Completable subspaces

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ABSTRACT: What subsets of a metric space support a complete metric? [From my topology course.]

Intro

For two metrics $\mathsf{d}_0, \mathsf{d}_1$ on the same set $\mathbf{X}$, we use "$\mathsf{d}_0 \prec \mathsf{d}_1$" to mean that every d_0-open set is d_1-open. Let "$\mathsf{d}_0 \asymp \mathsf{d}_1$" mean that two metrics induce the same topology; we have previously proven that this means the two metrics have the same convergent sequences.

On a metric space $(\mathbf{\Omega}, \mu)$, say that a subset $\mathbf{X} \subset \mathbf{\Omega}$ is ***completable*** if there exists a *complete* metric d on $\mathbf{X}$, with $\mathsf{d} \asymp \mu\rfloor_{\mathbf{X}}$. It turns out that the completable subsets can be characterized.

1: Completability Theorem. *$(\mathbf{\Omega}, \mu)$ is a metric space and $\mathbf{X} \subset \mathbf{\Omega}$. Then*

a: $\mathbf{X}$ completable $\implies$ $\mathbf{X}$ is a $\mathcal{G}_\delta$-subset of $\mathbf{\Omega}$.

b: If μ is complete then: $\mathbf{X}$ a $\mathcal{G}_\delta$-subset of $\mathbf{\Omega}$ $\implies$ $\mathbf{X}$ is completable. ◊

Proof of (a). In $\mathbf{\Omega}$, let $\Omega_\varepsilon(\omega)$ denote the μ-ball of radius ε, centered at point ω. In $\mathbf{X}$, let $B_r(x)$ mean the d-ball about x, with radius-r. We make an *assumption* –to be removed later– that $\mathbf{X}$ is dense in $\mathbf{\Omega}$. We will construct a collection of closed sets $\{K_{1/j}\}_{j=1}^\infty$ whose union is $\mathbf{\Omega} \setminus \mathbf{X}$.

Defining K. Fix a positive number r. Let K be the set of $\omega \in \mathbf{\Omega}$ such that: For all positive ε,

2: $$\mathsf{d}\text{-Diam}(\mathbf{X} \cap \Omega_\varepsilon(\omega)) \;>\; 2r\,.$$

Since $\mu\rfloor_{\mathbf{X}} \asymp \mathsf{d}$, for each $x \in \mathbf{X}$ there is a sufficiently small ε such that

$$\mathbf{X} \cap \Omega_\varepsilon(x) \;\subset\; B_r(x)\,.$$

Thus $\mathsf{d}\text{-Diam}(\mathbf{X} \cap \Omega_\varepsilon(x)) \le r\cdot 2$ and so $x \notin K$. Conclusion: *K is disjoint from* $\mathbf{X}$.

K is $\mathbf{\Omega}$-closed. For each point ζ in the $\mathbf{\Omega}$-closure of K and for every ε, there is a point ω in $K \cap \Omega_\varepsilon(\zeta)$ for which (2). But $\Omega_\varepsilon(\omega) \subset \Omega_{2\varepsilon}(\zeta)$. Thus

$$\mathsf{d}\text{-Diam}(\mathbf{X} \cap \Omega_{2\varepsilon}(\zeta)) \;>\; r\,.$$

This holds for all ε and so ζ is in K.

Filling X complement. Indicate the dependence of K on r by calling it K_r. We have shown that

$$V := \bigcup_{j=1}^\infty K_{1/j}$$

is an $\mathcal{F}_\sigma$-subset of $\mathbf{\Omega}$ which is disjoint from $\mathbf{X}$.

We finish the proof by showing that *each* ω in $\mathbf{\Omega} \setminus \mathbf{X}$, is also in V. Since $\mathbf{X}$ is $\mathbf{\Omega}$-dense, there are points $x_n \to \omega$. But d is complete and thus $(x_n)_{n=1}^\infty$ is <u>not</u> d-Cauchy. That is,

$$r \;:=\; \inf_{N\ge1} \mathsf{d}\text{-Diam}(\{x_n\}_{n=N}^\infty)$$

is positive. Taking an integer j with $1/j < r$, we see that $\omega \in K_{1/j} \subset V$.

Removing the assumption. No longer assume that $\mathbf{X}$ is dense in $\mathbf{\Omega}$. Nonetheless, its $\mathbf{\Omega}$-closure, $\overline{\mathbf{X}}$, is a metric space and so the above shows that $\mathbf{X}$ is a $\mathcal{G}_\delta$-subset of $\overline{\mathbf{X}}$. This implies (exercise!) that $\mathbf{X} = \overline{\mathbf{X}} \cap G$, for some G, a $\mathcal{G}_\delta$-subset of $\mathbf{\Omega}$.

But $\overline{\mathbf{X}}$ is closed in $\mathbf{\Omega}$ and so it is also a $\mathcal{G}_\delta$-subset of $\mathbf{\Omega}$ (exercise: use that $\mathbf{\Omega}$ is a metric space). Consequently $\mathbf{X} \in \mathcal{G}_\delta(\mathbf{\Omega})$. ♦

Proof of (b). WLOG $\mathbf{X}$ is a proper subset of $\mathbf{\Omega}$. Thus we can write $\mathbf{X} = \bigcap_{k=1}^\infty U_k$, where each U_k is an open proper subset of $\mathbf{\Omega}$.

Building a pseudo-metric on an open set. Suppose U is an open proper subset of $\mathbf{\Omega}$. We want to "stretch" the metric μ near the boundary of U. Define a function $f{:}U \to [0,\infty)$ and pseudo-metric m by

$$f(x) \;:=\; \frac{1}{\mu(x\,,\,\mathbf{\Omega} \setminus U)} \qquad \text{and}$$

$$\mathsf{m}(x,z) \;:=\; |f(x), f(z)|\,.$$ (Here, $|\cdot,\cdot|$ is the usual metric on $\mathbb{R}$.)

This f is cts, so $\mathsf{m} \prec \mu\rfloor_U$. Moreover, if a sequence $\mathbf{x} := (x_n)_{n=1}^\infty \subset U$ is m-Cauchy then its image $f(\mathbf{x})$ is an m-bounded sequence. Consequently:

3: *If a sequence $\mathbf{x} \subset U$ is μ-Cauchy <u>and</u> is m-Cauchy, then its limit $\omega := \mu\text{-}\lim(\mathbf{x})$ must be in U.*

This, since $\mu(\omega, U^c)$ is necessarily positive.

Building a metric on a $\mathcal{G}_\delta$ set. From the above, for each k there is a pseudo-metric m_k on U_k satisfying (3). Moreover, a homework problem tells us that the ratio $\mathsf{m}_k/[1+\mathsf{m}_k]$ is a pseudo-metric which is Cauchy equivalent to m_k. Thus WLOG m_k is bounded. So WLOG the m_k-diameter of U_k is less than $1/2^k$.

Define an extended metric d on $\mathbf{X}$ by

$$4\mathbf{:} \qquad \mathsf{d} \quad := \quad \mu\rfloor_{\mathbf{X}} + \sum_{k=1}^{\infty} \mathsf{m}_k\rfloor_{\mathbf{X}} \,.$$

(Since μ distinguishes points, so does m.) This sum is everywhere finite, by the summable bound on diameters, and thus d is a metric. Moreover, each $\mathsf{m}_k \prec \mu\rfloor_{\mathbf{X}}$ and so, by the same homework problem, $\mathsf{d} \asymp \mu\rfloor_{\mathbf{X}}$.

To see that $(\mathbf{X}, \mathsf{d})$ is complete, suppose that sequence $\mathbf{x} := (x_n)_{n=1}^{\infty}$ is d-Cauchy. From (4), then, $\mathbf{x}$ is μ-Cauchy and so $\omega := \mu\text{-lim}(\mathbf{x})$ exists in $\mathbf{\Omega}$. But $\mathbf{x}$ is also m_k-Cauchy and so, by (3), this ω is in U_k. This holds for each k and so $\omega \in \mathbf{X}$. And since $\mathsf{d} \asymp \mu\rfloor_{\mathbf{X}}$ we know that $x_n \to \omega$ holds also in the d-metric. ♦

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