Polynomial preliminaries. A poly(nomial) such as 
\( f(x) = 8x^2 - 7x + 2 \) can be written as 
\( 8e_2 - 7e_1 + 2e_0 \), where \( e_j(x) := x^j \). As a linear combination, 
\[ f = \sum_{j=0}^{2} \alpha_j e_j, \]
where \( \alpha := (\alpha_0, \alpha_1, \alpha_2) = (2, -7, 8) \). The product of 
\( f \) with a degree-3 polynomial 
\[ g = \sum_{k=0}^{3} \beta_k e_k \]
is an 
\[ f \cdot g = \sum_{n=0}^{5} \gamma_n e_n, \]
where, summing over natnum-pairs \((j,k)\), each 
1a:
\[ \gamma_n = \sum_{(j,k): j+k=n} [\alpha_j \cdot \beta_k]. \]

This \( \gamma = (\gamma_0, \ldots, \gamma_5) \) is \( \alpha \circ \beta \), the convolution of \( \alpha \) 
with \( \beta \). An alternative notation to (1) is 
\[ f = \sum_{j=0}^{\infty} \alpha_j e_j, \]
where \( \alpha_j = 0 \) when \( j > 2 \). Such an \( \alpha = (\alpha_0, \alpha_1, \ldots) \) 
is an eventually-zero sequence.

Chromatic form. Looking ahead, consider a monic 
intopoly \( \mathcal{P}() \) with \( R \in \mathbb{N} \) many [not necessarily distinct] 
integer-roots \( Z_1, Z_2, \ldots, Z_R \). Writing \( \mathcal{P}(x) \) in chromatic form 
means writing 
\[ \mathcal{P}(x) = \prod_{j=1}^{R} [x - Z_j] \cdot f(x), \]
where [either “\( f(x) \)” is absent or] \( f \) is a monic intopoly 
with no integral roots. Indeed, courtesy the Gauss 
Lemma for polynomials, our \( f \) has no rational roots.

When \( \mathcal{P} \) is a chromatic polynomial, its integer-root 
part \( \prod_{j} [x - Z_j] \) should be written in form 
\[ \ast: \quad [x^{e_0} \cdot [x - 1]^{e_1} \cdot [x - 2]^{e_2} \cdots [x - [K-1]]^{e_{K-1}}, \]
only mixed with falling-factorials, e.g
\[ **: \quad [x \downarrow 2]^5 \cdot [x \downarrow 4] \cdot [x - 1]^7. \]
See (4d), the Chromatic Polynomial Corollary. 

Graph terminology. Use \( K_n \) for the complete 
graph on \( n \) vertices, and \( K_{j,k} \) for the complete \((j,k)\)- 
bipartite graph. Use \( C_n \) for the cyclic graph 
with \( n \) vertices and \( n \) edges \( \big| C_n \big| \) has a single self-edge; \( C_2 \) has 2 
edges between 2 vertices; for \( n \geq 3 \), our \( C_n \) is a simple graph]. 
Use \( \text{Emp}_n \) for the \( n \)-vertex graph with no edges; an “Empty graph”.

Coloring. Consider \( G = (V,E) \), a finite graph, 
[loops, multiple-edges ok] with \( N := |V| \), and \( L := |E| \). 
Use \( c(G) \) for the \# of connected components of \( G \).

For \( k \in \mathbb{N} \), a “\( k \)-coloring of \( G \)” means to assign a 
“color”, an element of \([1..k]\), to each vertex, so that 
no two neighbors [the end-vertices of an edge] have the same color. A \( k \)-coloring is full if it uses all \( k \) colors. The chromatic number of \( G \), written \( \chi(G) \), is 
the minimum number of colors needed. I.e, it is the unique \( k \) so that there is a \( k \)-coloring of \( G \), and every 
\( k \)-coloring of \( G \) is full.

For \( k = 0,1,2,\ldots \), let \( \mathcal{P}_k(G) \) be the number of \( k \)-colorings of \( G \). Note \( \mathcal{P}_{\text{Emp}_n}(k) = 1 \) since the void 
graph [no vertices] has exactly one \( k \)-coloring. Chromatic number \( \chi(G) \) is the smallest natnum \( k \) for which 
\( \mathcal{P}_k(G) \) is positive. \(^{\dagger} \)

\(^{\dagger}\)If \( G \) has a loop, then \( \mathcal{P}_0(G) = 0 \) and \( \chi(G) = \infty \).
If \( G \) has multiple edges between vertices \( u,v \), then replacing 
them by a single edge will not change the chromatic poly/number. Without announcement we will do this; effectivelly, we compute chromatic polys only of simple graphs.
Deletion-contraction. Consider an $N$-vertex graph $G$ and edge $\alpha \in E$. Let $G \setminus \{\alpha\}$ mean to delete edge $\alpha$; no vertices are removed, so $G \setminus \{\alpha\}$ still has $N$ vertices. In contrast, use $G/\alpha$ to mean the graph with $N-1$ vertices, where we have contracted $\alpha$, so that its two endpoints become a single vertex, and $\alpha$ is gone.

2: Deletion-contraction Thm. On $N$-vertex simple graph $G = (V, E)$, function $\mathcal{P}_G()$ is a degree-$N$ monic polynomial. For each $\alpha \in E$,

$$2a: \quad \mathcal{P}_G(x) = \mathcal{P}_{G \setminus \{\alpha\}}(x) - \mathcal{P}_{G/\alpha}(x),$$

as polynomials. Also, $\mathcal{P}_G(x)$ has no constant term, except when $G$ is the void graph. \hfill \lozenge

\textbf{Pf.} The only $N=0$ graph is void, and $\mathcal{P}_{\text{Emp}_0}(x)$ is constant-1, which is monic and of degree zero. Fixing $N \geq 1$, we induct on the number, $L$, of edges. The $L=0$ case is trivial, since $\mathcal{P}_{\text{Emp}_N}(x) = x^N$.

As $G \setminus \{\alpha\}$ has $L-1$ edges, poly $\mathcal{P}_{G \setminus \{\alpha\}}$ is monic of degree-$N$. And $G/\alpha$ has $N-1$ vertices, so $\mathcal{P}_{G/\alpha}$ is a degree-$[N-1]$ poly. Since their difference is a monic degree-$N$ poly, INDuce the (2a) recurrence. We will show that

$$2b: \quad \mathcal{P}_G(k) + \mathcal{P}_{G/\alpha}(k) = \mathcal{P}_{G \setminus \{\alpha\}}(k),$$

for each posint $k$. This implies equality as polynomials, since we will have equal outputs, for $N+1$ many values of $k$.

The endpoints of $\alpha$, call them $u$ and $v$. Consider a coloring of $G/\alpha$, but split apart the combined vertex back into separate vertices $u$ and $v$ [and don’t put in edge $\alpha$]. This is now a coloring of $G \setminus \{\alpha\}$ that gives $u$ and $v$ the same color. In contrast, each coloring of $G$ gives distinct colors to $u$ and $v$; so removing $\alpha$ gives a coloring of $G \setminus \{\alpha\}$ with $u$ and $v$ having different colors. Hence (2b). \hfill \lozenge

\textbf{Properties of $\mathcal{P}_G$.} Initially, $N \in \mathbb{N}$.

$$3a: \quad \mathcal{P}_{\text{Emp}_N}(x) = x^N.$$

Now, $N \geq 1$. The $N$-vertex path graph, $P_N$, is a special case of a tree. Below, $T_N$ is an arbitrary tree on $N$ vertices. Easily,

$$3b: \quad \mathcal{P}_{P_N}(x) = \mathcal{P}_{T_N}(x) = x^{[x-1]^{N-1}}.$$

For $N \geq 2$,

$$3c: \quad \mathcal{P}_{C_N}(x) = [x-1]^N + [-1]^N[x-1].$$

In particular,

$$3c': \quad \mathcal{P}_{C_4}(x) = x \cdot [x-1] \cdot [x^2 - 3x + 3] = x^4 - 4x^3 + 6x^2 - 3x.$$

\textbf{Proof of (3c)}. Note $C_2$ becomes the path $P_2$, after collapsing the multi-edge, hence has chrom-poly $x \cdot [x-1]$, which is what $(3c)_{N=2}$ equals; the base case.

Applying (2a) to $G := C_{N+1}$ and an edge $\alpha$, gives $G \setminus \{\alpha\} = P_{N+1}$ and $G/\alpha = C_N$. So $\mathcal{P}_G(x)$ equals

$$[x \cdot [x-1]^N] - [x-1]^N + [-1]^N[x-1].$$

And this reduces to $[x-1]^{N+1} + [-1]^{N+1}[x-1].$ \hfill \lozenge

From each vertex of $C_N$, attach an edge to a common vertex, $u_{N+1}$. This wheel graph $W_{N+1}$, has $N+1$ vertices and $2N$ edges. For $N \geq 2$, then,

$$3d: \quad \mathcal{P}_{W_{N+1}}(x) = x \cdot \mathcal{P}_{C_N}(x-1) =\begin{cases} x \cdot [x-2]^N + [-1]^N[x-2] \end{cases}.$$

(Fixing a posint $x$, there are $x$ choices to color vertex $u_{N+1}$, hence $x-1$ colors available for the embedded $C_N$. E.g., $(3c')$ gives $\mathcal{P}_{C_4}(x-1) = [x-1] \cdot [x-2] \cdot [x^2 - 5x + 7]$. So

$$3d': \quad \mathcal{P}_{W_5}(x) = [x \downarrow 3] \cdot [x^2 - 5x + 7].$$

\textbf{Trivial graphs.} Note $\mathcal{P}_{C_1} = x$ and $\mathcal{P}_{C_0} = \mathcal{P}_{\text{Emp}_0} = 1$; neither produced by $(3c)$. The wheel-recurrence $\mathcal{P}_{W_N}(x) = x \cdot \mathcal{P}_{C_{N-1}}(x-1)$ holds $\forall N \geq 1$. \hfill \Box

Exer E1: Suppose $\mathcal{P}_G(x) = x \cdot [x-1]^{N-1}$. Prove that $G$ is a tree.
Cone over a graph. Write this: Wheel graph $W_{N+1}$ is the cone over $C_N$. Double cone. And $K$-vertex cone gives clumsy formula involving bell numbers.

Alternatively, the $K$-vertex cone over $G$ is the full-product of $G$ with $\text{Emp}_K$. \hfill \checkmark

Defn. An alternating polynomial $h(x)$ has form

$$B_Nx^N - B_{N-1}x^{N-1} + B_{N-2}x^{N-2} - B_{N-3}x^{N-3} + \ldots + (-1)^j B_{N-j}x^{N-j} + \ldots + (-1)^{N-K} B_Kx^K,$$

where $N \geq K$ are natnums, and each $B_j > 0$. Call index $K$ the low-degree of $h$, written LD($h$). Here is an easy exercise. For $f$ and $g$ alternating-polys:

4a: Product $f \cdot g$ is alternating, and $\text{LD}(f \cdot g) = \text{LD}(f) + \text{LD}(g)$.

4b: If $\deg(f) = 1 + \deg(g)$, then $f - g$ is alternating, with $\text{LD}(f - g) = \text{Min}(\text{LD}(f), \text{LD}(g))$. \hfill \checkmark

4c: Chromatic polynomial Theorem. For a non-void simple graph $G = (V, E)$, write its chromatic polynomial $\mathcal{P}_G(x)$ in form (4). Then

\* $\mathcal{P}_G$ is a monic alternating intpoly, with $N = |V|, B_{N-1} = |E|$ and $K = c(G)$.

[This $c(G)$ is the number of connected-components.] \hfill \checkmark

Proof. [We have “monic” and “$N = |V|$” from (2).] First suppose $G$ decomposes into (non-void) disjoint subgraphs $H_1 \sqcup H_2$. Let $N_j := |V_{H_j}|, L_j := |E_{H_j}|, K_j := c(H_j)$ and $f_j := \mathcal{P}_{H_j}$. So $f_j$ has form

$$f_j(x) = x^{N_j} - L_j x^{N_j-1} + \ldots + C_j x^{K_j}$$

with $C_j \neq 0$. Easily, $\mathcal{P}_G = f_1 \cdot f_2$, hence is alternating, by (4a), with low-degree $K_1 + K_2$ monic $c(G)$. The penultimate coeff of $f_1 \cdot f_2$ is $-|L_1 + L_2|$, which indeed is the number of $G$-edges. So WLOG, $G$ is connected. \hfill \checkmark

When $G$ is connected. [Recall $N \geq 1$ since $G$ is non-void.] Pick a $G$-edge, $\alpha$, whose removal does not disconnect $G$; if there is none such, then $G$ is a tree [possibly the edgeless tree], where (*) evidently holds.

Hence both $G \setminus \{\alpha\}$ and $G/\alpha$ are connected, Thus $f := \mathcal{P}_{G \setminus \{\alpha\}}$ and $g := \mathcal{P}_{G/\alpha}$, each satisfy (*). So $f - g$ is alternating, by (4b), and $\text{LD}(f - g) = \text{Min}(1, 1) = 1$, which is indeed the number of connected-comps of $G$. \hfill \checkmark

Counting edges. Let $L := |E|$. Our $G \setminus \{\alpha\}$ has $L - 1$ edges, thus $f(x) = x^N - |L-1|x^{N-1} + \ldots$. And $g$ is monic, $g(x) = x^{N-1} - \ldots$. The difference thus has form $\mathcal{P}_G(x) = x^N - Lx^{N-1} + \ldots$, as desired. \hfill \checkmark

4d: Chromatic-polynomial Corollary. Polynomial $\mathcal{P}_G$ has no negative roots. Setting $K := \chi(G)$, we can therefore write $\mathcal{P}_G(x)$ in chromatic form as

$$x^{e_0} \cdot [x - 1]^{e_1} \cdot [x - 2]^{e_2} \cdots [x - \lfloor K - 1\rfloor]^{e_{K-1}} \cdot f(x),$$

with each $e_j \in \mathbb{Z}_+$. Moreover, $f$ is [either absent, or] a monic intpoly, with no negative real roots, and no rational roots. \hfill \checkmark

Proof. An alternating-poly evaluated at a negative real, yields a sum of posreals, hence is positive. Finally, since $f$ is primitive (the GCD of its coeffs is 1) each rational root must be integral, by the Gauss Lemma for polynomials.

Exer E3.1415: “The composition of two chromatic-polys is always a chromatic-poly.” Prove, or CEX. \hfill \checkmark

Gluing

The next result uses

\ldots two graphs $H_j$, for $j = 1, 2$, with $N_j$ many vertices and $L_j$ many edges. Let

$$h_j() := \mathcal{P}_{H_j}().$$

5a: Gluing lemma. When $G$ is built from non-void simple graphs $H_1$ and $H_2$ by

0: disjoint union, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)$.

1: picking a vertex $u_j$ in $H_j$ and identifying the two vertices, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)/x$. This $G$ has $N_1 + N_2 - 1$ many vertices and $L_1 + L_2$ many edges. Call this $G$ a point-gluing of $H_1$ and $H_2$.

2: picking an edge $\alpha_j$ in $H_j$ and identifying the two edges (choose an orientation), then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)/x[x-1].$$

This $G$ has $N_1 + N_2 - 2$ vertices and $L_1 + L_2 - 1$ edges. Call this $G$ an edge-gluing of $H_1$ with $H_2$. \hfill \checkmark

\footnote{However, $f$ can have complex roots with negative real-part.}
Let’s use the symbol \( G \) for both injections we’ll use a common symbol (usually \( \Phi \)) and write \( \Phi: \mathcal{M} \rightarrow \mathcal{H} \).

Consider graphs \( H_j \) as in (5), as well as a graph \( M \). Suppose \( \Phi_1: \mathcal{H}_1 \rightarrow \mathcal{M} \) and \( \Phi_2: \mathcal{M} \rightarrow \mathcal{H}_2 \). Define the gluing of \( H_1 \) with \( H_2 \), over \( (\Phi_1, \Phi_2) \) as the graph \( G \) which is the “union” of \( H_1 \) and \( H_2 \), where for each vertex \( u \) and edge \( \alpha \) of \( M \):

- Vertex \( \Phi_1(u) \) is identified with \( \Phi_2(u) \) and edge \( \Phi_1(\alpha) \) is identified with \( \Phi_2(\alpha) \).

So \( G \) has \( N_1 + N_2 - |\text{Vertices}(M)| \) many vertices, and \( L_1 + L_2 - |\text{Edges}(M)| \) many edges. When we don’t need the details of the gluing, we will refer to \( G \) as a gluing of \( H_1 \) with \( H_2 \), over \( M \).

**Notation:** I use symbol \( \sqcup \) for “disjoint union”, so let’s use \( H_1 \sqcup H_2 \) for gluing over the void graph. More generally, use

- \( H_1 \sqcup_M H_2 \) or \( H_1 \sqcup_{\Phi_2} H_2 \)

for gluing over \( M \); the latter, if the details are needed.

Say that \( M \) is **gluing-good** if, for all graphs \( H_1, H_2 \) having \( M \) as a subgraph, necessarily

\[
\Phi_G(x) = \frac{\Phi_{H_1}(x) \cdot \Phi_{H_2}(x)}{\Phi_M(x)},
\]

whenever \( G \) is a gluing of \( H_1 \) with \( H_2 \) over \( M \). \( \Box \)

**Exer E2.1:** Find an infinite family of connected graphs which are gluing-good.

**Exer E2.2:** “Each connected graph \( M \) is gluing-good”. Find a proof, or CEX.

**Bipartite graphs.** For natural numbers \( B \) and \( G \), the complete bipartite graph \( K_{B,G} \) has all edges between \( B := [1..B] \) and \( G := [1..G] \), the “Boys” and “Girls”, and no other edges.

For natural numbers \( B \) and \( \ell \), the “**Stirling number of the second kind**”, \( \mathcal{S}(B, \ell) \), is the number of partitions of \( [1..B] \) into \( \ell \) many non-empty atoms. (i.e., \( \mathcal{S}(B, \ell) \) is the number of equivalence relations on \( [1..B] \) that have \( \ell \) many [non-void] equivalence classes.)

**6a:** Let \( f_{B,G} \) abbreviate \( \Phi_{K_{B,G}} \). Then

\[
f_{B,G}(x) = \sum_{\ell=0}^{B} \mathcal{S}(B, \ell) \cdot \lfloor x \downarrow \ell \rfloor \cdot [x - \ell]^G.
\]

**Computing.** At \( B = 0 \), the \( \text{RhsS} \) is \( 1 \cdot [x - 0]^G \) which is \( x^G \), which is correct.

Once \( B \geq 1 \) we can start the sum at \( \ell = 1 \), since there are no partitions of the empty set into positively many atoms.

**Case:** \( B = 1 \). The \( \text{RhsS} \) is \( 1 \cdot [x - 1]^G \cdot [x - 1]^G \), i.e., \( x \cdot (x - 1)^G \), which is what (3b) says, as \( K_{1,G} \) is a tree.

**Case:** \( B = 2 \). At \( B = 2 \), our \( \text{RhsS}(6a) \) is

\[
\begin{align*}
\mathcal{S}(2, 1) \cdot \lfloor x - 1 \rfloor \cdot [x - 1]^G & \quad 1 \cdot [x - 1]^G \\
\mathcal{S}(2, 2) \cdot \lfloor x - 2 \rfloor \cdot [x - 2]^G & \quad 1 \cdot x \cdot [x - 1]^G \cdot [x - 2]^G.
\end{align*}
\]

So \( f_{2,G}(x) \) equals \( x \cdot (x - 1)^G + [x - 1] \cdot [x - 2]^G \).

Once \( G \geq 1 \), we have that

\[
f_{2,G}(x) = x \cdot (x - 1)^G \cdot [x - 1] \cdot [x - 2]^G \cdot \ldots
\]

Plugging in \( G = 2 \) yields \( \text{RhsS}(3c) \), which is reassuring seeing as \( K_{2,2} \) equals \( \mathcal{C}_4 \).

**Case:** \( B = 3 \). \( \text{RhsS}(6a) \) is a sum of three terms:

\[
\begin{align*}
\mathcal{S}(3, 1) \cdot \lfloor x - 1 \rfloor \cdot [x - 1]^G & \quad 1 \cdot [x - 1]^G \\
\mathcal{S}(3, 2) \cdot \lfloor x - 2 \rfloor \cdot [x - 2]^G & \quad 3 \cdot [x - 1] \cdot [x - 2]^G \\
\mathcal{S}(3, 3) \cdot \lfloor x - 3 \rfloor \cdot [x - 3]^G & \quad 1 \cdot x \cdot [x - 1] \cdot [x - 2] \cdot [x - 3]^G.
\end{align*}
\]

As soon as \( G \geq 1 \), ratio \( f_{3,G}(x)/x[x - 1] \) equals

\[
\]

Using colors orange&blue there are 2 colorings of \( K_{3,G} \) [since \( G > 0 \)], so (6c) at \( x=2 \) better equal 1. Does it?
When $G = 1$, the $K_{3,1}$ is a 4-vertex tree, so (6c) with $G = 1$ better darn well be $[x-1]^2$. Is it? FWIW, $(6c)_{G=2}$ is $[x^3 - 5x^2 + 10x - 7]$; irreducible. Evaluating, $(6c)_{G=3}$ gives
\[
\frac{f_{3,3}(x)}{x \cdot [x-1]} = x^4 - 8x^3 + 28x^2 - 47x + 31.
\]
This last is irreducible over the rationals. \(\square\)

**Graph-Stirling numbers**

For a generalization of “bipartite graph”, consider two graphs $H_j = (V_j, E_j)$ with $N_j$ vertices and $L_j$ edges. The **full product** $G := H_1 \circledast H_2$ is their disjoint union, augmented by an edge from each $H_1$-vertex, to each $H_2$-vertex. Thus

7a: $H_1 \circledast H_2$ has $N_1 + N_2$ many vertices, and $L_1 + L_2 + [N_1N_2]$ many edges.

We’ll get a formula for its chrom-poly, in terms of the following type of polynomial product.

Given two polynomials $f$ and $g$, we define their **falling product** $h := f \prec g$, as following.

1. Write each w.r.t the falling-factorial basis, i.e

\[
f = \sum_{j=0}^J \alpha_j t_j \quad \text{and} \quad g = \sum_{k=0}^K \beta_k t_k,
\]

as shown in (1b).

2. Compute $\vec{\gamma} := \vec{\alpha} \circ \vec{\beta} = (\gamma_0, \gamma_1, \ldots, \gamma_{J+K})$, the convolution.

3. Define $h := \sum_{n=0}^{J+K} \gamma_n t_n$.

**Defn.** A **Stirling partition** of graph $G = (V, E)$, is a partition of $V$ into non-empty subsets (called the **atoms** of the ptn) so that no two adjacent vertices are in the same atom. (I.e., each atom is an “independent set”.)

For natnum $\ell$, define the **graph-Stirling number**

7b: $S(G, \ell)$

to be the number of $\ell$-atom Stirling partitions of $V$.

7c: **Graph-Stirling Thm.** For an $N$-vertex\(^\ddagger\) graph $G$,

\[
\mathcal{P}_G(x) = \sum_{\ell=0}^N S(G, \ell) \cdot [x \downarrow \ell].
\]

**Proof.** Exercise. \(\uparrow\)

7d: **Full-product Thm.** Consider a graph $G := H_1 \circledast H_2$.

Then

\[
\mathcal{P}_G = \mathcal{P}_{H_1} \circ \mathcal{P}_{H_2}.
\]

**Proof idea.** Fix $\ell$. The $\ell$-atom Stirling partitions of $G$ are in 1-to-1 correspondence with: Pick natnums with $j_1 + j_2 = \ell$, then take a $j_1$-atom Stirling ptn of $H_1$, and a $j_2$-atom Stirling ptn of $H_2$. \(\uparrow\)

7e: **Example.** Let $H$ be $P_2 \cup P_1$. So $S(H, 1) = 0$, $S(H, 2) = 2$ and $S(H, 3) = 1$. Our (7c) asserts that

\[
\mathcal{P}_H = \sum_{j=0}^3 \alpha_j t_j,
\]

where $\vec{\alpha} = (0, 0, 2, 1)$. I.e.,

\[
\mathcal{P}_H(x) = 2 \cdot [x \downarrow 2] + [x \downarrow 3] = [x \downarrow 2][2 + [x - 2]] \text{ note } x^2 \cdot [x - 1].
\]

This agrees with (3b) and with (5a.0).

Let’s compute $G := H \circ K$, where $K := K_1$. Our $G$ is a $K_3$ with a new edge attached to a vertex. So (5a.3) says $\mathcal{P}_G(x) = [x \downarrow 3] \cdot [x - 1]$. What does (7d) say?

Our $K$ has chr-poly $x = [x \downarrow 1]$. We set $\vec{\beta} := (0, 1)$, then compute convolution $\vec{\gamma} := \vec{\alpha} \circ \vec{\beta} = (0, 0, 0, 2, 1)$.

Thm (7d) asserts that

\[
\mathcal{P}_G(x) = 2 \cdot [x \downarrow 3] + [x \downarrow 4] = [x \downarrow 3][2 + [x - 3]] \text{ note } [x \downarrow 3][x - 1].
\]

This agrees with our derivation via (5a.3). \(\square\)

7f: **Bipartite ex.** Graph $H := \textbf{Emp}_3$ has $S(H, 1) = 1$, $S(H, 2) = 3$, $S(H, 3) = 1$, and so $\vec{\alpha} = (0, 1, 3, 1)$. Thus

\[
\mathcal{P}_H(x) = [x \downarrow 1] + 3[x \downarrow 2] + [x \downarrow 3] = x \cdot \left[1 + [x - 1][3 + [x - 2]]\right] = x \cdot \left[1 + [x^2 - 1]\right] = x^3.
\]

\(^\ddagger\)We can start the sum at $\ell=1$ except when $G$ is the void graph. After all, when $N$ is positive then $S(G, 0)$ is zero.
This indeed agrees with (3a).

Since $K_{3,3}$ is the full-product of $H$ with $H$, we compute $\bar{\gamma} := \bar{\alpha} \circ \bar{\alpha} = \{0, 0, 1, 6, 11, 6, 1\}$. Polynomial $[x \downarrow 2] + 6[x \downarrow 3] + 11[x \downarrow 4] + 6[x \downarrow 5] + [x \downarrow 6]$ simplifies (thanks, Maple) to

$$x \cdot [x-1] \cdot [x^4 - 8x^3 + 28x^2 - 47x + 31].$$

Happily, this agrees with (6d).

Spanning subgraphs

Fix a $G = (V, E)$ with $N$ vertices and $L$ edges. Each subset $S \subseteq E$ can be interpreted as $(V, S)$, a spanning subgraph of $G$. [Thus $G$ has $2^L$ many spanning subgraphs.]

Let $c(S)$ denote the number of connected components of $S$.

8: CPSS Thm (Chromatic-Poly Spanning Subgraph). The chromatic polynomial of $G$ satisfies

$$\#: \quad \mathcal{P}_G(x) = \sum_{S: S \subseteq \mathcal{E}} (-1)^{|S|} \cdot x^{c(S)}. \quad \diamond$$

Exer E4: Prove this. Think Inclusion-Exclusion. [Hint: See the pamphlet on our Teaching Page.]

Comparison. We can paraphrase Theorems (8) and (7c) as saying: Spanning subgraphs express $\mathcal{P}_G$ w.r.t the standard basis $(e_j)_{j=0}^{|S|}$ [see (1b)], whereas Graph-Stirling numbers write $\mathcal{P}_G$ w.r.t the falling-factorial basis $(t_n)_{n=0}^\infty$.

9a: Orientations. An orientation of $G = (V, E)$ is putting a direction on each edge, creating a digraph; so $G$ has $2^{|E|}$ many orientations. An orientation of $G$ is acyclic if it has no directed-cycles. Use $\mathcal{A}(G)$ for the number of acyclic orientations of $G$.

An orientation of $K_N$ is called an $N$-tournament; there are $2^\left(\binom{N}{2}\right)$ of them. E.g., $K_4$ has $2^6 = 64$ orientations. Exer E5: Prove that $\mathcal{A}(K_N)$ equals $[N!]$. $\diamond$

9b: Acyclic-count Thm (Richard Stanley). On $N$-vertex $G$,

$$\mathcal{A}(G) = [-1]^N \cdot \mathcal{P}_G(-1). \quad \diamond$$

Sketch. The idea is to establish an analog of (2a),

$$9c: \quad \mathcal{A}(G) = \mathcal{A}(G \setminus \{\alpha\}) + \mathcal{A}(G \cup \alpha),$$

for each edge $\alpha$ of $G$, as follows. Let $M := G \setminus \{\alpha\}$.

• Each acyclic-orient of $M$ extends to at least one acyclic-orient of $G$.

• The number of acyclic-orient of $M$ which give rise to two acyclic-orient of $G$, is $\mathcal{A}(G/\alpha)$.

10: Generalizing full-product. Let $\mathcal{G}$ be the set of all graphs. Consider a graph $S = (V, E)$ and a fnc $F: V \to \mathcal{G}$; we’ll often write $F(u)$ as $F_u$. Use $N_u | L_u$ for the number of vertices/edges of $F_u$.

The “full product of $F$ over $S$”

$$G := \bigcirc_S (F)$$

is $S$, but where each $S$-vertex $u$ has been replaced by a copy of graph $F_u$. Moreover, for each $S$-edge $v \rightarrow w$:

Each $F_v$-vertex $v'$ and each $F_w$-vertex $w'$, are the endpoints of a $G$-edge.

Thus $\bigcirc_S (F)$ has $\sum_{u \in V} N_u$ many vertices. It has

$$\sum_{u \in V} L_u + \sum_{V \cdot W} [N_v \cdot N_w]$$

many edges, where the second sum is over all $S$-edges $v \rightarrow w$. In the case that $S$ is a single edge, $P_2$, we recover the full-product as defined above (7a).
Full-product from (7a) is associative, so expressions such as $H_1 \odot H_2 \odot \ldots \odot H_5$ make sense. If we define $\mathcal{F}(j) := H_j$, then

$$H_1 \odot H_2 \odot \ldots \odot H_5 = \bigodot_{K_5}(\mathcal{F}),$$

regarding $[1..5]$ as the vertex-set of $K_5$. □

**Exer:** Is there a formula for the chromatic number/polynomial of $\bigodot_S(\mathcal{F})$, in terms of corresponding information about $S$ and function $\mathcal{F}$? What about special cases, e.g. $S = P_3$? Or $S = C_{\text{Even}}$?

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