Counting spanning-trees

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/

9 April, 2018 (at *09:32*)

Entrance. Use ***spantree*** for "spanning-tree".

Our matrices are over some commutative ring Γ, and lowercase greek $\alpha,\beta,\ldots,\omega \in \Gamma$. Use $\mathsf{M}_{\nu,\,\delta;\,K}$ for the $K\times K$ matrix with each $m_{j,j} := \delta$ [δiagonal] and each $m_{i\neq j} := \nu$ [νot diagonal]. When the matrix-size K is understood, then I will just write $\mathsf{M}_{\nu,\,\delta}$ for $\mathsf{M}_{\nu,\,\delta;\,K}$.

1: Lemma. *For $\sigma,\omega,J \in \Gamma$ and matrix-size $K \in \mathbb{Z}_+$:*

$$\dagger: \quad \mathrm{Det}(\mathsf{M}_{\text{-}1,\,\sigma}) \;=\; [\sigma{+}1]^{K-1}\cdot[\sigma{+}1-K]\,.$$

$$\ddagger: \quad \mathrm{Det}(\mathsf{M}_{\text{-}J,\,\omega-J}) \;=\; \omega^{K-1}\cdot[\omega \,-\, [JK]]\,. \qquad \diamondsuit$$

Proof of ($\dagger$). Define β by $\beta + [K{-}1] := \sigma$. Adding the bottom $K{-}1$ rows of M to the top row, produces

$$\mathsf{H} \;:=\; \begin{bmatrix} \beta & \beta & \beta & \cdots & \beta \\ \text{-}1 & \sigma & \text{-}1 & \cdots & \text{-}1 \\ \text{-}1 & \text{-}1 & \sigma & \cdots & \text{-}1 \\ \text{-}1 & \text{-}1 & \text{-}1 & \ddots & \text{-}1 \\ \text{-}1 & \text{-}1 & \text{-}1 & \cdots & \sigma \end{bmatrix}, \quad \text{with } \mathrm{Det}(\mathsf{H}) \text{ equaling } \mathrm{Det}(\mathsf{M}).$$

If $\beta = 0$, then $[\sigma{+}1-K] = 0$ so RhS($\dagger$)=0; and the top row of H is $\mathbf{0}$ so Det(H)=0. Thus, WLOG $\beta \neq 0$.

Multiplying by β, each of the bottom $K{-}1$ rows of H, builds this matrix

$$\mathsf{H}' \;:=\; \begin{bmatrix} \beta & \beta & \cdots & \beta \\ \text{-}\beta & \sigma\beta & \text{-}\beta & \text{-}\beta \\ \text{-}\beta & \text{-}\beta & \ddots & \text{-}\beta \\ \text{-}\beta & \text{-}\beta & \cdots & \sigma\beta \end{bmatrix}, \quad \text{where } \mathrm{Det}(\mathsf{H}') \text{ equals } \beta^{K-1}{\cdot}\mathrm{Det}(\mathsf{M}).$$

Adding the top row to each of the others produces

$$\mathsf{H}'' \;:=\; \begin{bmatrix} \beta & \beta & \cdots & \beta \\ & \sigma\beta+\beta & & \\ & & \ddots & \\ & & & \sigma\beta+\beta \end{bmatrix}. \quad \text{And } \mathrm{Det}(\mathsf{H}'') \text{ equals } \beta^{K-1}{\cdot}\mathrm{Det}(\mathsf{M}).$$

So $\beta^{K-1}{\cdot}\mathrm{Det}(\mathsf{M}) = [\sigma\beta+\beta]^{K-1}{\cdot}\beta$. As $\beta{\neq}0$, dividing says that $\mathrm{Det}(\mathsf{M})$ equals $[\sigma{+}1]^{K-1}{\cdot}\beta \overset{\text{note}}{=\!=} \text{RhS}(\dagger)$. ♦

Proof of ($\ddagger$). WLOG $J \neq 0$. Dividing each M-row by J, produces

$$\mathsf{G} \;:=\; \begin{bmatrix} \frac{\omega}{J}-1 & \text{-}1 & \text{-}1 & \text{-}1 \\ \text{-}1 & \frac{\omega}{J}-1 & \text{-}1 & \text{-}1 \\ \text{-}1 & \text{-}1 & \ddots & \text{-}1 \\ \text{-}1 & \text{-}1 & \cdots & \frac{\omega}{J}-1 \end{bmatrix}, \quad \text{where } J^K{\cdot}\mathrm{Det}(\mathsf{G}) \text{ equals } \mathrm{Det}(\mathsf{M}).$$

By ($\dagger$), then, $\mathrm{Det}(\mathsf{G}) = [\frac{\omega}{J}]^{K-1}\cdot[\frac{\omega}{J}-K]$. Multiplying by J^K then shows that $\mathrm{Det}(\mathsf{M}) = \text{RhS}(\ddagger)$ ♦

Appl. Fixing $N \in \mathbb{Z}_+$, let's compute the number of spantrees of K_N, the complete graph on N vertices. Its Laplacian matrix is $\mathsf{L} := \mathsf{M}_{\text{-}1,\,N-1;\,N}$. Take a ***reduced Laplacian*** by removing its last row and column, giving $\mathsf{L}_0 := \mathsf{M}_{\text{-}1,\,N-1;\,N-1}$. The Matrix-tree thm says that K_N has Det(L_0) many spantrees.

Courtesy ($\dagger$), our Det(L_0) equals

$$N^{[N-1]-1}\cdot[N-[N{-}1]] \overset{\text{note}}{=\!=} N^{N-2}\,,$$

as Cayley's thm asserts. □

2.1: Block-UT-matrix Lemma. *For $\mathcal{A},\mathcal{B} \in \mathbb{N}$, consider an Upper-Triangular partitioned matrix*

$$2.2: \qquad \mathsf{M} \;=\; \begin{bmatrix} \mathsf{A}_{\mathcal{A}\times\mathcal{A}} & \mathsf{G}_{\mathcal{A}\times\mathcal{B}} \\ \mathbf{0}_{\mathcal{B}\times\mathcal{A}} & \mathsf{B}_{\mathcal{B}\times\mathcal{B}} \end{bmatrix}$$

Then $\mathrm{Det}(\mathsf{M}) \;=\; \mathrm{Det}(\mathsf{A}){\cdot}\mathrm{Det}(\mathsf{B})$. *In consequence, the char-poly $\wp_{\mathsf{M}}$ factors as*

$$2.2': \qquad \wp_{\mathsf{M}} \;=\; \wp_{\mathsf{A}}\cdot\wp_{\mathsf{B}}\,. \qquad \diamondsuit$$

Proof of $\mathrm{Det}(\mathsf{M}) = \mathrm{Det}(\mathsf{A})\cdot\mathrm{Det}(\mathsf{B})$. Since Det($\mathsf{M}$) is a sum of products taken over all transversals [generalized diagonals] of M, ISTShow that a transversal straying from the A,B blocks necessarily has product zero.

WLOG this misguided transversal hits G. It therefore misses some row of A hence (since A is square) some *column* of A. In this column, then, the transversal must hit the $\mathbf{0}_{\mathcal{B}\times\mathcal{A}}$ block.

Exer: *Why do the signs of the permutations work out correctly?* ♦

3: Lemma. *Define $[J{+}K]\times[J{+}K]$ block-diagonal matrix*

$$\mathsf{V} \;:=\; \left[\begin{array}{c|c} \begin{matrix}\alpha & & \\ & \ddots & \\ & & \alpha\end{matrix} & -1 \\ \hline -1 & \begin{matrix}\beta & & \\ & \ddots & \\ & & \beta\end{matrix} \end{array}\right],$$

where $\alpha,\beta \in \Gamma$ and $J,K \in \mathbb{Z}_+$. Furthermore, $\begin{bmatrix}\alpha & & \\ & \ddots & \\ & & \alpha\end{bmatrix}$ *is $J{\times}J$, block* $\begin{bmatrix}\beta & & \\ & \ddots & \\ & & \beta\end{bmatrix}$ *is $K{\times}K$, and the $J{\times}K$, and $K{\times}J$ rectangles are filled with -1. Then*

$$*: \qquad \mathrm{Det}(\mathsf{V}) \;=\; \alpha^{J-1}\beta^{K-1}\cdot[\alpha\beta \,-\, JK]\,. \qquad \diamondsuit$$

Proof. Exercise: Show the result holds if $\alpha = 0 = \beta$.

Suppose that $(\alpha,\beta) \neq (0,0)$. Since V is symmetric in α,β, WLOG $\boxed{\alpha \neq 0}$.

Multiply by α, each of the bottom K rows of V, producing

$$\mathsf{H} := \left[\begin{array}{c|c} {}^{\alpha}\ddots{}_{\alpha} & -1 \\ \hline -\alpha & {}^{\alpha\beta}\ddots{}_{\alpha\beta} \end{array}\right], \quad \text{where } \operatorname{Det}(\mathsf{H}) = \alpha^K \cdot \operatorname{Det}(\mathsf{V}).$$

Now add each of the top J rows to each of the bottom K rows. This makes matrix

$$\mathsf{H}' := \left[\begin{array}{c|c} {}^{\alpha}\ddots{}_{\alpha} & -1 \\ \hline 0 & \mathsf{M} \end{array}\right], \quad \text{where } \mathsf{M} := \mathsf{M}_{-J,\,\alpha\beta-J;\,K}.$$

By the Block-UT-matrix Lemma, $\operatorname{Det}(\mathsf{H}')$ equals

$$\alpha^J \cdot \operatorname{Det}(\mathsf{M}) \overset{\text{by }(\ddagger)}{=\!=} \alpha^J \cdot [\alpha\beta]^{K-1}[\alpha\beta - JK].$$

Recall that $\operatorname{Det}(\mathsf{H}') = \alpha^K \cdot \operatorname{Det}(\mathsf{V})$. Since $\alpha \neq 0$, we divide to conclude that $\operatorname{Det}(\mathsf{V})$ equals

$$\frac{\alpha^J}{\alpha^K} \cdot [\alpha\beta]^{K-1} \cdot [\alpha\beta - JK] \overset{\text{note}}{=\!=} \operatorname{RhS}(*). \quad \blacklozenge$$

4: Spantrees in complete bipartite. *For $G,B \in \mathbb{Z}_+$, the complete bipartite $K_{G,B}$ has*

$$4': \qquad \#\text{Spantrees}(K_{G,B}) = G^{B-1} \cdot B^{G-1}$$

many spanning trees. ♢

Proof. Listing the B many boys before the girls, the Laplacian of $K_{B,G}$ is the $[B+G]\times[B+G]$ block-diagonal matrix

$$\mathsf{L} := \left[\begin{array}{c|c} {}^{G}\ddots{}_{G} & -1 \\ \hline -1 & {}^{B}\ddots{}_{B} \end{array}\right].$$

Letting $K := G-1$, remove the last row and column to get reduced Laplacian

$$\mathsf{L}_0 := \left[\begin{array}{c|c} {}^{G}\ddots{}_{G} & \overbrace{-1}^{G-1 \text{ cols}} \\ \hline -1 & {}^{B}\ddots{}_{B} \end{array}\right].$$

We apply Lemma 3 with $\alpha := G$, $J := B$, $\beta := B$ and $K := G-1$. Notice that the $[\alpha\beta - JK]$ from $(*)$ equals

$$GB - B[G-1] \overset{\text{note}}{=\!=} B.$$

Thus $\operatorname{Det}(\mathsf{L}_0) = G^{B-1} B^{G-1-1} \cdot B \overset{\text{note}}{=\!=} \operatorname{RhS}(4')$. $\blacklozenge$

5: Diminished K_N. *For $N \geqslant 2$, let D_N be the complete graph K_N with one edge removed; so D_N has $\binom{N}{2}-1$ edges. Then*

$$5': \qquad \#\text{Spantrees}(D_N) = [N-2] \cdot N^{[N-3]}. \qquad ♢$$

Pf. For $N=2$, the above correctly asserts "$0 = 0$", so henceforth $N \geqslant 3$. Numbering the vertices $1,2,\ldots,N$ and removing the edge between vertices $N-1$ and N, the $N\times N$ Laplacian matrix $\mathsf{L}(D_N)$ equals

$$\mathsf{L}(D_N) := \begin{bmatrix} N-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & -1 & \cdots & -1 \\ -1 & -1 & -1 & N-1 & -1 & -1 \\ -1 & \cdots & -1 & -1 & N-2 & 0 \\ -1 & \cdots & -1 & -1 & 0 & N-2 \end{bmatrix}.$$

Removing the last row and column yields reduced-Laplacian

$$\mathsf{L}_0(D_N) := \begin{bmatrix} N-1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & \cdots & -1 & -1 \\ -1 & -1 & \ddots & \cdots & -1 \\ -1 & -1 & -1 & N-1 & -1 \\ -1 & \cdots & -1 & -1 & N-2 \end{bmatrix}.$$

Replacing the first row by the sum of all the rows gives

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ -1 & N-1 & \cdots & -1 & -1 \\ -1 & -1 & \ddots & \cdots & -1 \\ -1 & -1 & -1 & N-1 & -1 \\ -1 & \cdots & -1 & -1 & N-2 \end{bmatrix}.$$

Adding the first row to all the others, produces

$$\mathsf{U}(D_N) := \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & N & \cdots & 1 & -1 \\ & & \ddots & 1 & -1 \\ & & & N & -1 \\ & & & & N-2 \end{bmatrix}.$$

Hence $\operatorname{Det}(\mathsf{L}_0) = \operatorname{Det}(\mathsf{U}) = \operatorname{RhS}(5')$. $\blacklozenge$

Filename: `Problems/GraphTheory/counting-spantrees.latex`
As of: *Monday 19Mar2018.* Typeset: *9Apr2018* at *09:32.*