Notes in the key of $\mathbb{C}$

Jonathan L.F. King, *Univ. of Florida*
squash@ufl.edu http://squash.1gainesville.com/
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§Overview

Entrance. Use $\boldsymbol{i}$ for one of the sqroots of 1. Thus $\boldsymbol{i}^2 = 1 = [-\boldsymbol{i}]^2$. Henceforth, x, y, u, v denote reals, unless otherwise stated. A complex number can be written in form $[x\cdot 1] + [y\cdot\boldsymbol{i}]$. The ***real and imaginary parts*** of $z := [x\cdot 1] + [y\cdot\boldsymbol{i}]$ are

$$\operatorname{Re}(z) := x \quad \textit{and} \quad \operatorname{Im}(z) := y\,.$$

(N.B: We will usually write $[x\cdot 1] + [y\cdot\boldsymbol{i}]$ as $x + \boldsymbol{i}y$ or as $x + y\boldsymbol{i}$.)
The std picture of $\mathbb{C}$ is called the ***Argand plane***. It is useful to interpret algebraic operations, addition, multiplication, complex conjugation, geometrically on this plane.

The ***complex conjugate*** of $z := x + \boldsymbol{i}y$ is written as $\overline{z}$. It is

$$\overline{z} \;:=\; \operatorname{Re}(z) - \operatorname{Im}(z)\boldsymbol{i} \stackrel{\text{note}}{=\!=} x - y\boldsymbol{i}\,.$$

Evidently $\forall\zeta,\omega,z \in \mathbb{C}$, with $z = x + \boldsymbol{i}y$:

$$\overline{\zeta+\omega} \;=\; \overline{\zeta}+\overline{\omega} \quad \textit{and} \quad \overline{\zeta\cdot\omega} \;=\; \overline{\zeta}\cdot\overline{\omega}\,;$$
$$\operatorname{Re}(z) \;=\; [z+\overline{z}]/2 \quad \textit{and} \quad \operatorname{Im}(z) \;=\; [z-\overline{z}]/[2\boldsymbol{i}]\,;$$
$$z\overline{z} \;=\; |z|^2 \stackrel{\text{note}}{=\!=} x^2+y^2\,.$$

Sequence notation. A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_1, x_2, x_3, \dots)$. For a set Ω, expression "$\vec{\mathbf{x}} \subset \Omega$" means $[\forall n\colon\; x_n \in \Omega]$. Use $\operatorname{Tail}_N(\vec{\mathbf{x}})$ for the subsequence

$$(x_N,\, x_{N+1},\, x_{N+2}, \dots)$$

of $\vec{\mathbf{x}}$. Given a fnc $f\colon\Omega\to\Lambda$ and an Ω-sequence $\vec{\mathbf{x}}$, let $f(\vec{\mathbf{x}})$ be the Λ-sequence $(f(x_1), f(x_2), f(x_2), \dots)$.

Suppose Ω has an addition and multiplication. For Ω-seqs $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, then, let $\vec{\mathbf{x}}+\vec{\mathbf{y}}$ be the sequence whose n^{th} member is $x_n + y_n$. I.e

$$\vec{\mathbf{x}}+\vec{\mathbf{y}} \quad=\quad [n \mapsto [x_n + y_n]]\,.$$

Similarly, $\vec{\mathbf{x}}\cdot\vec{\mathbf{y}}$ denotes seq $[n \mapsto [x_n\cdot y_n]]$.

A glance at Metric Spaces

The usual metric on $\mathbb{C}$ is

$$\operatorname{Dist}(\zeta,\omega) \;:=\; |\zeta - \omega|\,.$$

We will need to handle at least four MSes [***metric spaces***]: The Reals, the Complexes, $\mathbb{C}\times\mathbb{C}$ and the Riemann Sphere. As such, let's simplify and look at general metric spaces.

A ***metric space*** [MS] is a pair $(\mathbf{X}, \mathsf{m})$ where $\mathbf{X}$ is a set, and $\mathsf{m}\colon\mathbf{X}\times\mathbf{X}\to[0,\infty)$ is a metric. A ***metric*** m satisfies that $\forall w,x,y,z \in \mathbf{X}$:

MS1a: $\mathsf{m}(w,w) = 0$.

MS1b: If $\mathsf{m}(w,x) = 0$ then $w = x$.

MS2: $\mathsf{m}(y,z) = \mathsf{m}(z,y)$. [Symmetry]

MS3: $\mathsf{m}(w,y) \leqslant \mathsf{m}(w,x) + \mathsf{m}(x,y)$. [$\triangle$-Inequality]

Fix a point $\mathbf{p} \in \mathbf{X}$ and a "radius" $r \in \mathbb{R}$. Define ***open ball***, ***closed ball***, ***sphere*** and ***punctured (open) ball*** as follows:

$$\begin{aligned}
\mathrm{Bal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) < r\}\,; \\
\mathrm{CldBal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) \leqslant r\}\,; \\
\mathrm{Sph}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid \mathsf{m}(w,\mathbf{p}) = r\}\,; \\
\mathrm{PBal}_r(\mathbf{p}) &:= \{w \in \mathbf{X} \mid 0 < \mathsf{m}(w,\mathbf{p}) < r\}\,.
\end{aligned}$$

[Chasing definitions: When r is negative then all four sets are empty. When $r = 0$ then $\mathrm{Bal}_0(\mathbf{p}) = \varnothing = \mathrm{PBal}_0(\mathbf{p})$. And $\mathrm{CldBal}_0(\mathbf{p}) = \{\mathbf{p}\} = \mathrm{Sph}_0(\mathbf{p})$.] For non-negative α and r, define the open ***annulus*** as [form is $\mathrm{Ann}^{Inner}_{Outer}()$]

$$\mathrm{Ann}^{\alpha}_{r}(\mathbf{p}) := \{w \in \mathbf{X} \mid \alpha < \mathsf{m}(w,\mathbf{p}) < r\}\,.$$

This is the emptyset unless $r > \alpha$, in which case the thickness of the annulus is $r - \alpha$. The superscript α and subscript r are, respectively, the ***inner-radius*** and ***outer-radius*** of annulus $\mathrm{Ann}^{\alpha}_{r}(\mathbf{p})$. An inner-radius of zero has $\mathrm{Ann}^{0}_{r}(\mathbf{p}) = \mathrm{PBal}_r(\mathbf{p})$. Note that $\mathrm{Ann}^{\alpha}_{\infty}(\mathbf{p})$ is the exterior of a closed-ball. I.e

$$\mathrm{Ann}^{\alpha}_{\infty}(\mathbf{p}) = \mathbf{X} \smallsetminus \mathrm{CldBal}_{\alpha}(\mathbf{p})\,.$$

Seq.-Limit. Seq $\vec{\mathbf{x}} \subset \mathbf{X}$ ***converges*** to a point $\mathbf{p} \in \mathbf{X}$ if $\mathsf{m}(x_n, \mathbf{p}) \to 0$ as $n \nearrow \infty$. I.e, if for each $\varepsilon > 0$, there exists index K st. $\forall n \geqslant K$, we have $\mathsf{m}(x_n, \mathbf{p}) < \varepsilon$. Equiv.: $\forall \varepsilon > 0$, $\exists K \in \mathbb{Z}_+$ st. $\mathrm{Tail}_K(\vec{\mathbf{x}}) \subset \mathrm{Bal}_{\varepsilon}(\mathbf{p})$.

We indicate this convergence by $\lim(\vec{\mathbf{x}}) = \mathbf{p}$, or as $\left[\lim_{n\to\infty} x_n\right] = \mathbf{p}$. *Let's now justify the equal-sign.*

1: Unique-limit Lemma. *In MS $(\mathbf{X}, \mathsf{m})$, suppose a sequence $\vec{\mathbf{x}}$ converges to points $\mathbf{p}$ and $\mathbf{q}$. Then $\mathbf{p} = \mathbf{q}$.* ◊

Pf. FTSOContradiction suppose $\mathbf{p} \neq \mathbf{q}$. By (MS1b), distance $\mathsf{m}(\mathbf{p}, \mathbf{q})$ is positive; let's call it $2H$. So it suffices to produce a point $b \in \mathbf{X}$ with

$$*: \qquad \mathsf{m}(b, \mathbf{p}) < H \quad \text{and} \quad \mathsf{m}(b, \mathbf{q}) < H\,.$$

For then, symmetry (MS2) yields $\mathsf{m}(\mathbf{p}, b) < H$. Now our Triangle Inequality chirps in with

$$2H \overset{\text{def}}{=} \mathsf{m}(\mathbf{p}, \mathbf{q}) \overset{\triangle\text{-Ineq}}{\leqslant} \mathsf{m}(\mathbf{p}, b) + \mathsf{m}(b, \mathbf{q}) \overset{\text{note}}{<} 2H\,,$$

i.e, that $2H < 2H$. ※

[Length H is half the distance, and b is close to both.]

Obtaining such a b. Of course, the only place we could get such a b is from $\vec{\mathbf{x}}$; we'll show, for a large enough index M, that $b := x_M$ satisfies $(*)$. To do that, we'll simply apply the defn of limit.

Since $\lim(\vec{\mathbf{x}}) = \mathbf{p}$, there exists index K such that $[n \geqslant K] \Rightarrow \mathsf{m}(x_n, \mathbf{p}) < H$. And $\exists$ an index L such that $[n \geqslant L] \Rightarrow \mathsf{m}(x_n, \mathbf{q}) < H$. Happily, $M := \mathrm{Max}(K, L)$ dominates both K and L, so $b := x_M$ fulfills $(*)$. ♦

Open/closed sets. A set $U \subset \mathbf{X}$ is ***open*** [in $\mathbf{X}$] if U is a union of open balls (possibly ∞ly many).

The ***complement*** [in $\mathbf{X}$] of an $\mathbf{X}$-subset S is $\mathbf{X} \smallsetminus S$. If $\mathbf{X}$ is understood, the complement may be written as S^c or $\complement(S)$.

A set $E \subset \mathbf{X}$ is ***closed*** [in $\mathbf{X}$] if its $\mathbf{X}$-complement is open.[♡1] If a set is both open and closed, then it is called ***clopen***. [In $\mathbb{C}$, the only clopen sets are the whole space, $\mathbb{C}$, and its complement $\varnothing$, the empty set. Some MSes, however, have non-trivial clopen subsets.]

For a subset $S \subset \mathbf{X}$, a pt $p \in S$ is **"an *interior-point* of S"** if there exists an open ball B with $p \in B \subset \mathbf{X}$. I.e, $\exists r > 0$ with $\mathrm{Bal}_r(p) \subset S$. Relations "*neighborhood of*" and "*interior-pt of*" are inverses: *Set S is a **"neighborhood of p"** IFF p is an interior-point of S.* Use ***nbhd*** to abbreviate "*neighborhood*".

The ***interior*** of S is

$$\mathrm{Itr}(S) := \{p \in S \mid p \text{ is an interior-pt of } S\}\,.$$

Equiv., the interior of S is the union of all open subsets of S. Equiv., $\mathrm{Itr}(S)$ is the largest open subset of S. Consequently, S is open IFF $\mathrm{Itr}(S) = S$.

The ***closure*** of S is

$$\mathrm{Cl}(S) := \{p \in \mathbf{X} \mid \forall r > 0, \text{ open ball } \mathrm{Bal}_r(p) \text{ hits } S\}\,.$$

Equiv., $\mathrm{Cl}(S)$ is the intersection of all closed supersets of S. Equiv., $\mathrm{Cl}(S)$ is the smallest closed superset of S. Consequently, S is closed IFF $\mathrm{Cl}(S) = S$.

Closure-of and *Interior-of* are dual notions in that $\complement(\mathrm{Cl}(E)) = \mathrm{Itr}(\complement(E))$.

The **"*boundary* of set S [in $\mathbf{X}$]"** is

$$\partial(S) := \left\{p \in \mathbf{X} \;\middle|\; \begin{array}{l} \forall r > 0, \text{ open ball } \mathrm{Bal}_r(p) \\ \text{hits both } S \text{ and } \mathbf{X} \smallsetminus S. \end{array}\right\}.$$

So $\partial(S) = \mathrm{Cl}(S) \cap \mathrm{Cl}(S^c)$.

[♡1] Typically, most sets in a MS are neither open nor closed.

A set $S \subset \mathbf{X}$ is ***limit-closed*** [in $\mathbf{X}$] if $\forall \vec{\mathbf{s}} \subset S$: *Whenever* $\mathbf{p} := \lim(\vec{\mathbf{s}})$ *exists in* $\mathbf{X}$, *then* $\mathbf{p} \in S$.

2: Limit-closed lemma. *Set $E \subset \mathbf{X}$ is closed IFF E is limit-closed.* ◊

Pf(⇒). Consider a seq $\vec{\mathbf{s}} \subset E$ and limit $\mathbf{p} := \lim(\vec{\mathbf{s}})$ in $\mathbf{X}$. Were $\mathbf{p}$ in the complement $U := \mathbf{X} \setminus E$, then $\exists r>0$ with $\mathrm{Bal}_r(\mathbf{p}) \subset U$. But this implies, for each n, that $\mathsf{m}(s_n, \mathbf{p}) \geqslant r$. And *that* contradicts the supposed convergence of $\vec{\mathbf{s}}$ to $\mathbf{p}$. ♦

Pf(⇐). FTSOC, suppose E fails to be closed. Then $U := \mathbf{X} \setminus E$ is not open, so $\exists \mathbf{q} \in U$ satisfying that *every* ball about $\mathbf{q}$ sticks out of U, that is, hits E.

Consequently, for $n = 1, 2, 3, \ldots$, the intersection

$$E \cap [\mathrm{Bal}_{1/n}(\mathbf{q})]$$

is non-void. Pick a point in that intersection, and call it, say, z_n. Then $[\lim_{n\to\infty} z_n]$ equals $\mathbf{q}$, contradicting that E was limit-closed. ♦

Defn. A set $E \subset \mathbf{X}$ is ***compact*** if each seq $\vec{\mathbf{s}} \subset E$ admits a subsequence $\vec{\mathbf{e}} \subset \vec{\mathbf{s}}$ which converges to a point <u>in E</u>. That is, there exist indices $n_1 < n_2 < \ldots$ and a point $\mathbf{p} \in E$ s.t $[\lim_{k\to\infty} s_{n_k}] = \mathbf{p}$.

The above Limit-closed lemma implies that compact sets are automatically[♡2] closed. □

Fnc limits. Consider MSes $(\mathbf{X}, \mathsf{m})$ and $(\mathbf{\Omega}, \mu)$, points $\mathbf{p} \in \mathbf{X}$ and $\boldsymbol{\omega} \in \mathbf{\Omega}$, and a fnc $h: [\mathbf{X} \setminus \{\mathbf{p}\}] \to \mathbf{\Omega}$. Expression

$$\Big[\lim_{z\to\mathbf{p}} h(z)\Big] = \boldsymbol{\omega}$$

means: $\forall \varepsilon>0, \exists \delta>0$ *such that*

$$\forall z \in \mathbf{X}\text{: If } 0 < \mathsf{m}(z, \mathbf{p}) < \delta \text{ then } \mu(h(z), \boldsymbol{\omega}) < \varepsilon\,.$$

3: Equiv.: $h\Big(\mathrm{PBal}_\delta(\mathbf{p})\Big) \subset \mathrm{Bal}_\varepsilon(\boldsymbol{\omega})\,.$

Equiv.: $\mathrm{PBal}_\delta(\mathbf{p}) \subset h^{-1}\Big(\mathrm{Bal}_\varepsilon(\boldsymbol{\omega})\Big)\,.$

These balls are in different spaces, with different metrics. To write, for example, this last property *precisely*, we'd write

$$\mathsf{m}\text{-}\mathrm{PBal}_\delta(\mathbf{p}) \subset h^{-1}\Big(\mu\text{-}\mathrm{Bal}_\varepsilon(\boldsymbol{\omega})\Big)\,.$$

3a: Unique fnc-limit Lemma. *With notation from above* [*WNFAbove*], *if*

$$\Big[\lim_{z\to\mathbf{p}} h(z)\Big] = \boldsymbol{\omega}_1 \quad \textit{and} \quad \Big[\lim_{z\to\mathbf{p}} h(z)\Big] = \boldsymbol{\omega}_2\,,$$

then $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$. *Pf.* See proof of Unique-limit Lemma. ◊

3b: *Defn.* Fnc $g: (\mathbf{X}, \mathsf{m}) \to (\mathbf{\Omega}, \mu)$ is ***continuous*** at a point $\mathbf{p} \in \mathbf{X}$ if $\lim_{z\to\mathbf{p}} g(z) = g(\mathbf{p})$. We say **"$g$ is *continuous*"** if g is cts at each point in $\mathrm{Dom}(g)$. □

3c: Thm. [*WNFAbove*]. *Fnc g is continuous at $\mathbf{p}$ IFF*

For each sequence $\vec{\mathbf{z}} \subset \mathbf{X}$, if $\lim(\vec{\mathbf{z}}) = \mathbf{p}$, then $\lim(g(\vec{\mathbf{z}})) = g(\mathbf{p})$.

Proof. Exercise. ◊

4a: Open pullback lemma. *Fnc $h: (\mathbf{X}, \mathsf{m}) \to (\mathbf{\Omega}, \mu)$ is* [*everywhere*] *cts IFF for each $\mathbf{\Omega}$-open set $\Lambda \subset \mathbf{\Omega}$, its pullback $h^{-1}(\Lambda)$ is open in $\mathbf{X}$.* *Proof.* Exercise. ◊

4b: *Example.* For a cts h, *pullbacks* preserve openness. However, push-forwards need not. E.g, the sine fnc $\sin: \mathbb{R} \to (-3, 3)$ is cts, and $U := (0, \frac{3\pi}{4})$ is open in $\mathbb{R}$. Yet the push-forward set $\sin(U)$, is the half-open interval $(0, 1]$, which is *not* an open [nor closed] subset of the output-space, $(-3, 3)$. □

Back home to $\mathbb{C}$

As a nice exercise, let's state and prove a fact about subsets of $\mathbb{C}$. [The same result holds in every MS.] Let $\mathsf{m}(z, w) := |z - w|$ denote the usual metric on $\mathbb{C}$.

5: Thm. *For an arbitrary $S \subset \mathbb{C}$, the set*

$$E := S \cup \partial(S)$$

is closed. ◊

[♡2] Our defn of *compact* is for MSes, and it generalizes to topological spaces. In a general topological space, is possible for a compact set to not be closed.

Set-up. ISTProve that $U := [\mathbb{C} \setminus E]$ is open.

FTSOC, suppose U not open. Then there exists a point $\mathbf{p} \in U$ such that $\mathbf{p} \notin \operatorname{Itr}(U)$. Imagine we could establish

5a: $\forall r > 0,\ \exists$ a point $q \in S$ with $\mathsf{m}(q, \mathbf{p}) < r$.

Then *every* ball about $\mathbf{p}$, hits S. But every ball also hits $\mathbb{C}\setminus S$, since the ball owns $\mathbf{p} \in U$. And this implies the contradiction that $\mathbf{p}$ is a boundary-pt of S. □

Proof of (5a). Fix an $r>0$. Since $\mathbf{p}$ is not a U-interior-point, $\exists b \in E$ with $\mathsf{m}(b, \mathbf{p}) < r$. If b is in S, then we are done.

Otherwise, b must be in $\partial(S)$. Recall that difference

$$r - \mathsf{m}(b, \mathbf{p})$$

is *positive*. Since $b \in \partial(S)$, there are points of S arbitrarily close to b. In particular, $\exists q \in S$ with

$$*:\qquad \mathsf{m}(q, b) < r - \mathsf{m}(b, \mathbf{p}).$$

Thus

$$\mathsf{m}(q, \mathbf{p}) \overset{\triangle\text{-Ineq}}{\leqslant} \mathsf{m}(q, b) + \mathsf{m}(b, \mathbf{p}) \overset{\text{by } (*)}{<} r$$

as desired. ♦

Polynomials over $\mathbb{C}$. An old theorem, slightly misnamed:

6: Fundamental Theorem of Algebra (Gauss and others). *Consider a monic $\mathbb{C}$-polynomial*

$$h(t) := t^N + B_{N-1}t^{N-1} + \ldots + B_1 t + B_0 .$$

Then h factors completely over $\mathbb{C}$ as

$$h(t) = [t - Z_1] \cdot [t - Z_2] \cdot \ldots \cdot [t - Z_N], \qquad \diamond$$

for a list $Z_1, \ldots, Z_N \in \mathbb{C}$, possibly with repetitions. This list is unique up to reordering.

If h is a **real** *polynomial, i.e $\overline{h} = h$, then h factors over $\mathbb{R}$ as a product of monic $\mathbb{R}$-irreducible linear and $\mathbb{R}$-irred. quadratic polynomials. The product is unique up to reordering.* *Proof.* See (16e, *P.10*).

[There is a proof in my *A Primer on Polynomials* pamphlet].

Cauchy-Riemann eqns. On an open set $D \subset \mathbb{C}$, consider a fnc $h{:}\,D \to \mathbb{C}$, which we have written as $h(x + \boldsymbol{i}y) = u(x, y) + \boldsymbol{i}v(x, y)$, giving names to its real and imaginary parts.

A point $x + \boldsymbol{i}y$ can also be written in polar coordinates as $re^{\boldsymbol{i}\theta}$, with $r,\theta \in \mathbb{R}$. So we can view u [and v] either as a fnc of (x, y) *or* as a fnc of (r, θ). Differentiability of $h()$ at a particular point z, forces equality of partial-derivs at z. The eqns are called the ***Cauchy-Riemann eqns***:

7a:	Cartesian:	$u_x = v_y$ and $u_y = -v_x$.
7b:	Polar:	$r\cdot u_r = v_\theta$ and $u_\theta = -r\cdot v_r$.

Proof of (7a). Firstly, for h to be diff'able at z means: *Our h is defined in a nhbd of z,* and $\lim_{\Delta z \to 0} \frac{h(z+\Delta z) - h(z)}{\Delta z}$ *exists in $\mathbb{C}$.*

Let $w := h(z)$ and $\Delta w := h(z + \Delta z) - h(z)$.

Case: Pure real: $\Delta z := \Delta x$ Computing, Δw equals

$$u(x + \Delta x,\, y) + \boldsymbol{i}v(x + \Delta x,\, y) - [u(x, y) + \boldsymbol{i}v(x, y)]$$
$$= [u(x + \Delta x,\, y) - u(x, y)] + \boldsymbol{i}[v(x + \Delta x,\, y) - v(x, y)].$$

Hence, $\frac{\Delta w}{\Delta z}$ equals

$$\frac{u(x + \Delta x,\, y) - u(x, y)}{\Delta x} + \boldsymbol{i}\cdot\frac{v(x + \Delta x,\, y) - v(x, y)}{\Delta x}.$$

Sending $\Delta x \to 0$ yields that

$$\dagger:\qquad \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x(x, y) + \boldsymbol{i} \cdot v_x(x, y).$$

Case: Pure imag: $\Delta z := \boldsymbol{i}\Delta y$ Our Δw equals

$$[u(x,\, y + \Delta y) - u(x, y)] + \boldsymbol{i}[v(x,\, y + \Delta y) - v(x, y)].$$

So $\frac{\Delta w}{\Delta z}$ equals $\frac{u(x, y+\Delta y) - u(x,y)}{\boldsymbol{i}\Delta y} + \boldsymbol{i}\cdot\frac{v(x, y+\Delta y) - v(x,y)}{\boldsymbol{i}\Delta y}$, i.e

$$-\boldsymbol{i}\cdot\frac{u(x,\, y + \Delta y) - u(x, y)}{\Delta y} + \frac{v(x,\, y + \Delta y) - v(x, y)}{\Delta y}.$$

Launching $\Delta y \to 0$ reveals that

$$\ddagger:\qquad \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = -\boldsymbol{i} \cdot u_y(x, y) + v_y(x, y).$$

Equating the real parts of (†) and (‡) gives LhS(7a). And equating the imaginary parts produces RhS(7a). ♦

Proof (7a) ⇒ (7b). The CoV from polar to cart coords is

$$\big(x,y\big) = \big(r\cos(\theta), r\sin(\theta)\big)\,.$$

Abbreviating $\mathbf{c} := \cos(\theta)$ and $\mathbf{s} := \sin(\theta)$, then,[♡3]

$$\frac{\partial u}{\partial r} = \left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}\right] + \left[\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\right] = [u_x\cdot\mathbf{c}] + [u_y\cdot\mathbf{s}]\,.$$

Computing all the first-partials gives

$$u_r = u_x\cdot\mathbf{c} + u_y\cdot\mathbf{s}\,; \qquad u_\theta = -r\big[u_x\cdot\mathbf{s} - u_y\cdot\mathbf{c}\big]\,;$$
$$v_r = v_x\cdot\mathbf{c} + v_y\cdot\mathbf{s}\,; \qquad v_\theta = r\big[v_y\cdot\mathbf{c} - v_x\cdot\mathbf{s}\big]\,.$$

Applying (7a) to write all the partials ITOf x, gives

$$\dagger\!: \quad u_r = u_x\cdot\mathbf{c} - v_x\cdot\mathbf{s}\,; \qquad u_\theta = -r\big[u_x\cdot\mathbf{s} + v_x\cdot\mathbf{c}\big]\,;$$
$$\ddagger\!: \quad v_r = v_x\cdot\mathbf{c} + u_x\cdot\mathbf{s}\,; \qquad v_\theta = r\big[u_x\cdot\mathbf{c} - v_x\cdot\mathbf{s}\big]\,.$$

Comparing LhS(†) with RhS(‡), and RhS(†) with LhS(‡), yields (7b). ♦

7c: *Caveat.* If z is not the origin, i.e $r \neq 0$, then the converse (7b) ⇒ (7a) holds. *However*, at the origin (7b) always holds, hence has no content. [E.g, $u_\theta(0)$ is always zero, since $[\theta \mapsto 0\cdot\exp(\boldsymbol{i}\theta)]$ necessarily has derivative zero.] So at the origin, (7b) *does not* imply (7a). □

8: Open-set Differentially-path-connected Thm. *Consider a path-connected subset $E \subset \mathbb{C}$. If E is open, then $\forall p,q \in E$, there exists a differentiable path $z\colon[0,1]\to E$ with $z(0) = p$ and $z(1) = q$.* ♢

Two consequences of the Cauchy-Riemann eqns.:

9a: Constancy theorem. *Consider a path-connected open $D \subset \mathbb{C}$, and holomorphic $h\colon D\to\mathbb{C}$.*

i: If $h' \equiv 0$, then h is constant on E.

ii: If h and $\overline{h}$ are holomorphic, then $h \equiv 0$.

iii: If $|h|$ is constant, then h is constant. ♢

[♡3] In Newton's notation, $u_x\cdot\mathbf{c}$ is $u_x\big(r\cos(\theta), r\sin(\theta)\big)\cdot\cos(\theta)$.

Pf of (i). Given $p,q \in D$, ISTProve $h(q) = h(p)$.

Our (8) gives a diff'able $z\colon[0,1]\to E$ with $z(0) = p$ and $z(1) = q$. So

$$0 = \int_0^1 h'(z(t))\cdot z'(t)\,\mathrm{d}t = \int_0^1 [h\circ z]'$$
$$= h(z(1)) - h(z(0))$$
$$= h(q) - h(p)\,.$$ ♦

Pf of (ii). Write h with real and imaginary parts, as $h = u + \boldsymbol{i}v$. So $\overline{h} = u + \boldsymbol{i}\cdot[-v]$. C-R eqns of h thus say $u_x = v_y$, and of $\overline{h}$ say $u_x = -v_y$. Hence $u_x \equiv 0$. The other C-R eqn shows $v_x \equiv 0$. Thus $h' \overset{\text{note}}{=\!=} u_x + \boldsymbol{i}v_x$ is identically zero. Now apply (i). ♦

Pf of (iii). If $|h| \equiv 0$, then $h \equiv 0$. So WLOG, number $\kappa := |h|^2 \neq 0$. As h is never zero, I may divide to conclude that $\overline{h}() = \frac{\kappa}{h()}$ is holomorphic. Now apply (ii). ♦

9b: Harmonic lemma. *Suppose h is holomorphic an open $D \subset \mathbb{C}$. Then $[\mathrm{Re}\circ h]$ and $[\mathrm{Im}\circ h]$ are each harmonic on D.* *Proof.* See **Brown&Churchill.** ♢

Path-independence and differentiability. Here is the non-trivial part of the thm from P.141 & P.146 of **Brown&Churchill, 9th-ed.**.

Say that fnc $f\colon D\to\mathbb{C}$ has the ***path-independence property*** [***PIP***] if for all closed-contours C: *The contour-integral $\int_{\mathsf{C}} f$ exists, and equals zero.*

10a: Path-indep theorem. *On an open $D \subset \mathbb{C}$, suppose $f\colon D\to\mathbb{C}$ is continuous. If f has the path-independence property, then there exists a differentiable function $g\colon D\to\mathbb{C}$, with $g' = f$.* ♢

Proof. WLOG D is non-void and connected, since we can argue for each path-connected component separately.

Fix a "base-point" $z_0 \in D$. For each $p \in D$ there exists a contour C from z_0 to p, since D is path-connected, and courtesy (8). Define $g(p) := \int_{\mathsf{C}} f$; this is well-defined because f has the PIP.

To show that g is diff'able at an arbitrary $\mathbf{p} \in D$, and that $g'(\mathbf{p}) = f(\mathbf{p})$, we fix an $\varepsilon > 0$. ISTProduce a $\delta>0$ such that for all $z \in \mathrm{PBal}_\delta(\mathbf{p})$:

$$\dagger\!: \qquad \left|\frac{g(\mathbf{p}+\Delta z) - g(\mathbf{p})}{\Delta z} - f(\mathbf{p})\right| \leqslant \varepsilon\,,$$

where we are writing z as $\mathbf{p}+\Delta z$.

Obtaining δ. Since D is open, $\exists r > 0$ such that $\text{Bal}_r(\mathbf{p}) \subset D$. And since f is cts at $\mathbf{p}$, there exists $\alpha{>}0$ so that each $z \in \text{Bal}_\alpha(\mathbf{p})$ has $|f(z) - f(\mathbf{p})| < \varepsilon$. Let $\delta := \text{Min}(r, \alpha)$, which we note is positive.

The Estimate. Fix a point $z \in \text{PBal}_\delta(\mathbf{p})$; so displacement $\Delta z := z - \mathbf{p}$ has $|\Delta z| < \delta$.

Let L denote the line-segment contour from $\mathbf{p}$ to z. We parametrize L as $w\colon [0,1] \to D$, by

$$\begin{aligned} w(t) &:= \mathbf{p} + [t \cdot \Delta z]. \quad \text{So} \\ w'(t) &= \Delta z. \quad \text{Thus} \\ g(\mathbf{p}{+}\Delta z) - g(\mathbf{p}) &= \int_0^1 f(w(t)) \cdot w'(t)\, \mathrm{d}t \\ &= \Delta z \cdot \int_0^1 f(\mathbf{p} + [t \cdot \Delta z])\, \mathrm{d}t. \end{aligned}$$

Dividing by Δz [Exer: Why is $\Delta z \neq 0$?], then subtracting

$$f(\mathbf{p}) \overset{\text{note}}{=\!=} \int_0^1 f(\mathbf{p})\, \mathrm{d}t$$

from both sides, yields that

$$\frac{g(\mathbf{p}{+}\Delta z) - g(\mathbf{p})}{\Delta z} - f(\mathbf{p}) = \int_0^1 \Big[f(\mathbf{p} + [t\,\Delta z]) - f(\mathbf{p}) \Big]\, \mathrm{d}t.$$

Taking abs.values and using our Triangle-Ineq-for-Integrals, yields

$$\ddagger\!: \quad \left| \tfrac{g(\mathbf{p}+\Delta z) - g(\mathbf{p})}{\Delta z} - f(\mathbf{p}) \right| \leqslant \int_0^1 \big| f(\mathbf{p} + [t\Delta z]) - f(\mathbf{p}) \big|\, \mathrm{d}t.$$

But each $\mathbf{p} + [t\Delta z]$ is in the δ-ball about $\mathbf{p}$. Hence the integrand in ($\ddagger$) is $\leqslant \varepsilon$. Thus $\text{RhS}(\ddagger) \leqslant \varepsilon \cdot [1 - 0] = \varepsilon$, yielding ($\dagger$), as desired. ♦

$\mathbb{C}$-exponential

For $z := x \cdot 1 + y \cdot \boldsymbol{i}$ with $x,y \in \mathbb{R}$, its ***complex conjugate*** $\overline{z}$ is $x \cdot 1 - y \cdot \boldsymbol{i}$. Its real and imaginary parts are

$$\text{Re}(z) := x = \frac{z + \overline{z}}{2}, \quad \text{Im}(z) := y = \frac{z - \overline{z}}{2\boldsymbol{i}}.$$

By the Pythagorean thm, $|z|^2 = x^2 + y^2 = z\overline{z}$.

For $\mu,\nu \in \mathbb{C}$, note, $\overline{\mu + \nu} = \overline{\mu} + \overline{\nu}$ and $\overline{\mu \cdot \nu} = \overline{\mu} \cdot \overline{\nu}$.

Let's extend the exponential fnc to the complex plane.

11a: Defn. *For $z \in \mathbb{C}$, define*

$$\exp(z) := \mathrm{e}^z := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^n = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \ldots;$$

$$\cos(z) := \sum_{j=0}^{\infty} \frac{[-1]^j}{[2j]!} \cdot z^{2j} = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 - \ldots;$$

$$\sin(z) := \sum_{k=0}^{\infty} \frac{[-1]^k}{[2k+1]!} \cdot z^{2k+1} = z - \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 - \ldots.$$

Each series has ∞-RoC. ♢

Since we have absolute convergence of each series at each z, we can re-order terms without changing convergence.

11b: Lemma. *Fix $\alpha,\beta \in \mathbb{C}$. Then*

$$\mathrm{e}^\alpha \cdot \mathrm{e}^\beta = \mathrm{e}^{\alpha+\beta}. \quad ♢$$

Proof. For natnum N, recall the Binomial thm which says that

$$*\!: \quad \sum_{j+k=N} \tbinom{N}{j,k} \cdot \alpha^j \beta^k = [\alpha + \beta]^N,$$

where the sum is over all ordered-pairs (j,k) of natnums. By its defn [and abs.convergence], $\mathrm{e}^\alpha \mathrm{e}^\beta$ equals

$$\Big[\sum_{j=0}^{\infty} \frac{1}{j!} \cdot \alpha^j\Big] \cdot \Big[\sum_{k=0}^{\infty} \frac{1}{k!} \cdot \beta^k\Big] = \sum_{N=0}^{\infty} \Big[\sum_{j+k=N} \frac{1}{j!} \frac{1}{k!} \cdot \alpha^j \beta^k\Big].$$

But $\frac{1}{j! \cdot k!}$ equals $\frac{1}{N!} \cdot \frac{N!}{j! \cdot k!}$. Hence $\mathrm{e}^\alpha \mathrm{e}^\beta$ equals

$$\sum_{N=0}^{\infty} \frac{1}{N!} \Big[\sum_{j+k=N} \tbinom{N}{j,k} \cdot \alpha^j \beta^k\Big] \overset{\text{by } (*)}{=\!=} \sum_{N=0}^{\infty} \frac{1}{N!} [\alpha + \beta]^N,$$

which is the defn of $\mathrm{e}^{\alpha+\beta}$. ♦

11c: Lemma. *For θ,x,y,z complex numbers:*

11.1: $e^{i\theta} = [\cos(\theta) + i\sin(\theta)] =: \text{cis}(\theta)$. *Hence*

11.2: $\dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta)$, $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta)$. *Also,*

11.3: $e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot [\cos(y) + i\sin(y)]$, *so*

11.4: $e^{x-iy} = e^x \cdot [\cos(y) - i\sin(y)]$,

since $\cos(-y) = \cos(y)$ *and* $\sin(-y) = -\sin(y)$. *When* θ *is real, then,*

11.5: $\text{Re}(e^{i\theta}) = \cos(\theta)$ *and* $\text{Im}(e^{i\theta}) = \sin(\theta)$.

Since the coefficients in their power-series expansions are all real, our exp(),cos(),sin() *fncs each commute with complex-conjugation, i.e*

11.6: $\overline{\exp(z)} = \exp(\overline{z})$, $\overline{\cos(z)} = \cos(\overline{z})$, $\overline{\sin(z)} = \sin(\overline{z})$;

Finally, the familiar translation-identities

11.7: $\cos(z - \frac{\pi}{2}) = \sin(z)$, $\sin(z + \frac{\pi}{2}) = \cos(z)$

extend to the complex plane. ♢

Proof. Exercise, using (11b). ♦

Examples from Fri.17Feb

Two examples from class.

12: cos-sin zeros Lemma. *All zeros of* [*complex*] cos() *lie on the real axis. In particular,* cos() *has only one period, that of* $2\boldsymbol{\pi}$. *Both stmts hold for* sin(). ♢

Proof for cos. Fix a $z = x + iy$ st. $\cos(z) = 0$. Thus

$$\begin{aligned} 0 = 2\cos(z) &= \exp(i \cdot [x + iy]) + \exp(-i \cdot [x + iy]) \\ &= \exp(-y + ix) + \exp(y - ix) \\ &= e^{-y}\text{cis}(x) + e^{y}\text{cis}(-x). \end{aligned}$$

Since these summands cancel, they must have equal abs.values. Thus, since x and y are real,

$$*: \quad e^{-y} = e^{-y} \cdot |\text{cis}(x)| = e^{y} \cdot |\text{cis}(-x)| = e^{y}.$$

But $\mathbb{R}$-exp() is 1-to-1, so ($*$) implies that $-y = y$. Hence $y = 0$, i.e z is real. ♦

Integration example. Fix a real $\alpha > 0$. To compute

$$J := \int_0^\alpha e^{it}\,dt,$$

we could directly use an antiderivative: So J equals

$$\dagger: \quad \frac{1}{i} e^{it}\Big|_{t=0}^{t=\alpha} = -i[e^{i\alpha} - 1].$$

Alternatively, we can decompose into real and imaginary parts, as $J = U + iV$, where

$$V := \int_0^\alpha \sin(t)\,dt = -\cos(t)\Big|_{t=0}^{t=\alpha} = -[\cos(\alpha) - 1]$$

and

$$U := \int_0^\alpha \cos(t)\,dt = \sin(t)\Big|_{t=0}^{t=\alpha} = \sin(\alpha).$$

With $\mathcal{S} := \sin(t)$ and $\mathcal{C} := \cos(t)$, then, $U + iV$ equals

$$\begin{aligned} \mathcal{S} - i[\mathcal{C} - 1] &= -i \cdot [i\mathcal{S} + \mathcal{C} - 1] \\ &= -i \cdot [\text{cis}(\alpha) - 1] \overset{\text{note}}{=\!=} \text{RhS}(\dagger), \end{aligned}$$

as expected. In this instance, direct integration was faster than breaking the integrand into real and imaginary parts. □

Cauchy-Goursat and friends

Let **SCC** mean "positively oriented simple-closed-contour". For a SCC C, have $\mathring{\mathsf{C}}$ be the (open) region C encloses, and let $\widehat{\mathsf{C}}$ mean C together with $\mathring{\mathsf{C}}$. So $\widehat{\mathsf{C}}$ is $\mathsf{C} \cup \mathring{\mathsf{C}}$; it is automatically simply-connected and is a closed bounded set.

CONVENTION: *Each circle mentioned, e.g* $\mathrm{Sph}_r(\mathbf{p})$, *is also viewed as an SCC, i.e, as positively oriented.*

13a: Cauchy-Goursat Theorem (C-Goursat). *Consider SCC C, and function f which is holomorphic on $\widehat{\mathsf{C}}$. Then* $\int_{\mathsf{C}} f = 0$. ♢

13b: Cauchy Integral Formula (CIF). *For a fnc f which is holomorphic on $\widehat{\mathsf{C}}$, where C is a SCC, then*

$$f(w) = \frac{1}{2\pi\boldsymbol{i}} \int_{\mathsf{C}} \frac{f(z)}{z-w}\,\mathrm{d}z\,,$$

for each point $w \in \mathring{\mathsf{C}}$. ♢

Proof outline. Take $r>0$ small enough that circle $S_r := \mathrm{Sph}_r(w)$ is enclosed by C. Since $h(z) := \frac{f(z)}{z-w}$ is holomorphic on the annulus bounded by C and S_r, our C-Goursat implies that $\int_{\mathsf{C}} h = \int_{S_r} h$. Now send $r \searrow 0$ and use that f is cts at w. Etc. ♦

13c: Generalized CIF (GCIF). *A function f which is holomorphic on open set D, is ∞ly-differentiable. Moreover, consider a SCC C with $\widehat{\mathsf{C}} \subset D$. Then for each point $w \in \mathring{\mathsf{C}}$, we have that*

$$f^{(n)}(w) = \frac{n!}{2\pi\boldsymbol{i}} \int_{\mathsf{C}} \frac{f(z)}{[z-w]^{n+1}}\,\mathrm{d}z\,,$$

for $n = 0, 1, 2, \ldots$. ♢

Pf sketch. For each n, verify that $\frac{f(z)}{[z-w]^{n+1}}$ satisfies the conditions for differentiating under the the integral-sign w.r.t w. Then differentiate. ♦

13d: Morera's theorem. *On open set D suppose cts f has path-independence property: $\int_{\mathsf{C}} f = 0$ for each closed contour $\mathsf{C} \subset D$. Then f is holomorphic.* ♢

Proof. By Path-indep thm (10a, P.5), our f has an antiderivative g. Courtesy GCIF, this g is ∞ly-differentiable, hence f is differentiable. ♦

14.1: *The set-up for multiple poles.* Consider simply-connected D, a SCC $\mathsf{C} \subset D$, and distinct points $w_1, \ldots, w_L$ in $\mathring{\mathsf{C}}$. Positive integers $J_1, \ldots, J_L$ determine a polynomial

$$*: \quad \mathbf{P}(z) := [z-w_1]^{J_1} \cdot [z-w_2]^{J_2} \cdot \ldots \cdot [z-w_L]^{J_L}\,.$$

For $k = 1, \ldots, L$, let $P_k(z)$ be product RhS($*$), *but omitting the k^{th}-term.* E.g,

$$P_3(z) := [z-w_1]^{J_1} \cdot [z-w_2]^{J_2} \cdot \prod_{k=4}^{L} [z-w_k]^{J_k}\,.$$

Lastly, consider SCCs $\mathsf{E}_1, \ldots, \mathsf{E}_L$ in D, which avoid all the w-points. Moreover, suppose E_k encloses point w_k, but none of the other w-points. □

14.2: Corollary. *Using notation from (14.1), suppose h is holomorphic on D. Then*

$$\dagger: \quad \int_{\mathsf{C}} \frac{h(z)}{\mathbf{P}(z)}\,\mathrm{d}z = \sum_{k=1}^{L} \int_{\mathsf{E}_k} \frac{h(z)}{\mathbf{P}(z)}\,\mathrm{d}z\,.$$

Further, defining $h_k(z) := \frac{h(z)}{P_k(z)}$ *then*

$$\ddagger: \quad \int_{\mathsf{E}_k} \frac{h(z)}{\mathbf{P}(z)}\,\mathrm{d}z = \int_{\mathsf{E}_k} \frac{h_k(z)}{[z-w_k]^{J_k}}\,\mathrm{d}z\,.$$

Since $h_k()$ is holomorphic on $\widehat{\mathsf{E}_k}$, the RhS(‡) can be computed by GCIF, theorem (13c) ♢

14.3: *CIF example.* [Problem **#2a,b**$^{P.}$*170*] Let C be the radius=2 circle $\mathrm{Sph}_2(\boldsymbol{i})$; it passes through points $-\boldsymbol{i}$ and $3\boldsymbol{i}$. We seek to compute

$$*\mathrm{a}: \quad J := \int_{\mathsf{C}} \frac{1}{z^2+4}\,\mathrm{d}z\,.$$

Soln a: Setting $\boldsymbol{\alpha} := 2\boldsymbol{i}$ and $\boldsymbol{\beta} := -2\boldsymbol{i}$, we factor z^2+4 as $[z-\boldsymbol{\alpha}]\cdot[z-\boldsymbol{\beta}]$. So point $\boldsymbol{\alpha}$ is enclosed by C, whereas point $\boldsymbol{\beta}$ is outside of C. Hence $f(z) := \frac{1}{z-\boldsymbol{\beta}}$ is holomorphic on $\widehat{\mathsf{C}}$. Writing the above integrand as $\frac{f(z)}{z-\boldsymbol{\alpha}}$, then, CIF (13b) yields

$$J = 2\pi\boldsymbol{i} \cdot f(\boldsymbol{\alpha}) = 2\pi\boldsymbol{i} \cdot \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} = 2\pi\boldsymbol{i} \cdot \frac{1}{4i} = \frac{\pi}{2}\,.$$ ♦

The second part of the problem asks us to compute

$$*b: \qquad J_b := \int_{\mathsf{C}} \frac{1}{[z^2+4]^2}\,dz\,.$$

Soln b: The integrand's denominator factors as $[z-\boldsymbol{\alpha}]^2 \cdot [z-\boldsymbol{\beta}]^2$. Rational fnc $h(z) := \frac{1}{[z-\boldsymbol{\beta}]^2}$ is holomorphic on $\widehat{\mathsf{C}}$. Writing the above integrand as $\frac{h(z)}{[z-\boldsymbol{\alpha}]^2}$, then, applying GCIF [thm (13c)] with n=1, gives

$$J_b = \frac{2\boldsymbol{\pi i}}{1!} \cdot h'(\boldsymbol{\alpha}) = 2\boldsymbol{\pi i} \cdot h'(\boldsymbol{\alpha})\,.$$

Note $h'(z) = \frac{-2}{[z-\boldsymbol{\beta}]^3}$, so $h'(\boldsymbol{\alpha}) = -2/[4\boldsymbol{i}]^3 = 1/[32\boldsymbol{i}]$. Consequently,

$$J_b = 2\boldsymbol{\pi i} \cdot \frac{1}{32\boldsymbol{i}} = \frac{\boldsymbol{\pi}}{16}\,. \qquad ♦$$

15a: Cauchy Inequality. *Fix $w \in \mathbb{C}$. For $r>0$, let $\mathsf{C}_r := \mathrm{Sph}_r(w)$. Consider an f which is holomorphic on $\widehat{\mathsf{C}_r}$ and let M_r be the maximum of $|f|$ on C_r. Then $\forall n \in \mathbb{N}$:*

$$*: \qquad |f^{(n)}(w)| \leqslant \frac{n!\,M_r}{r^n}\,. \qquad ♢$$

Proof. By GCIF, and Triangle-Ineq-for-Integrals,

$$\begin{aligned} |f^{(n)}(w)| &\leqslant \frac{n!}{2\boldsymbol{\pi}} \int_{\mathsf{C}_r} \frac{|f(z)|}{|z-w|^{n+1}}\,|dz| \\ &= \frac{n!}{2\boldsymbol{\pi} \cdot r^{n+1}} \int_{\mathsf{C}_r} |f(z)|\,|dz| \\ &\leqslant \frac{n!\,M_r}{2\boldsymbol{\pi} \cdot r^{n+1}} \int_{\mathsf{C}_r} |dz| \\ &= \frac{n!\,M_r}{2\boldsymbol{\pi} \cdot r^{n+1}} \cdot 2\boldsymbol{\pi} r\,. \end{aligned} \qquad ♦$$

15b: Liouville Thm. *Suppose f is entire and is bnded, i.e, there exists a number $\beta \geqslant 0$ with $|f| \leqslant \beta$ on $\mathbb{C}$. Then f is constant.* ♢

Proof. ISTShow that $f' \equiv 0$.

Applying Cauchy Inequality at n=1 gives

$$\forall w \in \mathbb{C}: \quad |f'(w)| \leqslant \tfrac{\beta}{r}\,,$$

for every $r>0$. Now send $r \nearrow \infty$. ♦

15c: Gauss mean value thm (Gauss-MVT). *"The arclength-average on a circle, of a holomorphic function, is its value at the center." Suppose f is holomorphic on region $\widehat{\mathsf{C}}$, where $\mathsf{C} := \mathrm{Sph}_r(\mathbf{p})$ is a circle. Then*

$$*: \qquad \frac{1}{\mathrm{Len}(\mathsf{C})} \int_{\mathsf{C}} f(z) \cdot |dz| = f(\mathbf{p})\,. \qquad ♢$$

Proof. Parametrize C by $z(t) := \mathbf{p} + r e^{it}$; so $z()$ maps $[0, 2\boldsymbol{\pi}]$ onto C. Noting $z'(t) = r\boldsymbol{i}e^{it}$, our CIF implies that $f(\mathbf{p})$ equals

$$\begin{aligned} \frac{1}{2\boldsymbol{\pi i}} \int_{\mathsf{C}} \frac{f(z)}{z-\mathbf{p}}\,dz &= \frac{1}{2\boldsymbol{\pi i}} \int_0^{2\boldsymbol{\pi}} \frac{f(z(t))}{r e^{it}} \cdot r\boldsymbol{i}e^{it}\,dt \\ &= \frac{1}{2\boldsymbol{\pi} r} \int_0^{2\boldsymbol{\pi}} f(z(t)) \cdot r\,dt \\ &= \frac{1}{\mathrm{Len}(\mathsf{C})} \int_0^{2\boldsymbol{\pi}} f(z(t)) \cdot r\,dt\,. \end{aligned}$$

Since $|z'(t)| = |r\boldsymbol{i}e^{it}| = r$, this last integral equals $\int_{\mathsf{C}} f(z) \cdot |dz|$. Hence (*). ♦

16a: Local-constancy lemma. *Suppose f is holomorphic on an open ball B with center point $\mathbf{p}$. If number $|f(\mathbf{p})|$ dominates $|f|$ on B, then f is constant on B.* ♢

Proof. Courtesy Constancy thm (9a, *P.5*), ISTShow $|f|$ constant on B. Fixing a circle $\mathsf{C} := \mathrm{Sph}_r(\mathbf{p})$ in B, then, ISTShow:

*: *The fnc $|f|$, on C, equals number $|f(\mathbf{p})|$.*

By hypothesis, difference $g(z) := \big[|f(\mathbf{p})| - |f(z)|\big]$ is non-negative on C, and is cts, since f is cts. We seek to show that g is identically-zero, which will follow from Non-neg Lemma (33, *P.21*) *if* we can establish that arclength-integral $\int_{\mathsf{C}} g(z)\,|dz|$ is zero.

Integrating. Recall $f(\mathbf{p}) = \frac{1}{\mathrm{Len}(\mathsf{C})} \int_{\mathsf{C}} f(z) \cdot |dz|$, courtesy the Gauss-MVT. Taking abs-values,

$$\begin{aligned} |f(\mathbf{p})| &\leqslant \frac{1}{\mathrm{Len}(\mathsf{C})} \int_{\mathsf{C}} |f(z)| \cdot |dz| \\ &\overset{\text{by hyp}}{\leqslant} \frac{1}{\mathrm{Len}(\mathsf{C})} \int_{\mathsf{C}} |f(\mathbf{p})| \cdot |dz| = |f(\mathbf{p})|\,. \end{aligned}$$

The ends are equal, so all three quantities are equal. In particular, the two integrals are equal, so their difference

$$\int_{\mathsf{C}} \Big[|f(\mathbf{p})| - |f(z)| \Big] \cdot |dz|$$

is zero. And that is the arclength-integral of g. ♦

16b: Maximum-modulus principle (MaxMP). *Suppose holomorphic f on domain D is such that $|f|$ attains a maximum on D. Then f is constant on D.* ◊

Proof. We use the "overlapping-ball argument".

Suppose $\mathbf{p} \in D$ is a point where $|f|$ attains a maximum on D. Fixing an arbitrary point $q \in D$, we seek to show that $f(q) = f(\mathbf{p})$.

Fix a polygonal path $\mathsf{C} \subset D$ going from $\mathbf{p}$ to q. Since D is open, and C is closed and bounded, there exists [this uses the completeness property of $\mathbb{R}$] a sufficiently small $\varepsilon > 0$ so that for every point $w \in \mathsf{C}$, ball $\mathrm{Bal}_{2\varepsilon}(w)$ lies in D. Pick a sequence of points

$$w_0 := \mathbf{p}, \quad w_1, \quad w_2, \quad \ldots, \quad w_{N-1}, \quad w_N := q$$

on C, so that each distance $|w_n - w_{n-1}| < \varepsilon$. Thus each ball $B_n := \mathrm{Bal}_{2\varepsilon}(w_n)$ owns the next point, w_{n+1}.

Applying Local-constancy, (16a), to B_0, says f is constant on B_0. So $f(w_1) = f(w_0) \stackrel{\text{note}}{=\!=} f(\mathbf{p})$. Thus $|f(w_1)|$ dominates $|f|$ on D, hence on B_1. We can now invoke Local-constancy on B_1, to conclude that $f(w_2) = f(\mathbf{p})$, since $w_2 \in B_1$. Iterating, we eventually show that $f(q) \stackrel{\text{note}}{=\!=} f(w_N) = f(\mathbf{p})$. ♦

16c: MaxMP corollary. *Suppose f is cts on a closed-bounded non-empty region $R \subset \mathbb{C}$ which is path-connected. If f is holomorphic and non-constant on the interior of R then:*

> *Fnc $|f|$ attains a maximum at at-least-one point of ∂R, and never on the interior of R.* ◊

16d: Minimum-modulus principle (MinMP). *Suppose h is non-constant and holomorphic on domain D. If h is never zero on D, then $|h|$ does not attain a minimum on D.* *Proof.* Apply MaxMP to $f := \frac{1}{h}$. ◊

16e: Fund. thm of Algebra. *Every non-constant polynomial h has a $\mathbb{C}$-root.* (*Consequently, h **splits** i.e, a monic h factors completely as $h(z) = [z - \mathbf{r}_1] \cdot \ldots \cdot [z - \mathbf{r}_N]$.*) ◊

Proof. WLOG h is monic. Since h is non-constant, its high-order term has form z^N for some $N \geqslant 1$. As $|z| \nearrow \infty$, this term dominates all the other terms in h. So $|h(z)| \to \infty$ as $|z| \nearrow \infty$. Hence there is a sufficiently large closed ball $B := \mathrm{CldBal}_r(0)$ so that:

∗: *There is strict inequality $|h(z)| > |h(0)|$, for each $z \in \mathbb{C} \smallsetminus B$.*

Now, FTSOC suppose h has no root, i.e, $|h|$ is never zero. Fix a B satisfying (∗). Since B is closed-bounded and $|h|$ is cts, our $|h|$ attains a minimum on B, hence, courtesy (∗), on all of $\mathbb{C}$. But this contradicts the Minimum-modulus principle. ♦

16f: Cone-boundedness Lemma. *For a holomorphic f on the unit ball $B := \mathrm{Bal}_1(0)$, suppose*

$$f(0) = 0 \quad \text{and} \quad \forall z \in B\colon |f(z)| \leqslant 1\,.$$

Then

†: $|f'(0)| \leqslant 1$. *On B, furthermore:* $|f(z)| \leqslant |z|$.

Conversely, if $|f'(0)| = 1$ or there exists a non-zero $w \in B$ with $|f(w)| = |w|$, then f is linear. I.e, f has form $f(z) = M \cdot z$, for some $M \in \mathbb{C}$ with $|M| = 1$. ◊

Proof. It follows from later work [Taylor's thm and friends] that

$$g(z) := \begin{Bmatrix} f(z)/z & \text{, if } z \neq 0 \\ f'(0) & \text{, if } z = 0 \end{Bmatrix}$$

is holomorphic on B. On circle $\mathsf{C}_r := \mathrm{Sph}_r(0)$, note that $|g|$ is (upper-)bnded by $\frac{1}{r}$, since $|f|$ is bnded by 1.

Obtaining (†). Fix $w \in B$ and radius with $|w| \leqslant r < 1$. Our g is holomorphic on $\widehat{\mathsf{C}_r}$. Applying MaxMP, (16b), to g on $\widehat{\mathsf{C}_r}$ shows that $|g(w)| \leqslant \frac{1}{r}$. Sending $r \nearrow 1$ implies that $|g(w)| \leqslant 1$. At $w{=}0$ this says $|f'(0)| \leqslant 1$, and at non-zero w it asserts $|f(w)| \leqslant |w|$.

The converse. A non-zero w with $|f(w)| = |w|$ says $|g(w)| = 1$. And $|f'(0)| = 1$ is equiv to $|g(0)| = 1$. If either happens, then $|g|$ attains a maximum at an interior point of B, so MaxMP implies that g is some constant; say, M, of abs.value 1. Thus $f(z) = M \cdot z$. ♦

Taylor's thm

The "K^{th} ***Taylor polynomial*** for f, centered at Q" is

17: $\mathbf{T}_{f,Q,K}(z) := \sum_{n=0}^{K-1} c_n \cdot [z-Q]^n$, where $c_n := \frac{f^{(n)}(Q)}{n!}$.

The K^{th} ***remainder term*** is defined by

$$f(z) \quad = \quad \mathbf{T}_{f,Q,K}(z) + \mathbf{R}_{f,Q,K}(z).$$

Sometimes the f, Q or z is dropped from the notation, when it is understood.

18a: Taylor-series thm. *Suppose f is holomorphic on open ball B centered at $Q \in \mathbb{C}$. Define coefficient*

$$c_n := \frac{f^{(n)}(Q)}{n!}.$$

Then power series

$$\widetilde{f}(z) := \sum_{n=0}^{\infty} c_n \cdot [z-Q]^n$$

converges to $f(z)$ on B, i.e $\widetilde{f}{\downharpoonright}_B = f$. ◊

Prelim. WLOG $Q = 0$. So $c_n = \frac{f^{(n)}(0)}{n!}$, and the K^{th} Taylor-polynomial is

$$\mathbf{T}_K(z) \quad := \quad \sum_{n=0}^{K-1} c_n \cdot z^n.$$

Fixing a point $\mathbf{p} \in B$, our goal is to establish

$$\sum_{n=0}^{\infty} c_n \cdot \mathbf{p}^n \quad \text{equals} \quad f(\mathbf{p}).$$

To accomplish this, we'll show that the K^{th} remainder term,

†: $$\mathbf{R}_K \quad := \quad f(\mathbf{p}) - \mathbf{T}_K(\mathbf{p})$$

goes to zero as $K \nearrow \infty$. The method is to integrate around a circle $\mathsf{C} := \text{Sph}_r(0) \subset B$ that encloses $\mathbf{p}$; so $r > |\mathbf{p}|$. Below: Let $\int$ mean $\int_{\mathsf{C}}$.

For a complex $w \neq 1$ and posint K, easily (exercise)

∗: $$\frac{1}{1-w} \quad = \quad \frac{w^K}{1-w} + \sum_{n=0}^{K-1} w^n.$$ □

Proof. CIF says $f(\mathbf{p})$ equals $\frac{1}{2\pi i}\int \frac{f(z)}{z-\mathbf{p}}\,dz$. For a $z \in \mathsf{C}$, ratio $w := \mathbf{p}/z$ isn't 1. So (∗) applies, giving

$$\begin{aligned}\frac{1}{z-\mathbf{p}} &= \frac{1}{z}\cdot\frac{1}{1-[\mathbf{p}/z]} \\ &\overset{\text{by }(*)}{=} \frac{1}{z}\cdot\frac{[\mathbf{p}/z]^K}{1-[\mathbf{p}/z]} + \frac{1}{z}\cdot\sum_{n=0}^{K-1}[\mathbf{p}/z]^n \\ &= \frac{\mathbf{p}^K}{[z-\mathbf{p}]\cdot z^K} + \sum_{n=0}^{K-1}\mathbf{p}^n\frac{1}{z^{n+1}}.\end{aligned}$$

Multiplying by $f(z)$, then integrating, says $f(\mathbf{p})$ equals

$$\mathbf{p}^K\cdot\frac{1}{2\pi i}\int\frac{f(z)}{[z-\mathbf{p}]z^K}\,dz + \sum_{n=0}^{K-1}\mathbf{p}^n\cdot\frac{1}{2\pi i}\int\frac{f(z)}{z^{n+1}}\,dz.$$

But GCIF says $\frac{1}{2\pi i}\int\frac{f(z)}{z^{n+1}}\,dz = \frac{f^{(n)}(0)}{n!}$, which is c_n. So the righthand sum is simply $\mathbf{T}_K(\mathbf{p})$. This establishes that

‡: $$\mathbf{R}_K \quad = \quad \mathbf{p}^K\cdot\frac{1}{2\pi i}\int\frac{f(z)}{[z-\mathbf{p}]\cdot z^K}\,dz.$$

Upper-bnding $|\mathbf{R}_K|$. Recall z is on C, a circle of radius $r > |\mathbf{p}|$. As $|z-\mathbf{p}| > |z| - |\mathbf{p}| = r - |\mathbf{p}|$, we have that $\frac{1}{|z-\mathbf{p}|} < \frac{1}{r-|\mathbf{p}|}$. Letting M be the maximum of $|f|$ on C, then,

$$\begin{aligned}\left|\int\frac{f(z)}{[z-\mathbf{p}]\cdot z^K}\,dz\right| &\leqslant \int\frac{M}{[r-|\mathbf{p}|]\cdot r^K}\,|dz| \\ &= \frac{M\cdot 2\pi r}{[r-|\mathbf{p}|]\cdot r^K}.\end{aligned}$$

Happy, (‡) hands us

$$|\mathbf{R}_K| \quad \leqslant \quad \frac{M\cdot r}{r-|\mathbf{p}|}\cdot\left[\frac{|\mathbf{p}|}{r}\right]^K.$$

Since ratio $|\mathbf{p}|/r < 1$, the RhS $\searrow 0$ as $K \nearrow \infty$. ♦

18b: Taylor-remainder coro. *Suppose h is holomorphic on $\widehat{\mathsf{C}}$, where C is a circle centered at some point Q.*
Consider the Taylor decomposition

$$h(p) \quad = \quad \mathbf{T}_{h,Q,K}(p) + \mathbf{R}_{h,Q,K}(p)$$

at a point $p \in \mathring{\mathsf{C}}$. Then the (‡)-formula for the remainder term, is

$$\mathbf{R}_{h,Q,K}(p) \;=\; [p-Q]^K \cdot \mathbf{h}_K(p)\,, \quad \textit{where}$$

$$*: \qquad \mathbf{h}_K(p) \;:=\; \frac{1}{2\pi\boldsymbol{i}} \int_{\mathsf{C}} \frac{h(\zeta)\,\mathrm{d}\zeta}{[\zeta-p]\cdot[\zeta-Q]^K}\,.$$

Moreover, this $\mathbf{h}_K()$ is holomorphic [since $(*)$ satisfies the conditions for diff'ing under the integral sign w.r.t p]. ♢

Remark. The above shows that holomorphic fncs are analytic [locally have power-series expansions], and term-by-term differentiation shows that analytic fncs are holomorphic. Unfinished: as of 9May2017 □

18c: *Remark.* Using the above notation,

$$h(z) \;=\; \Bigg[\sum_{n=0}^{K-1} [z-Q]^n \cdot \tfrac{f^{(n)}(Q)}{n!}\Bigg] \;+\; [z-Q]^K \cdot \mathbf{h}_K(z)\,.$$

Now suppose that some-order h-derivative at Q is *not* zero. Let K now be the smallest index such that $h^{(K)}(Q) \neq 0$. Unfinished: as of 9May2017 □

19a: *Defn.* For an analytic $f\colon D^{\text{open}} \to \mathbb{C}$, in a general sense each point $Q \in \partial(D)$ is a ***singular point***; that is, each nbhd of Q has a point of analyticity of f [see P.74]. A Q is a ***removable singularity*** if f can be defined at Q so that now, f is analytic in a nbhd of Q.

A singularity Q is an ***isolated singularity*** if f is analytic in some punctured-ball $\mathrm{PBal}_r(Q)$.

An isolated singularity Q is a **"*pole* of f"** if $\lim_{z\to Q}|f(z)| = \infty$; otherwise, Q is an ***essential singularity*** of f.

The **"*residue* of f at an isolated singularity Q"** is the unique complex number $\mathcal{R}$ such that function

$$z \;\mapsto\; f(z) \;-\; \tfrac{\mathcal{R}}{z-Q}$$

has an antiderivative in some $\mathrm{PBal}_r(Q)$ with $r > 0$.

At an isolated singularity Q, suppose f is analytic on $\mathrm{PBal}_r(Q)$, where $r > 0$. The Laurent expansion of f has form

$$f(z) \;=\; \Big[\sum_{k=1}^{\infty} \frac{b_k}{[z-Q]^k}\Big] \;+\; \Big[\sum_{n=0}^{\infty} a_n \cdot [z-Q]^n\Big]$$

where $\mathsf{RoC}(\vec{\mathbf{a}}) \geqslant r$ and $\mathsf{RoC}(\vec{\mathbf{b}}) = \infty$. Consequently $\mathrm{Res}(f,Q) = b_1$. □

19b: Residue Thm. *For a SCC C, suppose f is analytic on $\widehat{\mathsf{C}}$ except at finitely many points $Q_1,\ldots,Q_L$, each in $\mathring{\mathsf{C}}$. Then*

$$\int_{\mathsf{C}} f(z)\,\mathrm{d}z \;=\; 2\pi\boldsymbol{i}\cdot\Big[\sum_{\ell=1}^{L}\mathrm{Res}(f,Q_\ell)\Big]\,.$$ ♢

19c: *Residue computation.* Let $f(z) := \sin(z)\cdot\mathrm{e}^z/z^7$. What is $\mathcal{R} := \mathrm{Res}(f,0)$?

Writing $g(z) := \sin(z)\cdot\mathrm{e}^z$ as PS $\sum_{n=0}^{\infty} a_n z^n$, our $\mathrm{Res}(f,0)$ is a_6. Recall

$$\sin(z) \;=\; \tfrac{z}{1} - \tfrac{z^3}{6} + \tfrac{z^5}{120} - \ldots \quad \text{and}$$
$$\mathrm{e}^z \;=\; \textstyle\sum_{k=0}^{\infty} \tfrac{z^k}{k!}\,.$$

So $a_6 = [\frac{1}{1}\cdot\frac{1}{120}] - [\frac{1}{6}\cdot\frac{1}{6}] + [\frac{1}{120}\cdot\frac{1}{1}] = \frac{1}{6}\cdot[\frac{1}{10}-\frac{1}{6}] = \frac{-1}{90}$. □

20: *Standing notation.* For $r>0$, let L_r be the line segment from $-r$ to $+r$, and let A_r be the semicircular arc from $+r$ through $\boldsymbol{i}r$ to $-r$. Glued together they make SCC, D_r, which looks like a ⌂, a horizontal capital D.
Let $\mathsf{U} := \mathrm{Sph}_1(0)$ be the unit circle. □

CoV of Definite-integral to contour-integral, 1

To compute

21a: $$W := \int_0^{2\pi} \frac{1}{4+\cos(\theta)}\,d\theta\,,$$

let's use CoV $z := e^{i\theta}$. So $\frac{dz}{d\theta} = \boldsymbol{i}e^{i\theta} = \boldsymbol{i}z$. Thus

$$d\theta = \frac{dz}{\boldsymbol{i}z} \quad\text{and}\quad \cos(\theta) = \frac{1}{2}[z+\tfrac{1}{z}] = \frac{z^2+1}{2z}\,.$$

So,

$$W = \int_{\mathsf{U}} \frac{1}{[4+\frac{z^2+1}{2z}]}\frac{dz}{\boldsymbol{i}z} = \frac{1}{\boldsymbol{i}}\int_{\mathsf{U}}\frac{1}{[4+\frac{z^2+1}{2z}]z}\,dz\,.$$

The integrand's denominator is $4z + \frac{z^2+1}{2} = \frac{q(z)}{2}$, where $q(z) := z^2+8z+1$. Hence $\boxed{W = \frac{2}{i}\cdot J}$, where

$$J := \int_{\mathsf{U}} \frac{1}{q(z)}\,dz\,.$$

Poles. Note $\mathrm{Discr}(q) = 8^2 - 4{\cdot}1{\cdot}1 = 2^2[4^2-1] = 2^2{\cdot}15$. So g factors as $g(z) = [z-\alpha][z-\beta]$, where

$$\alpha := -4+\sqrt{15} \quad\text{and}\quad \beta := -4-\sqrt{15}\,.$$

Easily, α is enclosed by U, whereas β is outside of U. Letting $h(z) := 1/[z-\beta]$, our J equals

$$\int_{\mathsf{U}}\frac{h(z)}{z-\alpha}\,dz \overset{\text{by CIF}}{=\!=} 2\pi\boldsymbol{i}\cdot h(\alpha) = \frac{\pi\boldsymbol{i}}{\sqrt{15}}\,. \quad\text{Hence}$$

21a′: $$W = \frac{2}{\boldsymbol{i}}\cdot\frac{\pi\boldsymbol{i}}{\sqrt{15}} = \frac{2\pi}{\sqrt{15}}\,.$$

Extending. For $M>1$, define

21b: $$W_M := \int_0^{2\pi}\frac{1}{M+\cos(\theta)}\,d\theta\,.$$

Our CoV $z := e^{i\theta}$ says that $\boxed{W_M = \frac{2}{i}\cdot J}$ where $J := \int_{\mathsf{U}}\frac{1}{q(z)}\,dz$, for quadratic $q(z) := z^2+2Mz+1$.

As before, $\mathrm{Discr}(q) = 2^2[M^2-1]$. Hence $g(z)$ equals $[z-\alpha][z-\beta]$ where

$$\alpha := -M+\sqrt{M^2-1} \quad\text{and}\quad \beta := -M-\sqrt{M^2-1}\,.$$

Since $M>1$, our α is enclosed by U, whereas β is outside. With $H(z) := 1/[z-\beta]$, then, J equals

$$\int_{\mathsf{U}}\frac{H(z)}{z-\alpha}\,dz \overset{\text{by CIF}}{=\!=} 2\pi\boldsymbol{i}\cdot H(\alpha) = \frac{\pi\boldsymbol{i}}{\sqrt{M^2-1}}\,. \quad\text{Thus}$$

21b′: $$W_M = \frac{2\pi}{\sqrt{M^2-1}}\,.$$

General method. The CoV $z := e^{i\theta}$ transforms $[0,2\pi]$ into U, the unit-circle. Moreover, for $\boxed{k\in\mathbb{Z}}$:

21c:
$$d\theta = \frac{dz}{\boldsymbol{i}z}\,,$$
$$\cos(\theta) = \tfrac{1}{2}[z+\tfrac{1}{z}] = \frac{z^2+1}{2z},\quad \cos(k\theta) = \frac{z^{2k}+1}{2z^k}\,,$$
$$\sin(\theta) = \tfrac{1}{2i}[z-\tfrac{1}{z}] = \frac{z^2-1}{2\boldsymbol{i}z},\quad \sin(k\theta) = \frac{z^{2k}-1}{2\boldsymbol{i}z^k}\,.$$

Thus a $\int_0^{2\pi}$ integral of a rational function of $\cos(k\theta)$ and $\sin(\ell\theta)$ is transformed, by the CoV, into a $\int_{\mathsf{U}}$ integral of a rational fnc of z. Factoring the denominator gives the poles of the integrand, so we can apply CIF, equivalently, the Residue thm.

As an example, consider $W := \int_0^{2\pi}\frac{1}{2+\sin(\theta)}\,d\theta$. Our CoV (21c) says that

$$W = \int_{\mathsf{U}}\frac{1}{2+\frac{z^2-1}{2iz}}\cdot\frac{dz}{\boldsymbol{i}z} \overset{\text{note}}{=\!=} \int_{\mathsf{U}}\frac{2}{q(z)}\,dz\,,$$

where $q(z) := z^2+4\boldsymbol{i}z-1$. Thus

$$\mathrm{Discr}(q) = [4\boldsymbol{i}]^2 - 4{\cdot}1{\cdot}[-1] = 2^2{\cdot}[-3]\,. \quad\textit{So,}$$
$$\mathrm{Roots}(q) = \tfrac{1}{2}[-4\boldsymbol{i} \pm 2\boldsymbol{i}\sqrt{3}] = [-2\pm\sqrt{3}]\boldsymbol{i}\,.$$

Consequently $q(z) = [z-\boldsymbol{\alpha}]\cdot[z-\boldsymbol{\beta}]$, where

$$\boldsymbol{\alpha} := [-2+\sqrt{3}]\boldsymbol{i} \quad\textit{and}\quad \boldsymbol{\beta} := [-2-\sqrt{3}]\boldsymbol{i}\,.$$

Easily, $\boldsymbol{\beta}$ is outside U and $\boldsymbol{\alpha}$ is inside, since $[-1<\boldsymbol{\alpha}] \Leftrightarrow [1<\sqrt{3}]$, which holds. Hence W equals

$$\int_{\mathsf{U}}\frac{2/[z-\boldsymbol{\beta}]}{z-\boldsymbol{\alpha}}\,dz \overset{\text{CIF}}{=\!=} 2\pi\boldsymbol{i}\cdot\frac{2}{\boldsymbol{\alpha}-\boldsymbol{\beta}} = 2\pi\boldsymbol{i}\cdot\frac{2}{2\boldsymbol{i}\sqrt{3}}\,.$$

21d: *I.e,* $$\int_0^{2\pi}\frac{1}{2+\sin(\theta)}\,d\theta = \frac{2\pi}{\sqrt{3}}\,.$$

Higher-order poles. The preceding examples had an order-1 pole, so let's go up. For natnum N, define

21e: $$J_N := \int_0^{2\pi} \cos(\theta)^N \, d\theta \, .$$

Of course, the symmetry of cos() forces J_{Odd} to be zero, but let's apply The Method, and see what transpires. ♡4

Our CoV says J_N equals

$$\int_{\mathsf{U}} \Big\lfloor \frac{z^2+1}{2z} \Big\rfloor^N \cdot \frac{dz}{\boldsymbol{i}z} \;=\; \frac{1}{\boldsymbol{i}\cdot 2^N} \int_{\mathsf{U}} \frac{[z^2+1]^N}{z^{N+1}} \, dz \, .$$

With $f(z) := [z^2+1]^N$, let C denote the coefficient of z^N in $f(z)$. Then $f^{(N)}(0) = [N! \cdot C]$. Our GCIF says $\int_{\mathsf{U}} \frac{[z^2+1]^N}{z^{N+1}} \, dz$ equals $\frac{2\pi \boldsymbol{i}}{N!} \cdot f^{(N)}(0) \stackrel{\text{note}}{=\!=} 2\pi\boldsymbol{i} \cdot C$. Thus

$$J_N = \frac{2\pi}{2^N} \cdot C \, .$$

When N odd then $C{=}0$, giving $J_{\text{Odd}}{=}0$, as expected.

When $N = 2H$ is even: The coefficient of z^{2H} in polynomial $f(z) = [z^2+1]^{2H}$ is binomial-coeff $\binom{2H}{H}$. So for $H = 0, 1, 2, \ldots$,

21e′: $$J_{2H} \;=\; \frac{2\pi}{2^{2H}} \cdot \binom{2H}{H} \;=\; 2\pi \cdot \frac{\binom{2H}{H}}{2^{2H}} \, .$$

This multiplier, $\binom{2H}{H}/2^{2H}$, we recognize as: *The Probability, in $2H$ flips of a fair coin, of getting exactly H heads.* [That probability indeed decreases monotonically to zero as $H \nearrow \infty$.]

We get the curiosity that the *average value* of the integral, $\frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)^N \, d\theta$, is a probability. *Hmm...*

♡4 For $N{=}2H$ even, we must have $J_N \searrow 0$ monotonically as $N \nearrow \infty$, since $\cos(\theta)^N$ goes to monotonically to zero, except when θ is a π-multiple.

Definite-integral from limit of contour-int., 2

For posint N, we seek

22a: $$V_N \;:=\; \int_{-\infty}^{\infty} \frac{1}{x^N + 1} \, dx$$

when N is **<u>even</u>**. [When N odd, then the integrand has a pole at $x = -1$.] Moreover, does this limit exist in $\mathbb{R}$?:

22b: $$\Lambda \;:=\; \lim_{\substack{N\to\infty \\ N \text{ even}}} V_N \, .$$

The Trick. Note that

$$\Big| \int_{\mathsf{A}_r} \frac{1}{z^N+1} \, dz \Big| \;\leqslant\; \int_{\mathsf{A}_r} \frac{1}{r^N - 1} \, |dz| \;=\; \frac{\pi r}{r^N - 1} \, ,$$

which goes to zero as $r \nearrow \infty$, since $N > 1$. Thus

$$V_N \stackrel{\text{def}}{=\!=} \lim_{r\nearrow\infty} \int_{\mathsf{L}_r} \frac{1}{z^N+1} \, dz \;=\; \lim_{r\nearrow\infty} \int_{\mathsf{D}_r} \frac{1}{z^N+1} \, dz \, .$$

The only zeros of $z^N + 1$ lie on the unit circle, and so all $r > 1$ yield the *same value* for the righthand integral. Thus its value <u>is</u> V_N, i.e

$$V_N \;=\; \int_{\mathsf{D}_r} \frac{1}{z^N+1} \, dz \, , \qquad \text{for each } r{>}1.$$

Henceforth, D denotes one of these contours; say, D_2 for specificity.

The poles. Let $\omega := \omega_N := \text{cis}(\pi/N)$. The N many N^{th}-roots of -1 are $\omega, \omega^3, \omega^5, \ldots, \omega^{2N-1}$. Those that lie in the upper half-plane, i.e, those enclosed by D, are in list

†: $$\mathcal{L} \;=\; \mathcal{L}_N \;:=\; \{\omega, \omega^3, \omega^5, \ldots, \omega^{N-3}, \, \omega^{N-1}\} \, ,$$

recalling that N is even. These are the poles of $\frac{1}{z^N+1}$ that are enclosed by D.

Fix a pole $\mathbf{p}$ in this list and define

$$f_{\mathbf{p}}(z) \;:=\; \frac{z - \mathbf{p}}{z^N + 1} \, .$$

The contour integral on a contour C that goes around only pole $\mathbf{p}$ is

$$\int_{\mathsf{C}} \frac{f_{\mathbf{p}}(z)}{z - \mathbf{p}} \, dz \, ,$$

which, by CIF, equals $2\pi\boldsymbol{i} \cdot f_{\mathbf{p}}(\mathbf{p})$.

Computing $f_{\mathbf{p}}(\mathbf{p})$. We *could* factor z^N+1, but simpler is to use l'Hôpital's rule. Our $f_{\mathbf{p}}(z)$ has a removable discty at $z{=}\mathbf{p}$, so

$$f_{\mathbf{p}}(\mathbf{p}) = \lim_{z\to\mathbf{p}} \frac{z-\mathbf{p}}{z^N+1} \stackrel{\text{l'H}}{=\!=} \lim_{z\to\mathbf{p}} \frac{1}{Nz^{N-1}} = \frac{1}{N\mathbf{p}^{N-1}}.$$

As $\mathbf{p}^N = \text{-}1$, our $\frac{1}{\mathbf{p}^{N-1}} = \text{-}\mathbf{p}$, thus $f_{\mathbf{p}}(\mathbf{p}) = \text{-}\mathbf{p}/N$.

Adding over the poles. We've now shown that

$$V_N = \frac{\text{-}2\pi i}{N}\cdot\sum_{\mathbf{p}\in\mathcal{L}}\mathbf{p} = \frac{2\pi}{Ni}\cdot\sum_{\mathbf{p}\in\mathcal{L}}\mathbf{p}.$$

Writing our even N as $N{=}2H$ [H for Half] gives the delightfully cheerful formula

‡: $$V_N = \frac{\pi}{Hi}\cdot\sum(\mathcal{L}_N).$$

Interlude. Using *Actual Numbers...*

$$V_2 = \frac{\pi}{1\cdot i}\cdot i = \pi.$$

$$V_4 = \frac{\pi}{2i}\cdot\Big[\frac{i-1}{\sqrt2} + \frac{i+1}{\sqrt2}\Big] = \frac{\pi}{2i}\cdot\frac{2i}{\sqrt2} = \pi/\sqrt2.$$

$$V_6 = \frac{\pi}{3i}\cdot\Big[\frac{i-\sqrt3}{2} + i + \frac{i+\sqrt3}{2}\Big] = \frac{\pi}{3i}\cdot 2i = \frac{2}{3}\pi.$$

Computing $\sum(\mathcal{L}_N)$. The poles of (†) can be paired, allowing us to cancel out the cosines and express this sum ITOf sines. [Discussed in class. In particular, $\Lambda = [\int_0^\pi \sin] = 2$.]

Alternatively, we can sum a finite geometric series. Note that $\frac{1}{\omega}\cdot\mathcal{L} = \{1,\omega^2,\omega^4,\ldots,\omega^{N-4},\omega^{N-2}\}$. Thus

$$\sum\big(\tfrac{1}{\omega}\cdot\mathcal{L}\big) = \sum_{j=0}^{H-1}[\omega^2]^j = \frac{1-[\omega^2]^H}{1-\omega^2}.$$

Recall that $\omega^{2H} = \omega^N = \text{-}1$, so

$$\sum(\mathcal{L}) = \omega\cdot\frac{2}{1-\omega^2} = 2\cdot\frac{\omega}{1-\omega^2}.$$

The reciprocal of $\frac{\omega}{1-\omega^2}$ is $\frac{1-\omega^2}{\omega} = \frac{1}{\omega} - \frac{\omega}{1} = \overline{\omega} - \omega$; this last, because ω is on the unit-circle. And $\overline{\omega}-\omega$ equals $\text{-}2i\cdot\mathrm{Im}(\omega)$, i.e, $2\cdot\mathrm{Im}(\omega)/i$. We get the nifty

22a′: $$\sum(\mathcal{L}_N) = i/\mathrm{Im}(\omega_N) = i/\sin(\tfrac{\pi}{N}), \quad\text{thus}$$
$$V_N = 2\cdot\frac{\pi/N}{\sin(\pi/N)}.$$

Easily, for $\theta \leqslant \frac{\pi}{2}$: *As $\theta\searrow 0$, ratio $\frac{\theta}{\sin(\theta)}$ strictly decreases to* 1. This proves that $V_2 > V_4 > V_6 > \ldots$ and that $V_N \searrow 2$.

Redoing, $V_6 = 2\cdot\frac{\pi/6}{1/2} = \frac{2}{3}\pi$, as before. To compute V_8, the half-angle (...that $\sin(\theta)^2 = \frac{1}{2}[1-\cos(2\theta)]$) formula tells us that $\sin(\frac{\pi}{8}) = \frac{\sqrt{2-\sqrt2}}{2}$. Thus

$$V_8 = \frac{\pi}{2\cdot\sqrt{2-\sqrt2}}.$$

This extended example hints at the power of the residue-calculus. In particular, it handles all...

...Integrals $\int_{-\infty}^{\infty}\frac{f(x)}{q(x)}\,\mathrm{d}x$ with f and q polynomials with $\mathrm{Deg}(q) - \mathrm{Deg}(f) \geqslant 2$, *and q having no real roots.*

Example: Squared outside. Contemplate

22c: $$Z := \int_{-\infty}^{\infty}\frac{1}{[x^2+1]^2}\,\mathrm{d}x.$$

As usual, the integral of $f(z) := \frac{1}{[z^2+1]^2}$ over arc A_r goes to zero as $r\nearrow\infty$, so $Z = \int_{\mathsf{D}} f$, where $\mathsf{D} := \mathsf{D}_2$. As i is the only upper half-plane singularity of f, we have

$$\int_{\mathsf{D}} f = \int_{\mathsf{D}}\frac{g(z)}{[z-i]^2}\,\mathrm{d}z, \text{ where } g(z) := [z+i]^{\text{-}2}.$$

Thus $\mathrm{Res}(f,i) = \frac{g'(i)}{1!} = \text{-}2[z+i]^{\text{-}3}\rfloor_{z=i} = \frac{\text{-}2}{2^3 i^3} = \frac{1}{4i}$. Hence,

22c′: $$\int_{-\infty}^{\infty}\frac{1}{[x^2+1]^2}\,\mathrm{d}x = 2\pi i\cdot\frac{1}{4i} = \frac{\pi}{2}.$$

Generalizing. For K a natnum, integral

22d: $$Z_K := \int_{-\infty}^{\infty}\frac{1}{[x^2+1]^{K+1}}\,\mathrm{d}x$$

equals $\int_{\mathsf{D}}\frac{g(z)}{[z-i]^{K+1}}\,\mathrm{d}z$, where $g(z) := [z+i]^{\text{-}[K+1]}$. Now $\mathrm{Res}(f,i) = \frac{g^{(K)}(i)}{K!}$. Doing the arithmetic yields

22d′: $$\int_{-\infty}^{\infty}\frac{1}{[x^2+1]^{K+1}}\,\mathrm{d}x = \pi\cdot\frac{\binom{2K}{K}}{2^{2K}}.$$

This looks a lot like (21e′). Again, *Hmm...*

Jordan's Lemma

We need an estimate to show that certain integrals are bounded on our A_r arcs. But first...

23a: Proposition. *Fix $T > 0$. Then*

$$\int_0^{\pi} |\exp(\boldsymbol{i}T\cdot\text{cis}(\theta))|\, d\theta \;\leqslant\; \frac{\pi}{T}\,. \qquad \diamondsuit$$

Proof. Since sin() is convex-down on $[0, \frac{\pi}{2}]$, its graph lies above the line-segment connecting $(0,0)$ to $(\frac{\pi}{2}, 1)$. Thus

$$\dagger: \quad \forall\theta \in [0, \tfrac{\pi}{2}]: \quad \sin(\theta) \geqslant \frac{\theta}{\pi/2}, \text{ so } \; -\sin(\theta) \leqslant \frac{-2}{\pi}\cdot\theta\,.$$

For S,$B > 0$, note $\int_0^B e^{-S\theta}\, d\theta = \frac{1}{S}[1 - e^{-SB}]$. Hence

$$\ddagger: \qquad \int_0^B e^{-S\theta}\, d\theta \quad\leqslant\quad 1/S\,.$$

Estimate. Since $\boldsymbol{i}T\cdot\text{cis}(\theta) = \boldsymbol{i}T\cos(\theta) - T\sin(\theta)$ and T is real, we have that

$$|\exp(\boldsymbol{i}T\cdot\text{cis}(\theta))| \;=\; \exp(-T\sin(\theta))\,.$$

On interval $[0, \pi]$, fnc sin() is symmetric about $\frac{\pi}{2}$. Thus

$$\begin{aligned}\int_0^{\pi} |\exp(\boldsymbol{i}\,T\cdot\text{cis}(\theta))|\, d\theta \;&=\; 2\int_0^{\pi/2} \exp(-T\sin(\theta))\, d\theta \\ &\leqslant\; 2\int_0^{\pi/2} \exp\Big(\frac{-2T}{\pi}\theta\Big)\, d\theta\,,\end{aligned}$$

courtesy (†) and that $T{>}0$, as well as that exp() is order-preserving on $\mathbb{R}$.

Applying (‡) with $B := \pi/2$ and $S := \frac{2T}{\pi}$ now yields that

$$\int_0^{\pi} |\exp(\boldsymbol{i}\,T\cdot\text{cis}(\theta))|\, d\theta \;\leqslant\; 2\Big/\frac{2T}{\pi} \;\overset{\text{note}}{=\!=}\; \frac{\pi}{T}\,. \qquad \blacklozenge$$

23b: Jordan Lemma. *Fix $P > 0$ and a fnc $g()$ which is continuous on the upper half-plane in $\mathbb{C}$. For each $r{>}0$, let M_r be the maximum of $|g|$ on A_r. Then every radius $r{>}0$ satisfies*

$$¥: \qquad \Big|\int_{A_r} e^{iPz}\cdot g(z)\, dz\Big| \;\leqslant\; \frac{\pi}{P}\cdot M_r\,. \qquad \diamondsuit$$

Pf. Note LhS(¥) $\leqslant \int_{A_r} |e^{\boldsymbol{i}Pz}|\cdot M_r\,|dz|$. So ISTShow that

$$\int_{A_r} |e^{\boldsymbol{i}Pz}|\,|dz| \;\overset{?}{\leqslant}\; \frac{\pi}{P}\,.$$

CoV $z = re^{\boldsymbol{i}\theta} \overset{\text{note}}{=\!=} r\,\text{cis}(\theta)$ has $\frac{dz}{d\theta} = \boldsymbol{i}re^{\boldsymbol{i}\theta}$. Thus,

$$\begin{aligned}\int_{A_r} |e^{\boldsymbol{i}Pz}|\,|dz| \;&=\; \int_0^{\pi} |\exp(\boldsymbol{i}Pr\cdot\text{cis}(\theta))|\cdot|\boldsymbol{i}re^{\boldsymbol{i}\theta}|\, d\theta \\ &=\; r\cdot\int_0^{\pi} |\exp(\boldsymbol{i}Pr\cdot\text{cis}(\theta))|\, d\theta \;\leqslant\; r\cdot\frac{\pi}{Pr}\,.\end{aligned}$$

This last inequality is courtesy Proposition 23a applied with $T := P\cdot r$. ♦

Appl. of Jordan Lemma. Consider

$$23c: \qquad Y \;:=\; \int_{-\infty}^{\infty} \frac{x\cdot\sin(x)}{x^2+1}\, dx\,.$$

The difference in the degrees of the denominator poly, $x^2 + 1$, and numer poly, x, is only 1. The positive and negative parts of the integrand each have infinite integral, hence $\int_{-\infty}^{\infty} \big|\frac{x\cdot\sin(x)}{x^2+1}\big|\, dx = \infty$; so the oscillations of sin() are crucial for convergence of (23c).

Fixing an $r > 1$, we seek to compute

$$Y_r \;:=\; \int_{-r}^{r} \frac{x\cdot\sin(x)}{x^2+1}\, dx\,.$$

Note $\int_{-r}^{r} \frac{x\cdot\cos(x)}{x^2+1}\, dx$ is zero, since cos() is an even fnc. Thus Y_r equals

$$\int_{-r}^{r} \frac{x\cdot[\sin(x) - \boldsymbol{i}\cos(x)]}{x^2+1}\, dx \;=\; -\boldsymbol{i}\int_{-r}^{r} \frac{x\cdot e^{\boldsymbol{i}x}}{x^2+1}\, dx\,.$$

Thus we'll have

$$*: \qquad Y \;=\; -\boldsymbol{i}\lim_{r\nearrow\infty} \int_{D_r} \frac{z\cdot e^{\boldsymbol{i}z}}{z^2+1}\, dz$$

if we can show that the contribution on arc A_r goes to zero.

Applying Jordan's Lemma (23b) with $g(z) := \frac{z}{z^2+1}$ and $P = 1$, gives

$$\Big|\int_{A_r} \frac{z\cdot e^{\boldsymbol{i}z}}{z^2+1}\, dz\Big| \;\leqslant\; \pi\cdot\frac{r}{r^2-1}\,.$$

This goes to zero as $r\nearrow\infty$. So Y equals $-\boldsymbol{i}\int_{D} \frac{z\cdot e^{\boldsymbol{i}z}}{z^2+1}\, dz$ where D is, say, D_2, since D_2 encloses all the upper half-plane singularities of the integrand.

Applying CIF to $f(z) := z \cdot e^{iz}/[z+i]$ gives

$$\int_{D} \frac{f(z)}{z-i}\,dz = 2\pi i \cdot f(i) = 2\pi i \cdot \frac{i \cdot e^{ii}}{[i+i]} = i \cdot \frac{\pi}{e}.$$

So (∗) says

23c′: $$\int_{-\infty}^{\infty} \frac{x \cdot \sin(x)}{x^2+1}\,dx = \frac{\pi}{e}.$$

Now *that* is pretty dang *Cool!*

Keyhole contours, 3

Some definite integrals can be neatly computed using a ***keyhole contour***. Here is an example:

> *Let* K *be the contour along* $\mathbb{R}$ *from* $1/r$ *to* r*, then CCW circle* $\mathrm{Sph}_r(0)$*, then along* $\mathbb{R}$ *from* r *to* $1/r$*, and finally CW circle* $\mathrm{Sph}_{1/r}(0)$*.*

Call the $1/r$ to r line-segment L_r. Call the r to $1/r$ line-segment $\widetilde{L}_r$; we need a different name because we will be integrating fncs with a branch-point at 0, and we have gone around that branch-point.

Computing Γ. Let's use our K to compute

24: $$\Gamma := \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$$

With $f(z) := \frac{\sqrt{z}}{z^2+1}$, observe that

$$\int_{\widetilde{L}_r} f = -[-1] \cdot \int_{L_r} f.$$

The negative-sign is because we traverse $\widetilde{L}_r$ in the opposite direction from L_r. The [-1] is what a square-root is multiplied-by, when we circle CCW once around the branch-point. Because of the form of our f, its value is multiplied by [-1] when circumnavigating the branch-point.

Easily the f-integral along the circles of radius r and $1/r$ go to zero as $r \nearrow \infty$. So

$$\lim_{r\nearrow\infty} \int_{K_r} f = \lim_{r\nearrow\infty} \Big[\int_{L_r} f + \int_{\widetilde{L}_r} f\Big] = \lim_{r\nearrow\infty} 2\int_{L_r} f = 2\Gamma.$$

The singularities of f are at $\pm i$. They are enclosed by $K := K_2$, whence

∗: $$2\Gamma = \int_K f = 2\pi i \cdot [\mathrm{Res}(f,i) + \mathrm{Res}(f,-i)].$$

Let □ mean a finite-value that we don't need to compute, because it will be multiplied by zero.

We could just factor z^2+1 and use CIF, but let's compute the residues at these order-1 poles. So $\mathrm{Res}(f,i)$ equals

$$\lim_{z\to i} [z-i]f(z) = \lim_{z\to i} \frac{[z-i]\sqrt{z}}{z^2+1} \overset{\text{L'H}}{=\!=} \lim_{z\to i} \frac{1\cdot\sqrt{z} + [z-i]\square}{2z} = \frac{\alpha}{2i},$$

where α is the sqroot of i for this branch of $\sqrt{\cdot}$.

Similarly, $\mathrm{Res}(f,-i)$ equals

$$\lim_{z\to -i} \frac{[z+i]\sqrt{z}}{z^2+1} \overset{\text{L'H}}{=\!=} \lim_{z\to -i} \frac{1\cdot\sqrt{z} + \square}{2z} = \frac{\beta}{-2i},$$

where β is the sqroot of $-i$ for this branch of $\sqrt{\cdot}$ fnc.

Computing the sqroots. For this branch of $\sqrt{\cdot}$, our $\alpha = \frac{i+1}{\sqrt{2}}$ and $\beta = \frac{i-1}{\sqrt{2}}$, whence $\alpha - \beta = \frac{2}{\sqrt{2}}$. So from (∗),

24′: $$\Gamma = \pi i \cdot \sum \mathrm{Res} = \pi i \cdot \frac{\alpha-\beta}{2i} = \frac{\pi}{\sqrt{2}}.$$

Cube-root. Our K also applies to

25: $$\Omega := \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$$

Let $g(z) := \frac{z^{1/3}}{z^2+1}$. As before,

$$\int_{L+\widetilde{L}} g = [1-M] \cdot \int_{L} g,$$

where M is what a cube-root is multiplied-by, when we circle CCW once around the branch-point. Because the form of our g, its value is multiplied by M. Here, $M = \frac{1}{2}[i\sqrt{3} - 1]$, the cube-root of 1 that circumnavigation brings us to. Looking ahead,

†: $$1 - M = \tfrac{1}{2}[3 - i\sqrt{3}] = \tfrac{\sqrt{3}}{2} \cdot [\sqrt{3} - i].$$

As before, the g-integral on the circles dies off, so

‡: $$[1-M]\cdot\Omega = [1-M]\cdot\int_{L_\infty} g \overset{\text{note}}{=\!=} \int_K g.$$

Computing residues. Our $\operatorname{Res}(g, \boldsymbol{i})$ equals

$$\lim_{z\to\boldsymbol{i}} [z-\boldsymbol{i}]g(z) = \lim_{z\to\boldsymbol{i}} \frac{[z-\boldsymbol{i}]\cdot z^{1/3}}{z^2+1} \stackrel{\text{L'H}}{=\!=} \lim_{z\to\boldsymbol{i}} \frac{1\cdot z^{1/3}+\square}{2z} = \frac{\boldsymbol{\alpha}}{2\boldsymbol{i}},$$

where $\boldsymbol{\alpha}$ is the cube-root of $\boldsymbol{i}$ for this branch of $\sqrt[3]{\cdot}$.

Similarly, $\operatorname{Res}(g, -\boldsymbol{i})$ equals $\lim_{z\to -\boldsymbol{i}} [z+\boldsymbol{i}]g(z)$, i.e

$$\lim_{z\to -\boldsymbol{i}} \frac{[z+\boldsymbol{i}]\cdot z^{1/3}}{z^2+1} \stackrel{\text{L'H}}{=\!=} \lim_{z\to -\boldsymbol{i}} \frac{1\cdot z^{1/3}+\square}{2z} = \frac{\boldsymbol{\beta}}{-2\boldsymbol{i}},$$

where $\boldsymbol{\beta}$ is the cube-root of $-\boldsymbol{i}$ for this branch of $\sqrt[3]{\cdot}$.

Here, $\boldsymbol{\alpha} = \frac{1}{2}[\sqrt{3} + \boldsymbol{i}]$ and $\boldsymbol{\beta} = \boldsymbol{i}$, so $\boldsymbol{\alpha} - \boldsymbol{\beta}$ equals $\frac{1}{2}[\sqrt{3} - \boldsymbol{i}]$. Consequently,

$$\int_{\mathsf{K}} g = 2\boldsymbol{\pi}\boldsymbol{i}\cdot[\textstyle\sum \operatorname{Res}] = 2\boldsymbol{\pi}\boldsymbol{i}\cdot\dfrac{\boldsymbol{\alpha}-\boldsymbol{\beta}}{2\boldsymbol{i}} = \dfrac{\boldsymbol{\pi}}{2}\cdot[\sqrt{3} - \boldsymbol{i}].$$

Thus

$$\frac{\boldsymbol{\pi}}{2}\cdot[\sqrt{3} - \boldsymbol{i}] \stackrel{\text{by (‡)}}{=\!=} [1-M]\Omega \stackrel{\text{by (†)}}{=\!=} \frac{\sqrt{3}}{2}\cdot[\sqrt{3}-\boldsymbol{i}]\Omega.$$

I.e, $\frac{\boldsymbol{\pi}}{2} = \frac{\sqrt{3}}{2}\Omega$. So

25′: $$\Omega = \frac{\boldsymbol{\pi}}{\sqrt{3}}.$$

The power of contour-integration, *At Your Service!*

Four failures

Part of understanding a technique is when it *doesn't* apply, or when it needs to be modified.

Consider

$$Y_1 := \int_0^\infty \frac{5+\sqrt{x}}{x^2+1}\,dx.$$

In going around the branch-point, we multiply $\sqrt{z}$ by −1, but that doesn't multiply the integrand by −1, as 5 is unchanged. In this instance, we could write Y_1 as a sum $\left[\int_0^\infty \frac{5}{x^2+1}\,dx\right] + \left[\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx\right]$ and compute each integral separately.

Now consider

$$Y_2 := \int_0^\infty \frac{\sin(5+\sqrt{x}\,)}{x^2+1}\,dx.$$

Going around the branch point, we multiply $\sqrt{x}$ by −1, but not 5, and so what happens to $\sin(5+\sqrt{x})$ is complicated. It is unclear how to proceed. [Does the formula for the sine of a sum, help?]

Our third example is

$$Y_3 := \int_0^\infty \frac{\sin(\sqrt{x})}{x^2+1}\,dx.$$

In going around the branch point, we multiply $\sqrt{x}$ by −1. This *happens* to multiply $\sin(\sqrt{x})$ by −1, since sin() is an odd fnc, but it is important to understand *why* the technique still works in this instance.

Our fourth example is the innocuous

$$Y_4 := \int_0^\infty h, \quad \textit{where} \quad h(z) := \frac{\cos(\sqrt{z})}{z^2+1}.$$

Here, the method fails in a novel way.

Going around the branch point multiplies $\sqrt{z}$ by −1. Since cos() is even, this leaves $\cos(\sqrt{x})$ unchanged. Thus

$$\int_{\mathsf{D}} h = \int_{\mathsf{L}+\widetilde{\mathsf{L}}} h = [1-1]\int_{\mathsf{L}} h = 0\cdot Y_4.$$

Unsurprisingly, $\int_{\mathsf{D}} h$ will be zero, yielding the useless eqn $0 = 0\cdot Y_4$, giving **no** information about Y_4.

Applications of Rouché's thm

Rouché's thm can be viewed as a special case of The Argument Principle.

26: Rouché's Thm. *Consider SCC* C. *Suppose both* $\boldsymbol{\alpha}()$ *and* $\boldsymbol{\beta}()$ *are analytic on* $\widehat{\mathsf{C}}$, *and* $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$ *on* C. *Then*

> $\boldsymbol{\alpha}$ *and* $\boldsymbol{\alpha}+\boldsymbol{\beta}$ *have the same number of zeros* [*counted with multiplicity*] *in* $\mathring{\mathsf{C}}$.

Note $\boldsymbol{\alpha}$ *and* $\boldsymbol{\alpha}-\boldsymbol{\beta}$ *also have the same number of zeros, in* $\mathring{\mathsf{C}}$, *since* $|\boldsymbol{\alpha}| > |-\boldsymbol{\beta}|$ *on* C. ◊

For real number K, a z-expression $E{=}\boldsymbol{\alpha}(z)$ and an arbitrary set C, let an expression such as "$|E|_{\mathsf{C}} \geqslant K$" or "$|\boldsymbol{\alpha}|_{\mathsf{C}} \geqslant K$" or "$|\boldsymbol{\alpha}()|_{\mathsf{C}} \geqslant K$" mean that $\forall z \in \mathsf{C} : |\boldsymbol{\alpha}(z)| \geqslant K$.

27: *Ex. R1.*

Soln R1.

28: *Ex. R1.*

Soln R1.

29: *Ex. R4.* Fix real $M{>}0$ and $T{>}2$. Prove that

30: $$Mz^3 - z + T \;=\; [z+2] \cdot \mathsf{e}^{-z}$$

has precisely 2 solns in $\mathbf{H} := \{z \mid \mathrm{Re}(z) > 0\}$. □

Soln R4. We will use $\boldsymbol{\alpha}(z) := Mz^3 - z + T$ and $\boldsymbol{\beta}(z) := [z+2] \cdot \mathsf{e}^{-z}$. When $\mathrm{Re}(z) \geqslant 0$, note $|\boldsymbol{\beta}(z)| \leqslant |z+2|$, since $|\mathsf{e}^{-z}| \leqslant |\mathsf{e}^0| = 1$.

For $z = \boldsymbol{i}y$ on the imaginary axis, note $\boldsymbol{\alpha}(\boldsymbol{i}y)$ equals $-\boldsymbol{i}y^3M - \boldsymbol{i}y + T$. Since y^3M and y have the *same sign*,

$$|\boldsymbol{\alpha}(\boldsymbol{i}y)| \;\geqslant\; |\boldsymbol{i}y + T| \;>\; |\boldsymbol{i}y + 2|\,,$$

since $T{>}2$ is real, hence *orthogonal to* $\boldsymbol{i}y$. Thus

†: *On the imaginary axis,* $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$.

[This argument needed the strict $T{>}2$. For if $T{=}2$, then $|\boldsymbol{\alpha}(0)| = |\boldsymbol{\beta}(0)|$.]

The contour. For $r{>}0$, let A_r be the radius-r semicircle from $-\boldsymbol{i}r$ through r to $\boldsymbol{i}r$.

For $r > \frac{1}{M}$, note $Mr^3 - r - 2 > r^2 - r - 2$. For $z \in \mathsf{A}_r$, then, $|\boldsymbol{\alpha}(z)| > r^2 - r - 2$.

If also $r > 4$, then $r^2 > 4r$. Hence

$$r^2 - r - 2 \;>\; 3r - 2 \;>\; 2r \;>\; r + 2 \;\geqslant\; |\boldsymbol{\beta}(z)|\,,$$

for $z \in \mathsf{A}_r$. Consequently,

‡: *For* $r{>}\mathrm{Max}(4, \frac{1}{M})$, *we have* $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$ *on* A_r.

For such r, then, our (†,‡) guarantee that $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$ on contour D_r, where D_r is arc A_r glued to the line-segment from $-\boldsymbol{i}r$ to $\boldsymbol{i}r$. Sending $r \nearrow \infty$, then,

> *In half-plane* $\mathbf{H}$, *expression* $[z+2] \cdot \mathsf{e}^{-z}$ *has the same number of zeros as polynomial* $\boldsymbol{\alpha}(z)$. ♦

Counting roots of $\boldsymbol{\alpha}()$**.** As $x \searrow -\infty$, remark that $\boldsymbol{\alpha}(x) \to -\infty$. Yet $\boldsymbol{\alpha}(0) = T > 0$. So IVT (Intermediate Value Thm) implies $\boldsymbol{\alpha}()$ has a negative real-root.

Unfinished: as of 9May2017

Notation Appendix

Use $\in$ for "is an element of". E.g, letting $\mathbb{P}$ be the set of primes, then, $5 \in \mathbb{P}$ yet $6 \notin \mathbb{P}$. Changing the emphasis, $\mathbb{P} \ni 5$ ("$\mathbb{P}$ *owns* 5") yet $\mathbb{P} \not\ni 6$.

For subsets A and B of the same space, Ω, the ***inclusion relation*** $A \subset B$ means:

$$\forall \omega \in A, \textit{ necessarily } B \ni \omega.$$

And this can be written $B \supset A$. Use $A \subsetneqq B$ for *proper* inclusion, i.e, $A \subset B$ yet $A \neq B$.

The *difference set* $B \smallsetminus A$ is $\{\omega \in B \mid \omega \notin A\}$. Employ A^c for the ***complement*** $\Omega \smallsetminus A$. Use $A \triangle B$ for ***symmetric difference*** $[A \smallsetminus B] \cup [B \smallsetminus A]$. Furthermore

$A \boxdot B$,	Sets A & B have *at least one* point in common; they intersect.
$A \sqcap B$,	The sets have *no* common point; disjoint.

The symbol "$A \boxdot B$" both asserts intersection and represents the set $A \cap B$. For a collection $\mathcal{C} = \{E_j\}_j$ of sets in Ω, let the ***disjoint union*** $\bigsqcup_j E_j$ or $\bigsqcup(\mathcal{C})$ represent the union $\bigcup_j E_j$ *and also* assert that the sets are pairwise disjoint.

For fncs on a set $\mathbf{\Omega}$, each subset $B \subset \mathbf{\Omega}$ has its corresponding "***indicator function*** of B", written $\mathbf{1}_B$. It is the fnc $\mathbf{\Omega} \to \{0,1\}$ which sends points in B to 1 and points in $\mathbf{\Omega} \smallsetminus B$ to 0. [So $\mathbf{1}_A + \mathbf{1}_{\complement(A)}$ is constant-1.] E.g, $\mathbf{1}_{\text{Primes}}(5) = 1$, and $\mathbf{1}_{\text{Primes}}(9) = 0$.

General Appendix

The ***discriminant*** of quadratic [i.e, $A \neq 0$] polynomial $q(z) := Az^2 + Bz + C$ is

31.1: $$\text{Discr}(q) \quad := \quad B^2 - 4AC\,.$$

The zeros ["roots"] of q are

31.2: $$\text{Roots}(q) \quad = \quad \frac{1}{2A}\Big[-B \pm \sqrt{\text{Discr}(q)}\,\Big].$$

Hence when A,B,C are *real*, then the zeros of q form a complex-conjugate pair. And q has a *repeated root* IFF Discr(q) is zero.

A monic $\mathbb{R}$-irreducible quadratic has form

31.3: $$q(x) \;=\; x^2 - Sx + P \;=\; [x - Z]\cdot[x - \overline{Z}]\,,$$

where $Z \in \mathbb{C} \smallsetminus \mathbb{R}$. Note $S = Z + \overline{Z} = 2\text{Re}(Z)$ is the *Sum* of the roots. And $P = Z \cdot \overline{Z} = |Z|^2$ is the *Product* of the roots. The discriminant of g, Discr(g), equals

31.4: $$S^2 - 4P \stackrel{\text{note}}{=\!=} [Z - \overline{Z}]^2 \;=\; -4\cdot[\text{Im}(Z)]^2\,.$$

Completing-the-square yields

31.5: $$q(x) \;=\; [x - \tfrac{S}{2}]^2 \;+\; F^2, \text{ where } F := |\text{Im}(Z)|\,,$$

which is easily checked. [Exercise]

Abbreviations. Use ***posreal*** for "positive real number". A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_1, x_2, x_3, \ldots)$. Use $\text{Tail}_N(\vec{\mathbf{x}})$ for the subsequence $(x_N,\, x_{N+1},\, x_{N+2}, \ldots)$ of $\vec{\mathbf{x}}$. □

32a: Addition-Cts thm. *The addition operation* $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ *is continuous.* Restated: *Suppose* $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{C}$ *with* $\lim(\vec{\mathbf{x}}) = \alpha$ *and* $\lim(\vec{\mathbf{y}}) = \beta$. *With* $p_n := x_n + y_n$, *then,* $\lim(\vec{\mathbf{p}}) = \alpha + \beta$. ◊

Proof. Fix a posreal ε. Take N large enough that

$$\text{Tail}_N(\vec{\mathbf{x}}) \;\subset\; \text{Bal}_{\frac{\varepsilon}{2}}(\alpha) \quad\text{and}\quad \text{Tail}_N(\vec{\mathbf{y}}) \;\subset\; \text{Bal}_{\frac{\varepsilon}{2}}(\beta)\,.$$

Each index k has $p_k - [\alpha + \beta] = [x_k - \alpha] + [y_k - \beta]$. For each $k \geqslant N$, then,

$$\big|p_k - [\alpha + \beta]\big| \;\leqslant\; |x_k - \alpha| + |y_k - \beta| \;\leqslant\; \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \;=\; \varepsilon ♦$$

Remark. The same thm and proof hold for addition on a normed vectorspace; simply replace $|\cdot|$ by the norm $\|\cdot\|$. □

Abbreviations. Use ***WELOG*** for "without essential loss of generality", and ***posint*** for "positive integer".

A *sequence* $\vec{\mathbf{x}}$ abbreviates $(x_1, x_2, x_3, \ldots)$. Use $\text{Diam}(\vec{\mathbf{x}})$ for the *diameter* of the set $\{x_n\}_{n=1}^{\infty}$. □

32b: Mult-Cts thm. *The multiplication operation* $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ *is continuous.* Restated: *Suppose* $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{C}$ *with* $\lim(\vec{\mathbf{x}}) = \alpha$ *and* $\lim(\vec{\mathbf{y}}) = \beta$. *With* $p_n := x_n \cdot y_n$, *then,* $\lim(\vec{\mathbf{p}}) = \alpha \cdot \beta$. ◊

Proof. WELOG $|\beta| \leqslant 7$. Since $\vec{\mathbf{x}}$ converges, necessarily the $\text{Diam}(\vec{\mathbf{x}})$ is finite; WELOG

$$\dagger: \qquad \forall \text{ posints } n: \quad |x_n| \leqslant 50.$$

For each posint n, adding and subtracting a term gives

$$\begin{aligned} x_n y_n - \alpha\beta &= x_n y_n - x_n\beta + x_n\beta - \alpha\beta \\ &= x_n[y_n - \beta] + [x_n - \alpha]\beta. \end{aligned}$$

Taking absolute-values, then upper-bounding, yields

$$\begin{aligned} \ddagger: \quad |x_n y_n - \alpha\beta| &\leqslant |x_n| \cdot |y_n - \beta| + |x_n - \alpha| \cdot |\beta| \\ &\leqslant 50 \cdot |y_n - \beta| + |x_n - \alpha| \cdot 7, \end{aligned}$$

by (†) and the first sentence.

Fix a posreal ε. Since $\lim(\vec{\mathbf{y}}) = \beta$ and $\lim(\vec{\mathbf{x}}) = \alpha$, we can take K large enough that for each n in $[K .. \infty)$:

$$|y_n - \beta| \leqslant \frac{\varepsilon/2}{50} \quad \textit{and} \quad |x_n - \alpha| \leqslant \frac{\varepsilon/2}{7}.$$

Plugging these estimates in to (‡) gives that

$$|x_n y_n - \alpha\beta| \quad \leqslant \quad 50 \cdot \tfrac{\varepsilon/2}{50} + \tfrac{\varepsilon/2}{7} \cdot 7 \quad \stackrel{\text{note}}{=\!=} \quad \varepsilon,$$

for *each* $n \geqslant K$.

As this holds for every ε positive, $\lim(\vec{\mathbf{x}} \cdot \vec{\mathbf{y}})$ indeed equals $\alpha\beta$. ♦

33: Non-neg Lemma. *On interval $J := [a, b]$ suppose continuous function h satisfies $h \geqslant 0$. If $\int_a^b h(t)\,\mathrm{d}t$ is zero, then $h()$ is identically zero.*

On a closed contour $\mathsf{C} \subset \mathbb{C}$, suppose a continuous $g\colon \mathsf{C} \to \mathbb{R}$ is non-negative; $g() \geqslant 0$. If the arclength integral

$$\int_{\mathsf{C}} g(z)\, |\mathrm{d}z|$$

is zero, then g is identically-zero on C. ♢

Pf for h. FTSOC, suppose $\exists \mathbf{p} \in J$ with $3\varepsilon := h(\mathbf{p})$ positive. Cty of h at $\mathbf{p}$ says there exists an interval $I \ni \mathbf{p}$ of positive length, so that every $x \in I$ satisfies

$$\big|h(x) - h(\mathbf{p})\big| \leqslant \varepsilon;$$

hence $h(x) \geqslant 3\varepsilon - \varepsilon = 2\varepsilon$. But $h()$ is non-negative on J, so

$$\int_J h \;\geqslant\; \int_I h \;\geqslant\; \int_I 2\varepsilon \;=\; 2\varepsilon \cdot \text{Len}(I).$$

This latter is positive, yielding a contradiction. ♦

Pf for g. Let $z\colon [0, 1] \to \mathsf{C}$ be a [cts, piecewise smooth] parametrization of C. Then $h(t) := g(z(t))$ is cts and non-negative. By above, $h \equiv 0$ whence $g \equiv 0$. ♦

Sufficient condition for differentiability

Consider an open subset $U \subset \mathbb{R}^N$ and a map $h{:}U{\to}\mathbb{R}$. Use abbreviation $\vec{\mathbf{x}}$ for the N-tuple $\vec{\mathbf{x}} := (x_1, x_2, \ldots, x_N)$, a point in $\mathbb{R}^N$. Let h_j mean $\frac{\mathrm{d}h}{\mathrm{d}x_j}$, that is, the partial-derivative of $h()$ w.r.t its j^{th} argument. Finally, have $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^N$: $\|\vec{\mathbf{x}}\| := \sqrt{\sum_{j=1}^N |x_j|^2}$.

34: Thm. *Fix a point $\vec{\mathbf{c}} \in U$. Suppose all partial-derivs $h_1, \ldots, h_N$ are defined in a nbhd of $\vec{\mathbf{c}}$, and are each continuous* at *$\vec{\mathbf{c}}$. Then h is differentiable at $\vec{\mathbf{c}}$.* ◊

Proof. Without loss of generality, $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. [Rename $h_{New}(\vec{\mathbf{x}}) := h(\vec{\mathbf{x}} - \vec{\mathbf{c}})$, and translate U.]

WLOG, $h(\vec{\mathbf{0}}) = 0$. [Rename $h_{New}(\vec{\mathbf{x}}) := h(\vec{\mathbf{x}}) - h(\vec{\mathbf{0}})$.]

WLOG, $\forall j$, partial-deriv $h_j(\vec{\mathbf{0}})$ is zero. Why? Rename

$$h_{New}(\vec{\mathbf{x}}) \quad := \quad h(\vec{\mathbf{x}}) - \sum_{j=1}^N [h_j(\vec{\mathbf{0}}) \cdot x_j] .$$

Now that all the partials are zero at the origin, differentiability at the origin is can be stated thusly:

> *For all $\varepsilon{>}0$, there exists $\delta{>}0$ so that each $\vec{\mathbf{p}} \in U$ with $0 < \|\vec{\mathbf{p}}\| < \delta$, satisfies $\frac{|h(\vec{\mathbf{p}})|}{\|\vec{\mathbf{p}}\|} < \varepsilon$.*

Of course, the "$<\varepsilon$" can be replaced by any zero-going fnc of ε, so ISTProduce a δ such that:

GOAL: *For all $\varepsilon{>}0$, $\exists\delta{>}0$ so that each $\vec{\mathbf{p}} \in U$ with $0 < \|\vec{\mathbf{p}}\| < \delta$, has $|h(\vec{\mathbf{p}})| < \varepsilon \cdot K_N \cdot \|\vec{\mathbf{p}}\|$,*

for some positive constant K_N; that is, does not depend on ε, nor on $\vec{\mathbf{p}}$.

Continuity at $\vec{\mathbf{0}}$. Cty of the partials at $\vec{\mathbf{0}}$ admits a $\delta{>}0$ small enough that the open ball $\mathbf{B} := \mathrm{Bal}_\delta(\vec{\mathbf{0}})$ has this property:

†: *For each $j = 1, \ldots, N$ and $\forall \vec{\mathbf{x}} \in \mathbf{B}$, we have that*

$$|h_j(\vec{\mathbf{x}})| \stackrel{\text{note}}{=\!=} |h_j(\vec{\mathbf{x}}) - h_j(\vec{\mathbf{0}})| < \varepsilon .$$

Using MVT. Fix an $\varepsilon{>}0$, and consider a point $\vec{\mathbf{p}}{\in}\mathbf{B}$. We'll apply MVT at each index j for which $p_j \neq 0$; so for notational simplicity, assume every j has $p_j \neq 0$.

For $k = 0, 1, \ldots, N$ define

$$\vec{\mathbf{y}}^{\langle k \rangle} \quad := \quad (p_1, \ldots, p_k, \overbrace{0, 0, \ldots, 0}^{N-k}) .$$

And for $j = 1, \ldots, N$, let S_j denote the line-segment from $\vec{\mathbf{y}}^{\langle j-1 \rangle}$ to $\vec{\mathbf{y}}^{\langle j \rangle}$.

As $\|\vec{\mathbf{p}}\| \geqslant \|\vec{\mathbf{y}}^{\langle j \rangle}\|$, each $\vec{\mathbf{y}}^{\langle j \rangle} \in \mathbf{B}$. Hence, since ball $\mathbf{B}$ is convex, *each line-segment lies in* $\mathbf{B}$.

Apply the MVT to $h\rfloor_{S_j}$; that is, to h restricted to S_j. Our MVT guarantees a point, call it $\vec{\mathbf{x}}^j$, in S_j st.

$$|h_j(\vec{\mathbf{x}}^j)| \quad = \quad \frac{|h(\vec{\mathbf{y}}^{\langle j \rangle}) - h(\vec{\mathbf{y}}^{\langle j-1 \rangle})|}{\|\vec{\mathbf{y}}^{\langle j \rangle} - \vec{\mathbf{y}}^{\langle j-1 \rangle}\|} .$$

Note $\|\vec{\mathbf{y}}^{\langle j \rangle} - \vec{\mathbf{y}}^{\langle j-1 \rangle}\|$ is simply $|p_j|$. And $|h_j(\vec{\mathbf{x}}^j)| < \varepsilon$, courtesy (†), since $\vec{\mathbf{x}}^j \in S_j \subset \mathbf{B}$. Consequently,

$$|h(\vec{\mathbf{y}}^{\langle j \rangle}) - h(\vec{\mathbf{y}}^{\langle j-1 \rangle})| \leqslant \varepsilon \cdot |p_j| .$$

Using the Triangle Ineq., summing over $j = 1, \ldots, N$ yields that $|h(\vec{\mathbf{y}}^{\langle N \rangle}) - h(\vec{\mathbf{y}}^{\langle 0 \rangle})|$ is upper bounded by $\varepsilon \cdot \sum_{j=1}^N |p_j|$. By defn, $\vec{\mathbf{y}}^{\langle N \rangle} = \vec{\mathbf{p}}$ and $\vec{\mathbf{y}}^{\langle 0 \rangle} = \vec{\mathbf{0}}$, so

‡: $$|h(\vec{\mathbf{p}})| < \varepsilon \cdot \sum_{j=1}^N |p_j| ,$$

where we have used that $h(\vec{\mathbf{y}}^{\langle 0 \rangle})$ is zero.

Lastly, each $|p_j| = \sqrt{|p_j|^2} \leqslant \|\vec{\mathbf{p}}\|$. Summing over j gives $\sum_{j=1}^N |p_j| \leqslant N \cdot \|\vec{\mathbf{p}}\|$. This and (‡) together, yield (GOAL) with $K_N := N$. ♦

34a: *Remark.* The purists among you can use Jensen's Inequality [or Hölder's Inequality] to conclude the stronger $\sum_{j=1}^N |p_j| \leqslant \sqrt{N} \cdot \|\vec{\mathbf{p}}\|$. [For the above proof, however, this improvement is irrelevant.] □

Cauchy-Goursat for a rectangle

Here, a ***rectangle*** has form

$$\mathbf{R} \;=\; \{x + \boldsymbol{i}y \mid x \in [a\,..\,b] \text{ and } y \in [c\,..\,d]\}$$

where $a < b$ and $c < d$. Let $\partial\mathbf{R}$ denote the boundary of $\mathbf{R}$, both as a set and as a SCC, and let

$$\mathcal{I}_{\mathbf{R}} \;:=\; \int_{\partial\mathbf{R}} f(z)\,\mathrm{d}z\,.$$

Note that $\int_{\partial\mathbf{R}} 1\,\mathrm{d}z$ and $\int_{\partial\mathbf{R}} z\,\mathrm{d}z$ are each zero, since fncs $[z\mapsto 1]$ and $[z\mapsto z]$ each have an antiderivative. So for arbitrary constants J,K,L, we have that

35a: $$\int_{\partial\mathbf{R}} f(z)\,\mathrm{d}z = \int_{\partial\mathbf{R}} \Big[[f(z) - J] \,-\, [z-K]L\Big]\,\mathrm{d}z.$$

Splitting. Rectangle $\mathbf{R}$ splits into 4 congruent subrectangles, A,B,C,D each with half the width and height of $\mathbf{R}$. Note

$$\mathcal{I}_{\mathbf{R}} \;=\; \mathcal{I}_A + \mathcal{I}_B + \mathcal{I}_C + \mathcal{I}_D\,,$$

since each internal edge is traversed twice, once in each direction, cancelling. Hence

$$|\mathcal{I}_{\mathbf{R}}| \;\leqslant\; |\mathcal{I}_A| + |\mathcal{I}_B| + |\mathcal{I}_C| + |\mathcal{I}_D|\,.$$

So at least one of the subrectangles has its abs-value at least as large as $\frac{1}{4}|\mathcal{I}_{\mathbf{R}}|$. Pick one according to some definite rule (e.g, first one in CCW order) and call it $\mathbf{R}'$.

Pf of C-G for a rectangle. Consider a rectangle $\mathbf{R}_0$ and a fnc f holomorphic on $\widehat{\mathbf{R}_0}$. Use the preceding paragraph to define a sequence of rectangles

†: $$\mathbf{R}_0 \supset \mathbf{R}_1 \supset \mathbf{R}_2 \supset \ldots$$

by $\mathbf{R}_{n+1} := \mathbf{R}'_n$. Since $|\mathcal{I}_{\mathbf{R}_n}| \leqslant \frac{1}{4}|\mathcal{I}_{\mathbf{R}_{n+1}}|$, induction gives

‡: $$|\mathcal{I}_{\mathbf{R}_0}| \;\leqslant\; \tfrac{1}{4^n}\cdot|\mathcal{I}_{\mathbf{R}_n}|\,.$$

Letting D_n and P_n denote the diameter and perimeter of $\mathbf{R}_n$, note

∗: $$D_n \;=\; \tfrac{1}{2^n}\cdot D_0 \quad\text{and}\quad P_n \;=\; \tfrac{1}{2^n}\cdot P_0\,.$$

The intersection point. The rectangles are closed and bounded, and nested, so they converge to a point; call it $\mathbf{q}$. [Point $\mathbf{q}$ could be on $\partial\mathbf{R}_0$, which is fine.]

For future reference: Given an arbitrary rectangle $\mathbf{R}$, we can replace the constants J,K,L in (35a) by $f(\mathbf{q})$, $\mathbf{q}$ and $f'(\mathbf{q})$, respectively, to get

35b: $$\mathcal{I}_{\mathbf{R}} \;=\; \int_{\partial\mathbf{R}} \Big[f(z) - f(\mathbf{q}) \,-\, [z-\mathbf{q}]f'(\mathbf{q})\Big]\,\mathrm{d}z\,.$$

Using differentiability. Fix an $\varepsilon>0$. Since f is differentiable at $\mathbf{q}$, there exists $\delta>0$ so that every z with $0 < |z-\mathbf{q}| < \delta$ satisfies

$$\Big|\frac{f(z) - f(\mathbf{q})}{z - \mathbf{q}} \,-\, f'(\mathbf{q})\Big| \;\leqslant\; \varepsilon\,.$$

Multiply by $z-\mathbf{q}$, then take abs.values, to get

35c: $$\Big|f(z) - f(\mathbf{q}) \,-\, [z-\mathbf{q}]f'(\mathbf{q})\Big| \;\leqslant\; \varepsilon\cdot|z-\mathbf{q}|\,,$$

and this latter holds also for $z = \mathbf{q}$, hence holds for all z in $\mathrm{Bal}_\delta(\mathbf{q})$.

Picking index K. The rectangles of (†) all own $\mathbf{q}$, and their diameters shrink to zero, so we can choose an K large enough that $\mathbf{R}_K \subset \mathrm{Bal}_\delta(\mathbf{q})$.

Now (35b) and the Triangle-Ineq-for-Integrals gives that

$$|\mathcal{I}_{\mathbf{R}_K}| \;\leqslant\; \int_{\partial\mathbf{R}_K} \Big|f(z) - f(\mathbf{q}) \,-\, [z-\mathbf{q}]f'(\mathbf{q})\Big|\cdot|\,\mathrm{d}z|\,.$$

Courtesy (35c), then,

$$|\mathcal{I}_{\mathbf{R}_K}| \;\leqslant\; \varepsilon\cdot\int_{\partial\mathbf{R}_K} |z-\mathbf{q}|\cdot|\,\mathrm{d}z|\,.$$

Each $|z-\mathbf{q}| \leqslant D_K$, so

$$|\mathcal{I}_{\mathbf{R}_K}| \;\leqslant\; \varepsilon\,D_K\int_{\partial\mathbf{R}_K} |\,\mathrm{d}z| \;=\; \varepsilon\cdot D_K\cdot P_K\,.$$

Multiplying by 4^K, our (∗) and (‡) produce

‡‡: $$|\mathcal{I}_{\mathbf{R}_0}| \;\leqslant\; \varepsilon\cdot D_0\cdot P_0\,.$$

Happily, the RhS goes to zero as $\varepsilon\searrow 0$. ♦

Radius of Convergence

Series notations. Customs about how "series" is used in the context of "convergence of a series" are a bit strange. A **"*series* $\vec{\mathbf{e}}$"** is a *sequence* $\vec{\mathbf{e}} = (e_k)_{k=0}^{\infty}$, but[♡5] where the word "series" hints to the reader our interest in its *sum* $\sum(\vec{\mathbf{e}})$. This sum is the limit –when it exists– of the corresponding **"partial-sum sequence"** $\vec{\mathbf{s}}$, where

36: $$s_N := \sum_{k\in[0\,..\,N)} e_k$$

Use $\boxed{\vec{\mathbf{s}} = \mathrm{P}\Sigma(\vec{\mathbf{e}})}$ to indicate this partial-sum relation between sequences. Phrase **"series $\vec{\mathbf{e}}$ is convergent"** means that $\lim(\vec{\mathbf{s}})$ exists and is finite. So $\sum(\vec{\mathbf{e}}) := \lim(\vec{\mathbf{s}})$.

To clarify, the n^{th} partial sum means the sum of the first n terms, regardless of the initial index. For example, suppose $\vec{\mathbf{b}} = (b_\ell)_{\ell=5}^{\infty}$, and $\vec{\mathbf{e}} = \mathrm{P}\Sigma(\vec{\mathbf{b}})$. Then $e_3 = b_5+b_6+b_7$, and $e_0 = 0$.

Example: Let $\vec{\mathbf{b}} := (k^2)_{k=1}^{\infty}$ and $\vec{\mathbf{a}} := \mathrm{P}\Sigma(\vec{\mathbf{b}})$. Then $a_n = \frac{1}{6}\cdot[2n^3 + 3n^2 + n]$. □

37: Root-test lemma. *Given a series $\vec{\mathbf{e}} \subset \mathbb{C}$, define*

$$*: \quad \Lambda := \limsup_{n\to\infty} \sqrt[n]{|e_n|} \overset{\text{note}}{\in} [0, +\infty]\,.$$

If $\Lambda < 1$ then $\vec{\mathbf{e}}$ is an absolutely-convergent series.

If $\Lambda > 1$ then $\vec{\mathbf{e}}$ is "magnificently divergent" Not only $|e_n| \not\to 0$, but indeed $\limsup_{n\to\infty} |e_n| = +\infty$. ◇

Proof. Let $a_n := |e_n|$,

$\boxed{\text{CASE: When } \Lambda < 1.}$ ISTShow that $\vec{\mathbf{a}}$ is a convergent series. Pick ρ with $\Lambda < \rho < 1$. Take K large enough that $\sup_{n\geqslant K} \sqrt[n]{a_n} \leqslant \rho$. Hence $\sum_{n\geqslant K} a_n \leqslant \sum_{n\geqslant K} \rho^n < \infty$. And $\sum_{n\in[1\,..\,K]} a_n < \infty$.

$\boxed{\text{CASE: When } \Lambda > 1.}$ Pick ρ with $1 < \rho < \Lambda$. By $(*)$, the set $J := \{n \mid \sqrt[n]{a_n} > \rho\}$ is infinite. And each $n \in J$ has $a_n > \rho^n$. ♦

[♡5] The index will usually start at zero, but it doesn't have to. The sequence $\vec{\mathbf{e}}$ might be $(e_k)_{k=24}^{\infty}$, or $(e_k)_{k=-5}^{\infty}$.

A function $f{:}\mathbb{R}\to\mathbb{R}$ is ***eventually positive*** if $\boxed{\exists K \text{ s.t } \forall x \geqslant K{:}\ f(x) > 0}$. Thus a degree-$k$ poly,

$$f(x) := C_k x^k + \cdots + C_1 x + C_0\,,$$

is eventually positive IFF f has positive leading-coeff, $C_k > 0$.

Power-series notation. A sequence $\vec{\mathbf{c}} \subset \mathbb{C}$ and point $Q \in \mathbb{C}$ determine a ***power series***

38a: $$\mathrm{PS}_{\vec{\mathbf{c}},Q}(z) := \sum_{n=0}^{\infty} c_n \cdot [z - Q]^n\,. \qquad □$$

From the notation we sometimes drop the the center of expansion, just writing $\mathrm{PS}_{\vec{\mathbf{c}}}$. This is especially true when the center of expansion is $0{\in}\mathbb{C}$.

Use "PS" to abbreviate the phrase "power series". Use McS to abbrev ***Maclaurin Series***; a PS centered at $Q{=}0$. E.g $\mathrm{McS}_{\vec{\mathbf{c}}}(z) = \sum_{n=0}^{\infty} [c_n \cdot z^n]$.

Radius of Convergence. The set of $z \in \mathbb{C}$ for which RhS(38a) converges is called the **"*set-of-convergence*"**. We write it $\mathrm{SoC}(\vec{\mathbf{c}}, Q)$

It will turn out that the SoC comprises an open ball, possibly of radius 0 or ∞, together with some of the points on the boundary of this ball. This open ***ball of convergence*** is written $\mathrm{BoC}(\vec{\mathbf{c}}, Q)$. Its radius is the ***radius of convergence*** of RhS(38a), and is written $\mathrm{RoC}(\vec{\mathbf{c}})$.[♡6] So $\mathcal{R} := \mathrm{RoC}(\vec{\mathbf{c}})$ is always a value in $[0, +\infty]$, and $\mathrm{BoC}(\vec{\mathbf{c}}, Q) = \mathrm{Bal}_{\mathcal{R}}(Q)$. □

38b: RoC Lemma (Cauchy, 1821. Hadamard, 1888.) *Contemplate power series $\mathrm{PS}_{\vec{\mathbf{c}},Q}$, as in (38a). Let*

$$\Omega := \limsup_{n\to\infty} \sqrt[n]{|c_n|} \overset{\text{note}}{\in} [0, +\infty].$$

Then $\mathrm{RoC}(\vec{\mathbf{c}}) = 1/\Omega$ where, here, we interpret $\frac{1}{0}$ as $+\infty$ and $\frac{1}{+\infty}$ as 0. ◇

Proof sketch. Set $a_n := |c_n|$. ISTConsider convergence at a non-negative $x{\in}\mathbb{R}$. Applying the Root-test,

$$\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = \limsup_{n\to\infty} [x \cdot \sqrt[n]{a_n}\,]$$
$$= x \cdot \limsup_{n\to\infty} \sqrt[n]{a_n} = x \cdot \Omega =: \Lambda\,.$$

So Λ is less/greater than 1, as x is less/greater than $\frac{1}{\Omega}$. ♦

[♡6] The argument to RoC is a *sequence*. So we can write the RoC of PS $f(x) := \sum_{n=0}^{\infty} n^2 x^n$ as $\mathrm{RoC}(n \mapsto n^2)$, but **not** as $\mathrm{RoC}(n^2)$. nor as $\mathrm{RoC}(f)$.

39: *Three examples.* [ASIDE: For fncs on a set $\boldsymbol{\Omega}$, each subset $B \subset \boldsymbol{\Omega}$ has its corresponding **"*indicator function* of B"**, written $\mathbf{1}_B$. It is the fnc $\boldsymbol{\Omega} \to \{0,1\}$ which sends points in B to 1 and points in $\boldsymbol{\Omega} \setminus B$ to 0. [So $\mathbf{1}_A + \mathbf{1}_{\complement(A)}$ is constant-1.] E.g, $\mathbf{1}_{\text{Primes}}(5) = 1$, and $\mathbf{1}_{\text{Primes}}(9) = 0$.]

Let's apply the above (38b). Define

$$\mathbb{P} := \text{Primes};\ D := \text{Odds};\ S := \{1+n^2 \mid n \in \mathbb{N}\}.$$

Consider this power series:

39a: $$\sum_{n=0}^{\infty} 3^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot x^n = 9x^2 + 27x^3 + 243x^5 + \ldots.$$

Its RoC is 1/3, since there are ∞ly many primes.

A funkier PS, centered at 8, is

39b: $$\sum_{k=0}^{\infty} \Big[3^k \cdot \mathbf{1}_D(k) + 4^k \cdot \mathbf{1}_S(k)\Big] \cdot [x-8]^k.$$

Since $\sqrt[n]{3^n + 4^n} \xrightarrow{n} 4$, and $|S| = \infty$, the RoC is $\frac{1}{4}$.

Even more interesting is this PS:

39c: $$\sum_{n=0}^{\infty} \left[5^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot \mathbf{1}_S(n)\right] \cdot x^n.$$

As of March2017, its RoC is unknown. If there are ∞ly many primes[♡7] of form $1+n^2$ (conjectured, but unproven) then RoC $= \frac{1}{5}$; otherwise RoC $= \infty$, and the PS is a polynomial. □

40: Lemma. *For each $K \in \mathbb{R}$: $\lim_{x \nearrow \infty} \sqrt[x]{x^K} = 1$.*

Moreoever, for each rational function $h() := \frac{p()}{q()}$ which is eventually positive, $\lim_{n \nearrow \infty} \sqrt[n]{h(n)} = 1$.

Proof. Use L'Hôpital's rule. Etc. ♢

41: Same-RoC lemma. *Consider a sequence $\vec{\mathbf{c}} = (c_0, c_1, \ldots) \subset \mathbb{C}$, and let $\mathcal{R} := \text{RoC}(\vec{\mathbf{c}})$. For each natnum K, and for each rational function $g \neq$ Zip, these coefficient sequences*

i: $(0, \overset{K}{\ldots}, 0, c_K, c_{K+1}, c_{K+2}, \ldots)$

ii: $(c_K, c_{K+1}, c_{K+2}, \ldots)$

iii: $(g(n) \cdot c_n)_{n=0}^{\infty}$

give rise to power-series with RoC $= \mathcal{R}$. ♢

[♡7] For the curious, see Wikipedia on Landau's problems.

Proof sketch. Parts (i) and (ii) follow from (38b). Part (iii) follows from (40) and (38b). ♦

42: *Diff/Integrate a PS.* We differentiate and integrate, term-by-term, the $G := \text{PS}_{\vec{\mathbf{c}},0}$ power-series:

42a:
$$F(x) = \sum_{j=1}^{\infty} b_j \cdot x^j, \text{ where } b_j := \tfrac{1}{j} \cdot c_{j-1}.$$
$$G(x) = \sum_{k=0}^{\infty} c_k \cdot x^k.$$
$$H(x) = \sum_{\ell=0}^{\infty} d_\ell \cdot x^\ell, \text{ where } d_\ell := [\ell+1] \cdot c_{\ell+1}.$$

Lemma (41) tells us that the three PSes have the same RoC.

Observe that $\text{PS}_{\vec{\mathbf{d}}}$ is the term-by-term derivative of $\text{PS}_{\vec{\mathbf{c}}}$. And $\text{PS}_{\vec{\mathbf{b}}}$ is the term-by-term integral of $\text{PS}_{\vec{\mathbf{c}}}$. Does the same relation hold between the *functions* that these PSes determine? □

42b: Term-by-term PS Theorem. *Given a sequence $\vec{\mathbf{c}} \subset \mathbb{R}$, define sequences/fncs $\vec{\mathbf{b}}, \vec{\mathbf{d}}, F, G, H$ by (42a) and let $\mathcal{R} := \text{RoC}(\vec{\mathbf{c}})$. Then*

†: $$\text{RoC}(\vec{\mathbf{b}}) = \mathcal{R} = \text{RoC}(\vec{\mathbf{d}}).$$

With $B := \text{BoC}(\vec{\mathbf{c}})$, moreover,

‡: $$\forall z \in B:\quad F(z) = \int_0^z G.$$

And G is in $\mathbf{C}^{\infty}(B \to \mathbb{R})$, with $G' = H$. ♢

42c: Coro. *Suppose PS $G(x) := \sum_{j=0}^{\infty} c_j \cdot [x-Q]^j$ has positive RoC. Then this PS is the Taylor series of G, centered at Q.* ♢

Pf of (42b). We'll establish that $G' = H$; the integral result (‡) follows analogously. ISTo fix a posreal $\rho < \mathcal{R}$, let $U := \text{Bal}_\rho(0)$, and prove $G' = H$ when *restricted to U*. We will apply the DUC Thm (Derivative uniform-convergence) from *notes-AdvCalc.pdf* to these fncs (defined *only* on U)

$$f_n(x) := \sum_{j \in [0..n]} c_j x^j.$$

By definition of coeff-sequence $\vec{\mathbf{d}}$ from (42a),

$$f_n'(x) = \sum_{k \in [0..n)} d_k x^k.$$

In order to show that seq $(f'_n)_{n=1}^{\infty}$ is sup-norm Cauchy, pick a number V with $\rho < V < \mathcal{R}$.

Now $\frac{1}{V} > \limsup_{n\to\infty} \sqrt[n]{|d_n|}$ since, by (41), $\mathsf{RoC}(\vec{\mathbf{d}})$ equals $\mathcal{R}$. Thus there is an index K with

$$\forall n \geqslant K: \quad \sqrt[n]{|d_n|} < \tfrac{1}{V}.$$

We henceforth only consider indices n dominating K. For each $k \geqslant n$, then,

42d: $$|d_k| \quad \leqslant \quad 1/V^k.$$

Sup-norm. For $x \in U$ and indices $\ell > n$,

$$f'_\ell(x) - f'_n(x) \;=\; \sum\nolimits_{k\in[n\,..\,\ell)} d_k x^k.$$

From (42d), then,

$$|f'_\ell(x) - f'_n(x)| \;\leqslant\; \sum_{k=n}^{\infty} \frac{|x|^k}{V^k}.$$

Since U owns x,

$$|f'_\ell(x) - f'_n(x)| \;\leqslant\; \sum_{k=n}^{\infty} \frac{\rho^k}{V^k} \;=\; \Big[\frac{\rho}{V}\Big]^n \cdot C,$$

where C is the positive constant $1/[1-\frac{\rho}{V}]$.

Taking a supremum over all $x \in U$ yields

42e: $$\|f'_\ell - f'_n\| \;\leqslant\; \Big[\frac{\rho}{V}\Big]^n \cdot C,$$

for each pair $\ell > n \geqslant K$. Sending $n \nearrow \infty$ sends $\mathrm{RhS}(42e) \to 0$.

The limit $\lim_n f_n(0)$ exists, equaling c_0. Now apply the DUC Thm. ♦

A power-series with a new center. We show that a function defined by a PS is analytic in its entire ball-of-convergence.

43: *The setting.* We have a point $P \in \mathbb{C}$ and a sequence $\vec{\mathbf{a}} \subset \mathbb{C}$ such that $\alpha \in (0, +\infty]$, where $\alpha := \mathsf{RoC}(\vec{\mathbf{a}})$. This engenders a $\mathbf{C}^\infty$-fnc from $\mathrm{Bal}_\alpha(P) \to \mathbb{C}$, by

43a: $$\mathcal{F}(z) \;:=\; \sum\nolimits_{k=0}^{\infty} a_k \cdot [z-P]^k.$$

Fix a new center $Q \in \mathbb{C}$ with $|Q-P| < \alpha$. Thus

43b: $\beta \in (0, +\infty]$, *where* $\beta := \alpha - |Q-P|$. *Moreoever,* $\mathrm{Bal}_\beta(Q) \subset \mathrm{Bal}_\alpha(P)$. □

44: New-center theorem. *Take $P, Q, \alpha, \beta, \vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from (43). For each natnum k, this summation is absolutely convergent:*

44a: $$b_k \;:=\; \sum\nolimits_{N=k}^{\infty} a_N \cdot \tbinom{N}{k} \cdot Q^{N-k} \;\in\; \mathbb{C}.$$

Moreoever, $\mathsf{RoC}(\vec{\mathbf{b}}) \geqslant \beta > 0$. *This value*

44b: $$\mathcal{G}(z) \;:=\; \sum\nolimits_{k=0}^{\infty} b_k \cdot [z-Q]^k,$$

44c: *agrees with $\mathcal{F}(z)$, for each $z \in \mathrm{Bal}_\beta(Q)$.*

Lastly, for each natnum k,

44d: $$b_k \;=\; \tfrac{1}{k!} \cdot \mathcal{F}^{(k)}(Q).$$

In other words, $\mathrm{RhS}(44b)$ is the Taylor series for $\mathcal{F}$, centered at Q. ♢

Proof. WLOG $P = 0$. Fix a point $Z \in \mathrm{Bal}_\beta(Q)$. Writing $Z = Q + [Z-Q]$, its N^{th}-power is

$$Z^N \;=\; \sum\nolimits_{k=0}^{N} \tbinom{N}{k} \cdot Q^{N-k} \cdot [Z-Q]^k.$$

Thus, since $Z \in \mathrm{Bal}_\alpha(P)$,

$$f(Z) \;=\; \sum\nolimits_{N=0}^{\infty} a_N \cdot Z^N$$
$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \underbrace{a_N \cdot \tbinom{N}{k} \cdot Q^{N-k} \cdot [Z-Q]^k}_{h_{N,k}}.$$

This is a sum, in a certain order, over the set $H := \{(N,k) \in \mathbb{N}\times\mathbb{N} \mid N \geqslant k\}$. We need this sum to be absolutely convergent. The sum $\sum_{N=0}^{\infty}\sum_{k=0}^{N} |h_{N,k}|$ equals

$*$: $$\sum_{N=0}^{\infty} \sum_{k=0}^{N} |a_N| \cdot \tbinom{N}{k} \cdot |Q|^{N-k} \cdot |Z-Q|^k \;=\; \sum_{N=0}^{\infty} |a_N| \cdot Y^N,$$

where $Y := |Q| + |Z-Q|$. From $Z \in \mathrm{Bal}_\alpha(0)$ and (43b), we conclude that $Y < \alpha$. From the proof of Root-test lemma (37, P.24), the righthand side of $(*)$ is finite.

Since $\mathbf{S} := \sum_{N=0}^{\infty}\sum_{k=0}^{N} |h_{N,k}|$ is finite, we can reverse the order of summation and conclude that

$$\mathbf{S} \;=\; \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} |h_{N,k}|$$
$$= \sum_{k=0}^{\infty} \Big[\sum_{N=k}^{\infty} |a_N| \cdot \tbinom{N}{k} \cdot |Q|^{N-k} \Big] \cdot |Z-Q|^k.$$

We *could* have chosen our $Z \neq Q$, thus allowing division by $|Z-Q|^k$. Hence, each bracketed sum is finite. So each sum in (44a) is *absolutely convergent*, and we have a well-defined number b_k.

For a general $Z \in \mathrm{Bal}_\alpha(0)$, reversing the original sum gives

$$f(Z) = \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} h_{N,k}$$
$$= \sum_{k=0}^{\infty} \Big[\sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \Big] \cdot [Z-Q]^k ,$$

which equals $\sum_{k=0}^{\infty} b_k \cdot [Z-Q]^k$.

Establishing (44d). Corollary 42c tells us that

$$k! \cdot b_k \stackrel{\text{by (42c)}}{=\!=} \mathcal{G}^{(k)}(Q) \stackrel{\text{by (44c)}}{=\!=} \mathcal{F}^{(k)}(Q) . \qquad \blacklozenge$$

45: Prop'n. *Power-series*

$$*: \qquad \mathcal{F}(z) := \sum_{n=0}^{\infty} a_n \cdot [z-Q]^n$$

has positive RoC. *Suppose* $\vec{\mathbf{y}}$ *is a sequence of distinct complex numbers converging to* Q, *such that*

$$\forall j \in \mathbb{Z}_+: \quad \mathcal{F}(y_j) = 0 .$$

Then $\vec{\mathbf{a}}$ *is all-zero, and* $\mathcal{F}$ *is the zero function.* ♢

Proof. WLOG, each $y_j \neq Q$. FTSOC, suppose $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$; let L be the smallest index with $a_L \neq 0$. Formally dividing $(*)$ by $[z-Q]^L$ gives PS

$$\mathcal{G}(z) := \sum_{k=0}^{\infty} b_k \cdot [z-Q]^k ,$$

where each $b_k := a_{L+k}$. Since each $y_j - Q \neq 0$,

$$\mathcal{G}(y_j) = \mathcal{F}(y_j) / [y_j - Q]^L = 0 .$$

But $\mathrm{RoC}(\vec{\mathbf{b}}) = \mathrm{RoC}(\vec{\mathbf{a}}) > 0$, so $\mathcal{G}$ is cts in a nbhd of Q, and thus $\mathcal{G}(Q) = \lim(\mathcal{G}(\vec{\mathbf{y}})) = 0$. This contradicts that $\mathcal{G}(Q) = b_0 = a_L \neq 0$. ♦

46: PS Uniqueness Thm. *Imagine power-series*

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} a_n \cdot [z-P]^n \quad \text{and}$$
$$\mathcal{G}(z) := \sum_{n=0}^{\infty} b_n \cdot [z-P]^n$$

where $B := \mathrm{BoC}(\vec{\mathbf{a}}) \cap \mathrm{BoC}(\vec{\mathbf{b}})$ *is non-void. Suppose there is a set* $Y \subset B$ *st.* $\mathcal{F}\lfloor_Y = \mathcal{G}\lfloor_Y$, *and* Y *has a cluster point,* Q_0, *in* B. *Then* $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, *so* $\mathcal{F} = \mathcal{G}$. ♢

Remark. It does <u>not</u> suffice for Y to have a cluster-point on the *boundary* of B: Distinct functions $\mathcal{F}(z) := \sin(\frac{1}{z-7})$ and $\mathcal{G} := -\mathcal{F}$ have Taylor series with RoC $= 7$. Yet

$$\mathcal{F}(y_k) = 0 = \mathcal{G}(y_k) , \quad \text{for each posint } k,$$

where $y_k := 7 + \frac{1}{2\pi k}$. □

Proof of (46). Subracting PSes gives us a PS

$$f(z) := \sum_{n=0}^{\infty} c_n \cdot [z-P]^n$$

so that $f\lfloor_Y \equiv 0$, making $\boxed{\vec{\mathbf{c}} \stackrel{?}{=} \vec{\mathbf{0}}}$ our goal.

For each $q \in B := \mathrm{BoC}(\vec{\mathbf{c}})$, let $U(q)$ denote the largest open ball (centered at q) which fits inside B. By the New-center thm, the Taylor-series for f, centered at q, converges to f on all of $U(q)$.

Pick a Y-cluster-point $Q_0 \in B$. By (45), f is identically zero on $U(Q_0)$.

On the line-segment running between Q_0 and P, we can pick a (finite) list of points

$$Q_0, Q_1, \ldots, Q_{K-1}, Q_K := P ,$$

such that each $Q_k \in U(Q_{k-1})$. Arguing inductively, since f is identically zero on $U(Q_{k-1})$, the the Taylor-series at Q_k has all-zero coeffs. This therefore holds at P. So $\vec{\mathbf{c}} = (0, 0, 0, \ldots)$. ♦

47: Coro. *Suppose* $\mathcal{F}$ *and* $\mathcal{G}$ *are analytic functions on some* <u>*connected*</u> *open set* $V \subset \mathbb{C}$. *If*

$$\{z \in V \mid \mathcal{F}(z) = \mathcal{G}(z)\}$$

has a cluster point <u>*in*</u> V, *then* $\mathcal{F} = \mathcal{G}$. ♢

§Index, with symbols at the beginning