A denumerable Hausdorff space which is connected

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1 February, 2022 (at *19:37*)

(This example is from Moe Hirsch; the details are routine. After working this out, I found this example in the book "Counterexamples in Topology". It is the *Relatively Prime Integer Topology*, example #**60** on p.82, which includes a lot more than I have here. A quite different example is #**75**, p.93, the *Irrational Slope Topology*.)

Remark. For a connected space Ω, any continuous image is connected. So if Ω is countable, then any continuous map $f:\Omega \to \text{MetricSpace}$ is constant. Conversely, were Ω disconnected it would have a continuous image in a two point space.

The inevitable conclusion is that the space Ω of the title cannot be compact –for then it would be normal, and Urysohn's lemma would provide a non-constant real-valued function. □

Construction

Our space, Ω, is $\mathbb{Z}_+$. Below, let x, y, a, b, c range over $\mathbb{Z}_+$. Define $\mathcal{B}$ to consist of the arithmetic progressions in Ω: Determined by a relatively-prime pair $x \perp a$, an ***arithmetic progression*** is of the form $x\mathbb{N} + a$; the set of all $xn + a$ with $n \in \mathbb{N}$. Let $\mathcal{B}$ generate the topology.

$\mathcal{B}$ is a basis; it is closed under pairwise intersection

Suppose $U := x\mathbb{N} + a$ and $V := y\mathbb{N} + b$ are not disjoint. Let $c := \text{Min}(U \cap V)$. Evidently

1: $$U \cap V \supset \text{LCM}(x, y)\mathbb{N} + c\,.$$

We now show equality in (1). Suppose $d \in U \cap V$; necessarily $d \geq c$. Since $d, c \in x\mathbb{N} + a$ then $d - c \in x\mathbb{Z}$ and so $x \mid [d - c]$; x divides $d - c$. Similarly $y \mid [d - c]$. Hence $\text{LCM}(x, y)$ divides $d - c$. Thus $d \in \text{LCM}(x, y)\mathbb{N} + c$. This establishes the reverse inclusion in (1) and hence equality.

The last step in showing that $U \cap V$ is in $\mathcal{B}$.

Note that $c \in x\mathbb{N} + a \implies c \perp x$, since $a \perp x$. Similarly $c \perp y$. Thus $c \perp \text{LCM}(x, y)$; hence $\text{LCM}(x, y)\mathbb{N} + c$ is indeed a member of $\mathcal{B}$. In particular, $\mathcal{B}$ is a basis for the topology.

The topology is Hausdorff

Given distinct $a, b \in \mathbb{Z}_+$, choose a prime number, x, exceeding both. Then $x\mathbb{N} + a$ and $x\mathbb{N} + b$ are necessarily disjoint.

Ω is connected

It suffices to show, for any non-void open U and V, that their closures, $\overline{U}$ and $\overline{V}$, intersect. Since $\mathcal{B}$ is a basis, without loss of generality $U, V \in \mathcal{B}$. So it certainly suffices to show that

2: $$\text{Cl}(x\mathbb{N} + a) \supset x\mathbb{Z}_+\,.$$

For then $\text{Cl}(x\mathbb{N} + a) \cap \text{Cl}(y\mathbb{N} + b) \supset x\mathbb{Z}_+ \cap y\mathbb{Z}_+$, which includes $\text{LCM}(x, y)\mathbb{Z}_+$ and hence is non-empty.

To prove (2), let $U := x\mathbb{N} + a$ and fix a point $x' \in x\mathbb{Z}_+$. To show that x' is in $\overline{U}$, we fix a basic neighborhood W of x' and prove that

2′: *Each neighborhood $W = y\mathbb{N} + b$ of x' indeed intersects U.*

Observe that $y \perp b \implies y \perp x'$, which implies $y \perp x$. Hence –by the Euclidean algorithm– there exist integers m, n such that

$$kn - jm = a\,.$$

Moreover, without loss of generality m is positive. (Just replace "m" by $m + kk$, for a sufficiently large k, and replace "n" by $n + kj$.) Consequently

$$W \supset x' + y\mathbb{N} \ni x' + yn = x' + [a + jm]\,.$$

But this latter is congruenct modulo x to a. Thus W intersects $x\mathbb{N} + a$, which is U. ♦