined by

$$F_n \coloneqq \mathtt{1}^{L_n - 1}\mathtt{0}X_n \; \mathtt{1}^{L_{n+1}-1}\mathtt{0}X_{n+1} \; \mathtt{1}^{L_{n+2}-1}\mathtt{0}X_{n+2} \; \cdots$$

Since F_n starts with $L_n - 1$ occurrences of $\mathtt{1}$, its leftmost bit is always $\mathtt{1}$. Note that the values of $X_n, X_{n+1}, \ldots$ can be read from F_n.

Determined by the $\mathbf{X}$ process, say that a position $n \in \mathbb{Z}$ is ***marked*** if the leftmost bit of X_n is 0 and the leftmost bit of X_{n+1} is 1, i.e, if

$$\mathrm{Left}(X_n)\mathrm{Left}(X_{n+1}) = \text{“}\mathtt{01}\text{”}.$$

Define the sleeping stockbroker process $\mathbf{V}$ by

$$V_n := \begin{cases} X_n, & \text{if } n \text{ is \underline{un}marked;} \\ X_{n+1} \oplus F_{n+2}, & \text{if } n \text{ is marked,} \end{cases}$$

where the symbol “$X \oplus F$” denotes the bit-string whose length is that of the shorter of bit-strings X and F and which is the componentwise mod-2 sum of the strings.

Note that always $\mathrm{Left}(V_n) = \mathrm{Left}(X_n)$ since, if n is marked, then

$$\begin{aligned} \mathrm{Left}(V_n) &= \mathrm{Left}(X_{n+1}) \oplus \mathrm{Left}(F_{n+2}) \\ &= \mathtt{1} \oplus \mathtt{1} = \mathtt{0} = \mathrm{Left}(X_n)\,. \end{aligned}$$

Thus we can determine from the $\mathbf{V}$ process which positions are marked.

Process V is non-consecutively independent. It is certainly enough to show V_0 independent of $(V_2, V_3, \dots)$. Since $\mathbf{V}_{[2\,..\,\infty)}$ is determined by $\mathbf{X}_{[2\,..\,\infty)}$, it suffices to establish that

$$V_0 \ \text{ is independent of } \ \mathbf{X}_{[2\,..\,\infty)} \tag{1}$$

Condition on $\mathbf{X}_{[2\,..\,\infty)}$ and note that F_2 is now determined.

In the case position 0 is marked, $V_0 = X_1 \oplus F_2$. The leftmost bit of word X_1 is 1. The rest of word, $X_1[2\,..\,L_1]$, is uniformly distributed over all words of length $L_1 - 1$ and consequently $V_0[2\,..\,L_1]$ is likewise uniformly distributed. Thus V_0 is uniformly distributed subject to the “marked” constraint on its leftmost bit.

Conversely, when position 0 is unmarked, V_0 equals X_0 and is uniformly distributed over words of length L_0, subject to the “not marked” constraint on its leftmost bit.

In each case the distribution of V_0 does not depend on $\mathbf{X}_{[2\,..\,\infty)}$. Since the event “position 0 is marked” is itself independent of $\mathbf{X}_{[2\,..\,\infty)}$, this proves (1).

§D Determinism

Assume for the moment that the distribution of L_0 can be chosen so that the following “Look-ahead Condition” holds.

Almost-surely, there are infinitely many k such that

$$L_{-k} \geq 2L_{-(k-1)} + 2L_{-(k-2)} + \cdots + 2L_{-1} + 2L_0\,. \tag{2}$$

Since (2) is independent of which positions are marked, there are infinitely many such k such that $-k-1$ is marked; call such a k "good".

The $\mathbf{V}$ process tells us the marked positions and so we can determine which k are good. For a good position k,

$$\begin{aligned} V_{-k} \oplus V_{-k-1} &= X_{-k} \oplus \big(X_{-k} \oplus F_{-k+1}\big) \\ &= F_{-k+1}[1 \mathinner{..} L_{-k}]\,. \end{aligned}$$

And by (2), this latter string reveals the values of $X_{-k+1}, X_{-k+2}, \ldots, X_0$.

The upshot is that the Look-ahead Condition insures that the arbitrarily remote past, the tail-field of $\mathbf{V}$, determines X_0. By stationarity, then, Tail($\mathbf{V}$) also determines $X_1, X_2, \ldots$. Thus it determines V_0.

Constructing the Look-ahead process. To produce an iid process $\mathbf{L}$ satisfying (2) we need to construct L_0 with sufficiently infinite expectation. We claim it suffices that

$$\sum_{k=1}^{\infty} \mathcal{P}\big(L_0 > 3^k\big) \quad \text{is infinite.} \tag{2'}$$

To prove this, let $f(k) := 3^k$ and note that the above implies

$$\text{For all } M\text{:} \qquad \sum_{k=1}^{\infty} \mathcal{P}\Big(\tfrac{L_0}{f(k)} > M\Big) \;=\; \infty\,. \tag{3a}$$

This, together with a Borel-Cantelli argument imply

$$\sup_k \frac{L_k}{f(k)} \;=\; \infty\,. \tag{3b}$$

Therefore, there exist infinitely many k such that $\frac{L_k}{f(k)} > \frac{L_i}{f(i)}$ for $i = 0, 1, \ldots, k-1$. Consequently, letting R abbreviate the ratio $\frac{L_k}{f(k)}$,

$$\begin{aligned} L_k &= R \cdot f(k) \\ &> R \cdot 2\big[f(k-1) + f(k-2) + \cdots + f(0)\big]\,, \qquad \text{since } f(i) = 3^i \\ &> 2\big[L_{k-1} + L_{k-2} + \cdots + L_0\big]\,, \end{aligned}$$

as desired.

Remark: *Inequality of the tail- and future- fields.* The argument above established that the tail-field of $\mathbf{V}$ is the entire sigma-field. Conversely, the future of $\mathbf{V}$ is determined by the future of $\mathbf{X}$. Since the future-field of the iid $\mathbf{X}$ is trivial, the future-field of $\mathbf{V}$ is also trivial. Thus the tail-field and future-field of $\mathbf{V}$ are as different as they can be.

§E Entropy

We close with this question.

$$\textit{What distributions are possible for } V_0\textit{, when } \mathbf{V} \textit{ is a stockbroker process?} \tag{4}$$

The only obstruction known to the authors comes from entropy, which we tersely review now. An extensive discussion can be found in [2] or [3].

Consider a random variable V which takes on countably many values $a_1, a_2, \ldots$. Its ***distribution entropy***, here abbreviated as "distropy", is the sum

$$H(V) \coloneqq \sum_k \eta\big(\mathcal{P}(V = a_k)\big)\,,$$

where $\eta\colon [0,1] \to [0,\infty)$ by $\eta(x) \coloneqq x \ln(1/x)$ and $\eta(0) \coloneqq 0$. The ***entropy*** of the process $\mathbf{V}$ is the limit

$$h(\mathbf{V}) \coloneqq \lim_{N\to\infty} \frac{1}{N} H(V_1, V_2, \ldots, V_N)\,,$$

where the N-tuple $(V_1, \ldots, V_N)$ is viewed as a countably-valued random variable; this limit always exists as a value in $[0,\infty]$. Immediately from the definition,

$$h\big(\{V_{2n}\}_{n=-\infty}^{\infty}\big) \;\le\; 2 \cdot h(\mathbf{V})\,.$$

Determinism and entropy. If $h(\mathbf{V}) = 0$ then $\mathbf{V}$ is deterministic. We refer the reader either to [2, pp.59,60,41] or [3, pp.239,240] for this partial converse.

If V_0 has finite distropy, then process $\mathbf{V}$ is deterministic iff $\mathbf{V}$ has zero entropy.

Consequently, for a process $\mathbf{V}$ with $H(V_0) < \infty$ (in particular, every finitely-valued process), determinism implies entropy zero and so forces the "every second day" process $\{V_{2n}\}_{n=-\infty}^{\infty}$ to also have zero entropy; whence it is deterministic. So a partial answer to question (4) is that V_0 have infinite distropy.

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