A $\mathbf{C}^\infty$ function which is not everywhere analytic

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The goal of this note is to produce a $\mathbf{C}^\infty$-function $\mathbf{G}$:$\mathbb{R}\circlearrowleft$ whose Taylor series (centered at zero) converges to a *different* fnc —namely, to the zero-function.

On $U := \mathbb{R}\setminus\{0\}$, the following fnc $\mathbf{G}()$ is strictly positive; thus it differs from the zero-fnc on all of U.

1a: $\mathbf{F}(x) := \mathsf{e}^{-1/x^2}$; $\quad \mathbf{G}(x) := \begin{Bmatrix} 0\,, & \text{if } x = 0 \\ \mathbf{F}(x)\,, & \text{if } x \neq 0 \end{Bmatrix}$.

Generalizing, a degree-D poly $P(z) := \sum_{n=0}^{\mathrm{D}} \mathrm{C}_n z^n$, with each C_n real, defines a function $\mathbf{G}_P$ by

1b: $$\mathbf{G}_P(x) := \begin{Bmatrix} 0\,, & \text{if } x = 0 \\ \mathbf{F}(x)\cdot P(\frac{1}{x})\,, & \text{if } x \neq 0 \end{Bmatrix}.$$

[So the $\mathbf{G}$ from (??) is $\mathbf{G}_1$.] It may not be evident that $\mathbf{G}_P$ is differentiable at 0. But certainly at a *non-zero* x, we can use the Product Rule to compute as follows:

$$\begin{aligned}
[\mathbf{G}_P]'(x) &= \mathbf{F}'(x)P(\tfrac{1}{x}) + \mathbf{F}(x)\cdot P'(\tfrac{1}{x})\cdot \tfrac{-1}{x^2} \\
&= [\mathbf{F}(x)\tfrac{2}{x^3}]P(\tfrac{1}{x}) - \mathbf{F}(x)\cdot \tfrac{1}{x^2}P'(\tfrac{1}{x}) \\
&= \mathbf{F}(x)\cdot \Big[2[\tfrac{1}{x}]^3 P(\tfrac{1}{x}) - [\tfrac{1}{x}]^2 P'(\tfrac{1}{x})\Big].
\end{aligned}$$

This suggests defining an operation on polynomials. Given a poly P, define a new poly, $\widetilde{P}$, by

1c: $$\widetilde{P}(z) := 2z^3 P(z) - z^2 P'(z).$$

The computation above showed, for each $x \neq 0$, that $[\mathbf{G}_P]'(x)$ equals $\mathbf{G}_{\widetilde{P}}(x)$. Now let's finish the job.

2a: Theorem. *For each polynomial P, the function $\mathbf{G}_P$ from (??) is everywhere differentiable. Moreover, $[\mathbf{G}_P]' = \mathbf{G}_{\widetilde{P}}$.* ♢

Proof. What is left to show is that $[\mathbf{G}_P]'(0)$ equals 0. Happily, the definition of derivative tells us that

$$\begin{aligned}
[\mathbf{G}_P]'(0) &\overset{\text{def}}{=\!=} \lim_{x\to 0} \frac{\mathbf{G}_P(x) - \mathbf{G}_P(0)}{x - 0} \\
&= \lim_{x\to 0} \frac{\mathbf{F}(x)P(\frac{1}{x})}{x} \\
&= \lim_{x\to 0} \mathbf{F}(x)\cdot \tfrac{1}{x}P(\tfrac{1}{x}).
\end{aligned}$$

This latter equals zero, by Proposition ??, which is proved below. ♦

2b: Corollary. *Given an arbitrary polynomial P, define a sequence of polys by $P_0 := P$ and $P_{n+1} := \widetilde{P_n}$. Then $\mathbf{G}_P$ is ∞ly differentiable, and its n^{th} derivative satisfies*

$$[\mathbf{G}_P]^{(n)} = \mathbf{G}_{P_n}\,,$$

for each $n = 0, 1, 2, \ldots$. In particular,

$$\mathbf{G}^{(n)} = \mathbf{G}_{R_n}\,,$$

where $R_0() := 1$ and $R_{n+1} := \widetilde{R_n}$. ♢

3a: Lemma. *For each integer $N \geqslant 0$, the limit*

$$\ell_N := \lim_{u\nearrow\infty} \frac{u^N}{\mathsf{e}^{[u^2]}}$$

exists, and equals zero. ♢

Proof. Certainly ℓ_0 is zero. I now induct on N. By L'Hôpital's rule,

$$\begin{aligned}
\ell_N &= \lim_{u\nearrow\infty} \frac{Nu^{N-1}}{2u\cdot \mathsf{e}^{[u^2]}} \\
&= \tfrac{N}{2}\Big[\lim_{u\nearrow\infty} \tfrac{1}{u}\Big]\Big[\lim_{u\nearrow\infty} \tfrac{u^{N-1}}{\mathsf{e}^{[u^2]}}\Big], \quad \text{\textit{since the limit of a product is the product of the limits, if both limits exist,}} \\
&= \tfrac{N}{2}\cdot 0\cdot \ell_{N-1} = \tfrac{N}{2}\cdot 0\cdot 0 = 0, \quad \text{by induction.}
\end{aligned}$$

Rem. This lemma implies, by letting $u := 1/x$, that

$$\lim_{x\searrow 0} \exp(-\tfrac{1}{x^2})\cdot \tfrac{1}{x^N} = \lim_{u\nearrow\infty} \mathsf{e}^{-[u^2]}\cdot u^N = 0.$$

Indeed, we conclude that this holds for the two-sided limit,

3b: $$\lim_{x\to 0} \mathbf{F}(x)\cdot \tfrac{1}{x^N} = 0,$$

since $\big|\mathbf{F}(x)\frac{1}{x^N}\big|$ equals $\mathbf{F}(x)\cdot \frac{1}{|x|^N}$. □

3c: Polynomial proposition. *For an arbitrary polynomial Q, necessarily*

$$*: \qquad \lim_{x\to 0} \mathbf{F}(x)\cdot Q(\tfrac{1}{x}) \;=\; 0\,.$$

Proof. Write $Q(z) = \mathrm{C}_0 + \mathrm{C}_1 z + \cdots + \mathrm{C}_\mathrm{D} z^\mathrm{D}$. Then

$$\begin{aligned}\lim_{x\to 0} \mathbf{F}(x) Q(\tfrac{1}{x}) &= \sum_{n=0}^{\mathrm{D}} \mathrm{C}_n \cdot \Big[\lim_{x\to 0} \mathbf{F}(x)\tfrac{1}{x^n}\Big] \\ &= \sum_{n=0}^{\mathrm{D}} \mathrm{C}_n \cdot 0\,, \quad \text{by } (\mathbf{??}).\end{aligned}$$

And this last sum equals zero, as desired. ♦

Analyticity is not sealed under uniform limits

We now obtain the preceding fnc $\mathbf{G}$ as a uniform-limit of entire[♡1] functions $F_1, F_2, F_3, \ldots$, where

$$F_N(x) \quad := \quad \exp\Big(\frac{-N}{1+Nx^2}\Big)\,.$$

Rewriting $F_N(x) = \exp(\frac{-1}{[1/N]+x^2})$ shows that, pointwise, $F_1 \geqslant F_2 \geqslant F_3 \geqslant \ldots \geqslant 0$. Of course, the completeness of the reals implies that the pointwise limit $[\lim_{n\to\infty} F_n]$ exists; evidently, this limit is the $\mathbf{G}$ from (**??**). But this even stronger result holds:

4: Theorem. $F_n \xrightarrow[n\to\infty]{\text{uniformly}} \mathbf{G}$. ♢

Proof. Use $\|\cdot\|$ for the sup-norm. Letting $K{=}K(N)$ denote the cube-root of posint N, my goal is

$$5: \quad \|F_N - \mathbf{G}\| \quad\leqslant\quad \mathrm{Max}\Big\{\big[\mathrm{e}^{1/K}\big] - 1,\; \frac{1}{\mathrm{e}^{K/2}}\Big\}\,.$$

Both terms in Max{} go to zero as $K\nearrow\infty$, so this will establish (**??**).

Note $F_N(0) - \mathbf{G}(0) = \mathrm{e}^{-N} \leqslant \mathrm{e}^{-K/2}$, which is the righthand term of Max{}. So ISTShow for each *nonzero* x that $F_N(x) - \mathbf{G}(x) \leqslant \mathrm{RhS}(\mathbf{??})$.

The substitution $z := x^2$ reduces our task to establishing that

??′:
$$\Big[\sup_{z\in\mathbb{R}_+} \big[\mathrm{e}^{\frac{-N}{1+Nz}} - \mathrm{e}^{\frac{-1}{z}}\big]\Big] \overset{?}{\leqslant} \mathrm{Max}\Big\{\big[\mathrm{e}^{1/K}\big] - 1,\, \mathrm{e}^{-K/2}\Big\}\,.$$

To this end, let $H(z) := \big[\mathrm{e}^{\frac{-N}{1+Nz}} - \mathrm{e}^{\frac{-1}{z}}\big]$. Fix a positive z and perceive[♡2] that

$$6: \qquad H(z) \;\leqslant\; \mathrm{e}^{\frac{-N}{1+Nz}} \;=:\; I(z)\;.$$

We get an alternate inequality, (**??**), by factoring,

$$H(z) \quad=\quad \mathrm{e}^{\frac{-1}{z}} \cdot \Big[\mathrm{e}^{\frac{1}{z[1+Nz]}} - 1\Big]\,.$$

But $\mathrm{e}^{\frac{-1}{z}} \leqslant 1$, so $H(z) \leqslant \mathrm{e}^{\frac{1}{z[1+Nz]}} - 1$. Reducing the denominator $z[1+Nz]$ to Nz^2 gives

$$7: \qquad H(z) \;\leqslant\; \big[\mathrm{e}^{1/Nz^2}\big] - 1 \;=:\; D(z)\,.$$

Maximizing over two intervals. This $D(z)$ is a decreasing fnc of $z \in \mathbb{R}_+$. For each z in interval $[\frac{1}{K}, \infty)$, then, $D(z) \leqslant D(\frac{1}{K})$. Since $N \cdot [\frac{1}{K}]^2 = K$,

$$\mathbf{??}': \qquad H(z) \quad\leqslant\quad \big[\mathrm{e}^{1/K}\big] - 1\,.$$

The $I(z)$ from (**??**) is an increasing fnc of $z \in \mathbb{R}_+$. For each $z \in (0, \frac{1}{K}]$, then, $I(z) \leqslant I(\frac{1}{K})$. Thus

$$\begin{aligned} H(z) &\leqslant \mathrm{e}^{\frac{-K^3}{1+K^2}} \\ &\leqslant \mathrm{e}^{\frac{-K^3}{K^2+K^2}}\,, \quad \text{since } K \geqslant 1 \text{ because } N \geqslant 1, \\ &\leqslant \mathrm{e}^{-K/2}\,. \end{aligned}$$

This, together with (**??**), implies (**??**). ♦

Unfinished: as of 27Aug2015

A $\mathbf{C}^\infty$ function with compact support

Taking the $\mathbf{G}$ from (**??**) and zero-ing it out on the negative real axis, gives

$$8: \qquad \mathbf{H}(x) \;:=\; \begin{Bmatrix} 0\,, & \text{if } x \leqslant 0 \\ \exp(-\frac{1}{x^2})\,, & \text{if } x > 0 \end{Bmatrix}$$

This is a $\mathbf{C}^\infty$-function whose n^{th}-derivative is

$$\mathbf{H}^{(n)}(x) \;:=\; \begin{Bmatrix} 0\,, & \text{if } x \leqslant 0 \\ \exp(-\frac{1}{x^2}) \cdot R_n(\frac{1}{x})\,, & \text{if } x > 0 \end{Bmatrix}\,,$$

where polynomial R_n is from (**??**).

[♡1] I.e, describable by a single power-series with infinite RoC.

[♡2] Can "perceive" really be used in the imperative?

For $k = 1, 2, \ldots$, define a "bump fnc" or "test fnc"

$$\mathbf{B}_k(x) \quad := \quad \mathbf{H}(\tfrac{1}{k} + x) \cdot \mathbf{H}(\tfrac{1}{k} - x)\,.$$

This $\mathbf{C}^\infty$-function has two points of non-analyticity; the points $\pm\frac{1}{k}$. The support of $\mathbf{B}_k$ is open interval

$$\operatorname{Supp}(\mathbf{B}_k) \;=\; \left(-\tfrac{1}{k}, \tfrac{1}{k}\right),$$

which is bounded. Hence one possible definition of the Dirac-delta is the distributional-limit $\lim_{k\to\infty} \mathbf{B}_k$.

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