Cooling

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Autonomous-DiffyQ Example. Let $u{=}u(t)$ denote the temperature of a particular object at time t. Newton's law of cooling says there is a proportionality constant $\mathbf{k}$ [depending on the materials] so that

$$\text{1a:}\qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{k}\cdot[u - \boldsymbol{A}]\,,$$

where $\boldsymbol{A}$ denotes the ambient temperature.

Using abstract units ⓟ for temPerature, and ⓣ for Time, we have: $u,\boldsymbol{A}$ :: ⓟ and thus $\mathbf{k}$ :: 1/ⓣ. Note that $\mathbf{k}$ will be negative, as the object's tmp will tend toward the ambient tmp.

Scenarios. If $\boldsymbol{A}$ is constant, then the situation is time-invariant, so the DE is autonomous.

Alternatively, $\boldsymbol{A}()$ and $u()$ might be linked U.Fncs; e.g, $u(t)$ is the [decreasing] temperature of the ice-cube, melting into your orange juice which is at [increasing] temperature $\boldsymbol{A}(t)$, both changing. Our (1a) is still an autonomous [time-invariant] DE, but now it relates two unknown-functions.

Alternatively, $\boldsymbol{A}{=}\boldsymbol{A}(t)$ might be a known fnc of time; e.g, we are in the desert, so $\boldsymbol{A}()$ is periodic with a 24-hour period. Here, (1a) is *not* an autonomous-DE. □

When $\boldsymbol{A}$ is constant. Here, (1a) is a FOLDE with constant coeff and target fncs. It general soln is

$$\text{1b:}\qquad u_\alpha(t) = \boldsymbol{A} + \alpha\mathrm{e}^{\mathbf{k}t}\,.$$

Note that product $\mathbf{k}t$ is unitless ["dimensionless"] and so is $\mathrm{e}^{\mathbf{k}t}$. Thus α :: ⓟ. □

Worked example. We remove a cake from a 320 °F oven. Three minutes later we measure the cake's temperature; it has cooled to 220 °F.

When will the cake be at an edible[♡1] *tmp of* 100 °F, *given that the ambient tmp of the kitchen is* 70 °F*?*

[♡1] Apparently we can eat foods noticeably hotter than 100 °F.

Step 1. *Replace all the quantities by meaningful letters.*

Below, I'll employ greek letters Δ and τ for time, and roman letters for temperature. With italic boldface $\boldsymbol{0}$ denoting 0min, define the following:

$$\Delta := \begin{bmatrix}\text{Time-change from}\\ \text{Oven to Measured}\end{bmatrix} = 3\mathsf{min}\,.$$
$$\tau := [\text{Edible time}]\,.$$
$$\boldsymbol{A} := [\text{Ambient tmp}] = 70\,°\mathrm{F}.$$
$$V := \begin{bmatrix}\text{Oven-tmp distance}\\ \text{to ambient}\end{bmatrix} = u(\boldsymbol{0}) - \boldsymbol{A} = [320 - 70]\,°\mathrm{F} = 250\,°\mathrm{F}.$$
$$M := \begin{bmatrix}\text{Measured-tmp}\\ \text{dist. to ambient}\end{bmatrix} = u(\Delta) - \boldsymbol{A} = [220 - 70]\,°\mathrm{F} = 150\,°\mathrm{F}.$$
$$E := \begin{bmatrix}\text{Edible-tmp dist.}\\ \text{to ambient}\end{bmatrix} = u(\tau) - \boldsymbol{A} = [100 - 70]\,°\mathrm{F} = 30\,°\mathrm{F}\,.$$

So Δ,τ :: ⓣ and $\boldsymbol{A},V,M,E$ :: ⓟ.

Step 2. *Identify the unknowns, and solve for them.* Parameter $\mathbf{k}$, in the DE, and parameter α, in the soln, are unknown. [We'll get to time τ, later.]

Evaluating (1b) at $t = 0\mathsf{min}$ gives

$$\alpha = \alpha\cdot 1 = \alpha\cdot\mathrm{e}^{\mathbf{k}\cdot\boldsymbol{0}} = u(\boldsymbol{0}) - \boldsymbol{A} \stackrel{\text{def}}{=\!=} V\,.$$

Hence

$$\text{1b}\dagger\text{:}\qquad u(t) = \boldsymbol{A} + V\mathrm{e}^{\mathbf{k}t}\,.$$

To solve for $\mathbf{k}$, note the tmp-drop over time Δ is

$$V - M = u(\boldsymbol{0}) - u(\Delta) = V\cdot[1 - \mathrm{e}^{\mathbf{k}\Delta}]\,.$$

Hence $\mathrm{e}^{\mathbf{k}\Delta} = \frac{M}{V}$. Taking logs, then,

$$\text{1c:}\qquad \mathbf{k} = \frac{1}{\Delta}\cdot\log(M/V)\,.$$

Plugging this in to (1$b\dagger$) gives

$$\text{1d:}\qquad u(t) = \boldsymbol{A} + V\cdot\left[\tfrac{M}{V}\right]^{t/\Delta}$$

for *the* soln to (1a) satisfying the init.conditions.

Step 3. *Does the symbolic soln agree with known values?* Evaluating RhS(1d) at $t{=}\mathbf{0}$ gives $\boldsymbol{A}+V$, which indeed equals $u(\mathbf{0})$, the LhS(1d).

Evaluating RhS(1d) at $t{=}\Delta$ delivers

$$\boldsymbol{A} + V\cdot\left[\tfrac{M}{V}\right] = \boldsymbol{A} + M \stackrel{\text{note}}{=\!=} u(\Delta)\,.$$

Vary the letters in your symbolic soln. Does the resulting answer vary the way the physics of the situation says it should?

E.g, lowering M [retaining all other values] means the cake lost more heat in those same 3 minutes; hence $u()$ should drop in temperature faster. And indeed, a decrease of M makes the smaller-than-one positive number $\frac{M}{V}$ even smaller, making its powers decrease faster.

Step 4. *Compute the desired quantities from the symbolic soln, then plug in the actual values.*

By its definition, time τ satisfies

$$\boldsymbol{A}+E \stackrel{\text{def}}{=\!=} u(\tau) \stackrel{\text{by (1d)}}{=\!=\!=} \boldsymbol{A} + V\cdot\left[\tfrac{M}{V}\right]^{\tau/\Delta}.$$

Thus $\left[\frac{M}{V}\right]^{\tau/\Delta} = \frac{E}{V}$. Consequently,

$$\text{1e:}\qquad \tau = \Delta\cdot\frac{\log(E/V)}{\log(M/V)} \stackrel{\text{note}}{=\!=} \Delta\cdot\frac{\log(V/E)}{\log(V/M)}\,.$$

This τ is a function of E. So, for a temperature-difference x, define

$$*:\qquad \tau(x) \;:=\; \Delta\cdot\frac{\log(V/x)}{\log(V/M)}\,.$$

We expect...

...$\tau(V)$ to be $\mathbf{0}$, since the cake hasn't cooled at all.

...$\tau(M)$ to be Δ, since the cake is at the measured temperature.

...that the limit of $\tau(x)$, as x decreases toward 0 °F [i.e, as edible-tmp decreases toward $\boldsymbol{A}$], to be $+\infty$; this, since the cake never achieves ambient temperature in finite time.

Reassuringly, all three of these expectations are fulfilled by RhS($*$).

Finally, let's compute an approx. value for $\tau{=}\tau(E)$:

$$\tau = 3\text{min}\cdot\frac{\log(250/30)}{\log(250/150)} = 3\text{min}\cdot\frac{\log(25/3)}{\log(5/3)} \approx 3\text{min}\cdot 4.15 \approx 13\text{min}\,.$$

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