Chromatic polynomial of a graph

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Page 1 of 7

Polynomial preliminaries. A polynomial such as \( f(x) = 8x^2 - 7x + 2 \) can be written as 
\[ 8e_2 - 7e_1 + 2e_0, \]
where \( e_j(x) := x^j \). As a linear combination,
\[ f = \sum_{j=0}^{2} \alpha_j e_j, \]
where \( \alpha := (\alpha_0, \alpha_1, \alpha_2) = (2, -7, 8) \). The product of \( f \) 
with a degree-3 polynomial \( g = \sum_{k=0}^{3} \beta_k e_k \) is
\[ f \cdot g = \sum_{n=0}^{5} \gamma_n e_n, \]
where, summing over natnum-pairs \((j,k)\), each
\[ \gamma_n = \sum_{(j,k) \atop j+k=n} \alpha_j \beta_k. \]
This \( \gamma = (\gamma_0, \ldots, \gamma_5) \) is \( \alpha \odot \beta \), the convolution 
of \( \alpha \) with \( \beta \). An alternative notation to (\( \dagger \)) is
\[ f = \sum_{j=0}^{\infty} \alpha_j e_j, \]
where \( \alpha_j = 0 \) when \( j > 2 \). Such an \( \alpha = (\alpha_0, \alpha_1, \ldots) \) 
is an eventually-zero sequence.

We'll also need the falling-factorial polynomials,
\[ [x \downarrow N] := x \cdot [x-1] \cdot [x-2] \cdots [x-[N-1]] \]
(a product of \( N \) many terms), where \( N \in \mathbb{N} \). How to write 
a poly such as \( F(x) := 2x^3 + x^2 - 3x + 3 \) in terms of 
falling factorials? The coeff of \( x^3 \) is \( 2 \), so we subtract,
\[ F(x) - 2 \cdot [x \downarrow 3] = 5x^2 - 7x + 3. \]
From this we subtract \( 5 \cdot [x \downarrow 2] \), producing
\[ -2x + 3. \]
From this we subtract \( -2 \cdot [x \downarrow 1] \), yielding 3. From 3 
we subtract \( 3 \cdot [x \downarrow 0] \), ending with zero. Thus
\[ F(x) = 3 \cdot [x \downarrow 0] - [x \downarrow 1] + 5 \cdot [x \downarrow 2] + 2 \cdot [x \downarrow 3]. \]
Letting \( t_n(x) := [x \downarrow n] \), we have equality
\[ F = 3e_0 - 3e_1 + e_2 + 2e_3; \]
\[ = 3t_0 - 2t_1 + 5t_2 + 2t_3. \]
We have done a change-of-basis computation, from 
basis \( (e_j)_{j=0}^{\infty} \) to the \( (t_n)_{n=0}^{\infty} \) basis.

Chromatic form. Looking ahead, consider a monic intpoly \( P() \) 
with \( R \subseteq \mathbb{N} \) many \{ not necessarily distinct \} 
integer-roots \( Z_1, Z_2, \ldots, Z_R \). Writing \( P(x) \) in chromatic form 
means writing
\[ 1c: \ P(x) = \prod_{j=1}^{R} [x - Z_j] \cdot f(x), \]
where \{ either "\( f(x) \) is absent or \} \( f \) is a monic intpoly 
with no integral roots. Indeed, courtesy the Gauss Lemma for polynomials, 
our \( f \) has no rational roots.

When \( P \) is a chromatic polynomial, its integer-root part
\( \prod_{j} [x - Z_j] \) should be written in form
\[ *: \ x^{e_0} \cdot [x - 1]^{e_1} \cdot [x - 2]^{e_2} \cdots [x - [K-1]]^{e_K-1}, \]
often mixed with falling-factorials, e.g
\[ **: \ ([x \downarrow 2]^3 \cdot [x \downarrow 4] \cdot [x - 1]^7. \]
See (4d), the Chromatic-polynomial Corollary.

Graph terminology. Use \( K_n \) for the complete graph 
on \( n \) vertices, and \( K_{j,k} \) for the complete \((j,k)\)-bipartite graph. Use \( C_n \) 
for the cyclic graph with \( n \) vertices and \( n \) edges \( |C_n| \) has a single self-edge; \( C_2 \) has 2 
edges between 2 vertices; for \( n \geq 3 \), our \( C_n \) is a simple graph. 
Use \( \text{Emp}_n \) for the \( n \)-vertex graph with no edges; an "Empty graph".

Coloring. Consider \( G = (V, E) \), a finite graph, 
[loops, multiple-edges ok] with \( N := |V| \), and \( L := |E| \). Use \( c(G) \) 
for the \# of connected components of \( G \).

For \( k \in \mathbb{N} \), a "\( k \)-coloring of \( G \)" means to assign a "color", 
an element of \([1 \ldots k] \), to each vertex, so that no two neighbors 
[the end-vertices of an edge] have the same color. A \( k \)-coloring is full 
if it uses all \( k \) colors. The chromatic number of \( G \), written \( \chi(G) \), 
is the minimum number of colors needed. I.e, it is the 
unique \( k \) so that there is a \( k \)-coloring of \( G \), and every \( k \)-coloring of \( G \) is full.

For \( k = 0, 1, 2, \ldots \), let \( P_G(k) \) be the number of \( k \)-colorings of \( G \). Note \( P_{\text{Emp}}(k) = 1 \) since the void graph 
[no vertices] has exactly one \( k \)-coloring. Chromatic number \( \chi(G) \) is 
the smallest natnum \( k \) for which \( P_G(k) \) is positive.\(^1\)

\(^1\)If \( G \) has a loop, then \( P_G() \equiv 0 \) and \( \chi(G) := \infty \).

If \( G \) has multiple edges between vertices \( u,v \), then replacing 
them by a single edge will not change the chromatic poly/number. Without announcement we will do this; 
effectively, we compute chromatic polys only of simple graphs.
Deletion-contraction. Consider an \(N\)-vertex \(G\) and edge \(\alpha \in \mathbb{E}\). Let \(G \setminus \{\alpha\}\) mean to delete edge \(\alpha\); no vertices are removed, so \(G \setminus \{\alpha\}\) still has \(N\) vertices. In contrast, use \(G/\alpha\) to mean the graph with \(N-1\) vertices, where we have contracted \(\alpha\), so that its two endpoints become a single vertex, and \(\alpha\) is gone. Delete loops if the contraction creates such. If contraction creates a multi-graph, it is a matter of taste whether to delete extra edges.

2: Deletion-contraction Thm. On \(N\)-vertex loopless multi-graph \(G = (V, \mathbb{E})\), function \(\mathcal{P}_G()\) is a degree-\(N\) monic polynomial. For each \(\alpha \in \mathbb{E}\),

\[
\mathcal{P}_G(x) = \mathcal{P}_{G \setminus \{\alpha\}}(x) - \mathcal{P}_{G/\alpha}(x),
\]
as polynomials. Also, \(\mathcal{P}_G(x)\) has no constant term, except when \(G\) is the void graph.

\(\square\)

Pf. The only \(N=0\) graph is void, and \(\mathcal{P}_{\text{Emp}_0}(x)\) is constant-1, which is monic and of degree zero. Fixing \(N \geq 1\), we induct on the number, \(L\), of edges. The \(L=0\) case is trivial, since \(\mathcal{P}_{\text{Emp}_N}(x) = x^N\).

As \(G \setminus \{\alpha\}\) has \(L-1\) edges, \(\text{poly} \mathcal{P}_{G \setminus \{\alpha\}}\) is monic of degree-\(N\). As \(G/\alpha\) has at-most \(L-1\) (we might have deleted a loop) and \(N-1\) vertices, our \(\mathcal{P}_{G/\alpha}\) is a degree-\([N-1]\) poly. Since their difference is a monic degree-\(N\) poly, ISTEstablish the (2a) recurrence. We will show that

\[
\mathcal{P}_G(k) + \mathcal{P}_{G/\alpha}(k) = \mathcal{P}_{G \setminus \{\alpha\}}(k),
\]
for each posint \(k\). This implies equalit as polynomials, since we will have equal outputs, for \(N+1\) many values of \(k\).

The endpoints of \(\alpha\), call them \(u\) and \(v\). Consider a coloring of \(G/\alpha\), but split apart the combined vertex back into separate vertices \(u\) and \(v\) [and don’t put in edge \(\alpha\)]. This is now a coloring of \(G \setminus \{\alpha\}\) that gives \(u\) and \(v\) the same color. In contrast, each coloring of \(G\) gives distinct colors to \(u\) and \(v\); so removing \(\alpha\) gives a coloring of \(G \setminus \{\alpha\}\) with \(u\) and \(v\) having different colors. Hence (2b).

\(\square\)

Properties of \(\mathcal{P}_G\). Initially, \(N \in \mathbb{N}\).

\[
\begin{align*}
\mathcal{P}_{\text{Emp}_N}(x) &= x^N. \\
\mathcal{P}_{\text{K}_N}(x) &= \left[x \downarrow 1\right].
\end{align*}
\]

Now, \(N \geq 1\). The \(N\)-vertex path graph, \(P_N\), is a special case of a tree. Below, \(T_N\) is an arbitrary tree on \(N\) vertices. Easily,

\[
\begin{align*}
\mathcal{P}_{P_N}(x) &= \mathcal{P}_{T_N}(x) = x \cdot [x-1]^{N-1}. \\
\end{align*}
\]

For \(N \geq 2\),

\[
\begin{align*}
\mathcal{P}_{C_N}(x) &= [x-1]^N + [-1]^N[x-1]. \\
\end{align*}
\]

In particular,

\[
\begin{align*}
\mathcal{P}_{C_4}(x) &= x \cdot [x-1]^3 \cdot [x^2 - 3x + 3] \\
&= x^4 - 4x^3 + 6x^2 - 3x.
\end{align*}
\]

Proof of (3c). Note \(C_2\) becomes the path \(P_2\), after collapsing the multi-edge, hence has chrom-poly \(x \cdot [x-1]\), which is what (3c) is equal to; the base case.

Applying (2a) to \(G := C_{N+1}\) and an edge \(\alpha\), gives \(G \setminus \{\alpha\} = P_{N+1}\) and \(G/\alpha = C_N\). So \(\mathcal{P}_G(x)\) equals

\[
[x \cdot [x-1]^N - \left([x-1]^N + [-1]^N[x-1]\right)].
\]

And this reduces to \([x-1]^{N+1} + [-1]^{N+1}[x-1].\)

From each vertex of \(C_N\), attach an edge to a common vertex, \(u_{N+1}\). This wheel graph \(W_{N+1}\), has \(N+1\) vertices and \(2N\) edges. For \(N \geq 2\), then,

\[
\mathcal{P}_{W_{N+1}}(x) = x \cdot \mathcal{P}_{C_N}(x-1)
\]

\[
\begin{align*}
\mathcal{P}_{W_{N+1}}(x) &= x \cdot \mathcal{P}_{C_N}(x-1) \\
&= x \cdot \mathcal{P}_{C_N}(x-1) \\
&= x \cdot [x-2]^N + [-1]^N[x-2].
\end{align*}
\]

(Fixing a posint \(x\), there are \(x\) choices to color vertex \(u_{N+1}\), hence \(x-1\) colors available for the embedded \(C_N\).) E.g., (???) gives \(\mathcal{P}_{C_3}(x-1) = [x-1] \cdot [x-2] \cdot [x^2 - 5x + 7]\). So

\[
\begin{align*}
\mathcal{P}_{W_5}(x) &= \left[x \downarrow 3\right] \cdot [x^2 - 5x + 7].
\end{align*}
\]

Trivial graphs. Note \(\mathcal{P}_{C_1} = x\) and \(\mathcal{P}_{C_0} = \mathcal{P}_{\text{Emp}_0} = 1\); neither produced by (3c). The wheel-recurrence \(\mathcal{P}_{W_N}(x) = x \cdot \mathcal{P}_{C_{N-1}}(x-1)\) holds \(\forall N \geq 1\).

Exer E1: Suppose \(\mathcal{P}_G(x) = x \cdot [x-1]^{N-1}\). Prove that \(G\) is a tree.

\(\square\)

Footnote: If \(u, v\) have multiple edges, then contracting a \(u \rightarrow v\) edge creates a loop, hence a graph with \(P() \equiv 0\). This is ok, but inefficient; typically, first collapse each multi-edge to a single edge.
Cone over a graph. Write this: Wheel graph \( W_{N+1} \) is the cone over \( C_N \). Double cone. And \( K \)-vertex cone gives clumsy formula involving bell numbers.

Alternatively, the \( K \)-vertex cone over \( G \) is the full-product of \( G \) with \( \text{Emp}_K \).

**Defn.** An alternating polynomial \( h(x) \) has form
\[
B_Nx^N - B_{N-1}x^{N-1} + B_{N-2}x^{N-2} - B_{N-3}x^{N-3} + \ldots + (-1)^{N-K}B_K x^K,
\]
where \( N \geq K \) are natnums, and each \( B_i \geq 0 \). Call index \( K \) the low-degree of \( h \), written \( \text{LD}(h) \). Here is an easy exercise. For \( f \) and \( g \) alternating-pols:

4a: \[ \text{Product } f \cdot g \text{ is alternating, and } \text{LD}(f \cdot g) = \text{LD}(f) + \text{LD}(g). \]

4b: \[ \text{If } \text{Deg}(f) = 1 + \text{Deg}(g), \text{ then } f - g \text{ is alternating, with } \text{LD}(f - g) = \text{Min}(\text{LD}(f), \text{LD}(g)). \]

4c: Chromatic polynomial Theorem. For a non-void simple graph \( G = (\mathcal{V}, \mathcal{E}) \), write its chromatic polynomial \( \mathcal{P}_G(x) \) in form (4). Then
\[ \mathcal{P}_G \text{ is a monic alternating intpoly, with } N = |\mathcal{V}|, B_{N-1} = |\mathcal{E}| \text{ and } K = c(G). \]

[This \( c(G) \) is the number of connected-components.]

**Proof.** [We have “monic” and “\( N = |\mathcal{V}| \) from (2).] First suppose \( G \) decomposes into (non-void) disjoint subgraphs \( H_1 \sqcup H_2 \). Let \( N_j := |\mathcal{V}_{H_j}|, L_j := |\mathcal{E}_{H_j}|, K_j := c(H_j) \) and \( f_j := \mathcal{P}_{H_j} \). So \( f_j \) has form
\[
f_j(x) = x^{N_j} - L_jx^{N_j-1} + \ldots + C_jx^{K_j}
\]
with \( C_j \neq 0 \). Easily, \( \mathcal{P}_G = f_1 \cdot f_2 \), hence is alternating, by (4a), with low-degree \( K_1 + K_2 \equiv c(G) \). The penultimate coeff of \( f_1 \cdot f_2 = -[L_1 + L_2] \), which indeed is the number of \( G \)-edges. So WLOG, \( G \) is connected.

When \( G \) is connected. [Recall \( N \geq 1 \) since \( G \) is non-void.] Pick a \( G \)-edge, \( \alpha \), whose removal does not disconnect \( G \); if there is none such, then \( G \) is a tree [possibly the edgeless tree], where (*) evidently holds.

Hence both \( G \setminus \{\alpha\} \) and \( G/\alpha \) are connected, Thus \( f := \mathcal{P}_{G \setminus \{\alpha\}} \) and \( g := \mathcal{P}_{G/\alpha} \), each satisfy (*). So \( f - g \) is alternating, by (4b), and \( \text{LD}(f - g) = \text{Min}(1,1) = 1 \), which is indeed the number of connected-comps of \( G \).

**Counting edges.** Let \( L := |\mathcal{E}| \). Our \( G \setminus \{\alpha\} \) has \( L - 1 \) edges, thus \( f(x) = x^N - [L-1]x^{N-1} + \ldots \). And \( g \) is monic, \( g(x) = x^{N-1} - \ldots \). The difference thus has form \( \mathcal{P}_G(x) = x^N - Lx^{N-1} + \ldots \), as desired. \( \diamondsuit \)

4d: Chromatic-polynomial Corollary. Polynomial \( \mathcal{P}_G \) has no negative roots. Setting \( K := c(G) \), we can therefore write \( \mathcal{P}_G(x) = x^{c_0} \cdot [x - 1]^{c_1} \cdot [x - 2]^{c_2} \cdots [x - (K-1)]^{c_{K-1}} \cdot f(x) \), with each \( c_j \in \mathbb{Z}_{>0} \). Moreover, \( f \) is either absent, or a monic intpoly, with no negative real\(^3\) roots, and no rational roots. \( \diamondsuit \)

**Proof.** An alternating-poly evaluated at a negative real, yields a sum of posreals, hence is positive. Finally, since \( f \) is primitive (the GCD of its coeffs is 1) each rational root must be integral, by the Gauss Lemma for polynomials.

Exer E3.1415: “The composition of two chromatic-pols is always a chromatic-poly.” Prove, or CEX.

Gluing

The next result uses
\[
\ldots \text{two graphs } H_j, \text{ for } j = 1, 2, \text{ with } N_j \text{ many vertices and } L_j \text{ many edges. Let } h_j := \mathcal{P}_{H_j}.
\]

5a: Gluing lemma. When \( G \) is built from non-void simple graphs \( H_1 \) and \( H_2 \) by…

0: disjoint union, then \( \mathcal{P}_G(x) = h_1(x) \cdot h_2(x). \)

1: picking a vertex \( u_j \) in \( H_j \) and identifying the two vertices, then \( \mathcal{P}_G(x) = h_1(x) \cdot h_2(x)/x \). This \( G \) has \( N_1 + N_2 - 1 \) many vertices and \( L_1 + L_2 + 1 \) many edges. Call this \( G \) a point-gluing of \( H_1 \) and \( H_2 \).

2: picking an edge \( \alpha_j \) in \( H_j \) and identifying the two edges (choose an orientation), then
\[
\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)/x[x-1].
\]

This \( G \) has \( N_1 + N_2 - 2 \) vertices and \( L_1 + L_2 - 1 \) edges. Call this \( G \) an edge-gluing of \( H_1 \) with \( H_2 \).

\(^3\)However, \( f \) can have complex roots with negative real-part.
3: picking a vertex $u_j$ in $H_j$ and putting in a (new) edge between them, then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) \cdot \frac{x^{-1}}{x}.$$  

This $G$ has $N_1 + N_2$ many vertices and $L_1 + L_2 + 1$ many edges. This $G$ is a new-edge-attaching of $H_1$ and $H_2$.  

**Proof.** Exercise.  

5b: Gluing on a subgraph. Graph $M = (\mathcal{V}, \mathcal{E})$ is a **subgraph** of $H = (\mathcal{V}', \mathcal{E}')$ if there exist injections $\Phi: \mathcal{V} \leftarrow \mathcal{V}'$ and $\Psi: \mathcal{E} \to \mathcal{E}'$ so that:

For each $\alpha \in \mathcal{E}$ with endpoints $u, v \in \mathcal{V}$, necessarily, the endpoints of $\Psi(\alpha)$ are $\Phi(u)$ and $\Phi(v)$.

Henceforth, for both injections we’ll use a common symbol (usually $\Phi$) and write $(\Phi: M \leftarrow H)$.

Consider graphs $H_j$ as in (5), as well as a graph $M$. Suppose $\Phi_1: M \to H_1$ and $\Phi_2: M \to H_2$. Define the **gluing** of $H_1$ with $H_2$, over $(\Phi_1, \Phi_2)$ as the graph $G$ which is the “union” of $H_1$ and $H_2$, where for each vertex $u$ and edge $\alpha$ of $M$:

5c: Vertex $\Phi_1(u)$ is identified with $\Phi_2(u)$ and edge $\Phi_1(\alpha)$ is identified with $\Phi_2(\alpha)$.

So $G$ has $N_1 + N_2 - |\text{Vertices}(M)|$ many vertices, and $L_1 + L_2 - |\text{Edges}(M)|$ many edges. When we don’t need the details of the gluing, we will refer to $G$ as a gluing of $H_1$ with $H_2$, over $M$.

**Notation:** I use symbol $\sqcup$ for “**disjoint union**”, so let’s use $H_1 \sqcup H_2$ for gluing over the void graph. More generally, use

5d: $H_1 \sqcup_M H_2$ or $H_1 \sqcup_{\Phi_1} H_2$ for gluing over $M$; the latter, if the details are needed.

Say that $M$ is **gluing-good** if, for all graphs $H_1, H_2$ having $M$ as a subgraph, necessarily

5e: $\mathcal{P}_G(x) = \mathcal{P}_{H_1}(x) \cdot \mathcal{P}_{H_2}(x)/\mathcal{P}_M(x)$,

whenever $G$ is a gluing of $H_1$ with $H_2$ over $M$.  

**Bipartite graphs.** For natural numbers $B$ and $G$, the complete bipartite graph $K_{B,G}$ has all edges between $B := [1..B]$ and $G := [1..G]$, the “Boys” and “Girls”, and no other edges.

For natural numbers $B$ and $\ell$, the **Stirling number of the second kind**, $\mathcal{S}(B, \ell)$, is the number of partitions of $[1..B]$ into $\ell$ many non-empty atoms. (I.e., $\mathcal{S}(B, \ell)$ is the number of equivalence relations on $[1..B]$ that have $\ell$ many [non-void] equivalence classes.)

6: Sara’s Lemma. Let $f_{B,G}$ abbreviate $\mathcal{P}_{K_{B,G}}$. Then

6a: $f_{B,G}(x) = \sum_{\ell=0}^{B} \mathcal{S}(B, \ell) \cdot [x \downarrow \ell] \cdot [x - \ell]^G$. 

**Computing.** At $B = 0$, the RhS is $1 \cdot [x - 0]^G$ which is $x^G$, which is correct.

Once $B \geq 1$ we can start the sum at $\ell = 1$, since there are no partitions of the empty set into positively many atoms.

**Case:** $B = 1$. The RhS is $1 \cdot [x \downarrow 1] \cdot [x - 1]^G$, i.e, $x \cdot [x - 1]^G$, which is what (3b) says, as $K_{1,G}$ is a tree.

**Case:** $B = 2$. At $B = 2$, our RhS(6a) is

$$\mathcal{S}(2, 1) \cdot [x \downarrow 1] \cdot [x - 1]^G + 1 \cdot x \cdot [x - 1]^G$$

$$\mathcal{S}(2, 2) \cdot [x \downarrow 2] \cdot [x - 2]^G + 1 \cdot x \cdot [x - 1] \cdot [x - 2]^G.$$ 

So $f_{2,G}(x)$ equals $x \cdot [x - 1]^G + [x - 1] \cdot [x - 2]^G$.  

Once $G \geq 1$, we have that

6b: $f_{2,G}(x) = x \cdot [x - 1] \cdot [x - 1]^{G-1} + [x - 2]^G$. 

Plugging in $G=2$ yields RhS(??‘), which is reassuring seeing as $K_{2,2}$ equals $C_4$.

**Case:** $B = 3$. RhS(6a) is a sum of three terms:

**Case:** $B = 3$. RhS(6a) is a sum of three terms:

$$\mathcal{S}(3, 1) \cdot [x \downarrow 1] \cdot [x - 1]^G = 1 \cdot x \cdot [x - 1]^G;$$

$$\mathcal{S}(3, 2) \cdot [x \downarrow 2] \cdot [x - 2]^G = 3 \cdot x \cdot [x - 1] \cdot [x - 2]^G;$$

$$\mathcal{S}(3, 3) \cdot [x \downarrow 3] \cdot [x - 3]^G = 1 \cdot x \cdot [x - 1] \cdot [x - 2] \cdot [x - 3]^G.$$ 

As soon as $G \geq 1$, ratio $f_{3,G}(x)/x[x - 1]$ equals


Using colors orange&blue there are 2 colorings of $K_{3,G}$ [since G>0], so (6c) at $x=2$ better equal 1. Does it?
When \( G = 1 \), the \( K_{3,1} \) is a 4-vertex tree, so \((6c)\) with \( G=1 \) better durn well be \([x-1]^2\). Is it?

FWIW, \((6c)_{G=2} \) is \([x^3 - 5x^2 + 10x - 7] \); irre.

Evaluating, \((6c)_{G=3} \) gives

\[
6d: \quad \frac{f_{3,3}(x)}{x \cdot [x-1]} = x^4 - 8x^3 + 28x^2 - 47x + 31.
\]

This last is irreducible over the rationals. \(\square\)

**Graph-Stirling numbers**

For a generalization of “bipartite graph”, consider two graphs \( H_j = (V_j, E_j) \) with \( N_j \) vertices and \( L_j \) edges.

The **full product** \( G := H_1 \odot H_2 \) is their disjoint union, augmented by an edge from each \( H_1 \)-vertex, to each \( H_2 \)-vertex. Thus

\[
7a: \quad H_1 \odot H_2 \text{ has } N_1 + N_2 \text{ many vertices, and } L_1 + L_2 + [N_1 N_2] \text{ many edges.}
\]

We’ll get a formula for its chrom-poly, in terms of the following type of polynomial product.

Given two polynomials \( f \) and \( g \), we define their **falling product** \( h := f \downarrow g \), as following.

1. Write each w.r.t the falling-factorial basis, i.e

\[
f = \sum_{j=0}^{J} \alpha_j t_j \quad \text{and} \quad g = \sum_{k=0}^{K} \beta_k t_k,
\]

as shown in \((1b)\).

2. Compute \( \vec{\gamma} := \vec{\alpha} \circ \vec{\beta} = (\gamma_0, \gamma_1, \ldots, \gamma_{J+K}) \), the convolution.

3. Define \( h := \sum_{n=0}^{J+K} \gamma_n t_n \).

**Defn.** A **Stirling partition** of graph \( G = (\mathbb{V}, E) \), is a partition of \( \mathbb{V} \) into non-empty subsets (called the **atoms** of the ptn) so that no two adjacent vertices are in the same atom. [I.e, each atom is an “independent set”.]

For natnum \( \ell \), define the **graph-Stirling number**

\[
7b: \quad \mathcal{S}(G, \ell)
\]

to be the number of \( \ell \)-atom Stirling partitions of \( \mathbb{V} \).

\textbf{7c: Graph-Stirling Thm.} For an \( N \)-vertex\(^\text{74}\) graph \( G \),

\[
\mathcal{P}_G(x) = \sum_{\ell=0}^{N} \mathcal{S}(G, \ell) \cdot [x \downarrow \ell].
\]

**Proof.** Exercise.

\(\diamond\)

\textbf{7d: Full-product Thm.} Consider a graph \( G := H_1 \odot H_2 \).

Then

\[
\mathcal{P}_G = \mathcal{P}_{H_1} \downarrow \mathcal{P}_{H_2}.
\]

**Proof idea.** Fix \( \ell \). The \( \ell \)-atom Stirling partitions of \( G \) are in 1-to-1 correspondence with: Pick natnums with \( j_1 + j_2 = \ell \), then take a \( j_1 \)-atom Stirling ptn of \( H_1 \), and a \( j_2 \)-atom Stirling ptn of \( H_2 \).

\(\diamond\)

\textbf{7e: Example.} Let \( H \) be \( P_2 \cup P_1 \). So \( \mathcal{S}(H, 1) = 0 \), \( \mathcal{S}(H, 2) = 2 \) and \( \mathcal{S}(H, 3) = 1 \). Our \((7c)\) asserts that \( \mathcal{P}_H = \sum_{j=0}^{3} \alpha_j t_j \), where \( \vec{\alpha} = (0, 0, 2, 1) \). I.e,

\[
\mathcal{P}_H(x) = 2[x \downarrow 2] + [x \downarrow 3] = [x \downarrow 2][2 + [x - 2]] \underset{\text{note}}{=} x^2 \cdot [x - 1].
\]

This agrees with \((3b)\) and with \((5a.0)\).

Let’s compute \( G := H \odot K \), where \( K := K_1 \). Our \( G \) is a \( K_3 \) with a new edge attached to a vertex. So \((5a.3)\) says \( \mathcal{P}_G(x) = [x \downarrow 3] \cdot [x - 1] \). What does \((7d)\) say?

Our \( K \) has chr-poly \( x = [x \downarrow 1] \). We set \( \vec{\beta} := (0, 1) \), then compute convolution \( \vec{\gamma} := \vec{\alpha} \circ \vec{\beta} = (0, 0, 0, 2, 1) \).

Thm \((7d)\) asserts that

\[
\mathcal{P}_G(x) = 2[x \downarrow 3] + [x \downarrow 4] = [x \downarrow 3][2 + [x - 3]] \underset{\text{note}}{=} [x \downarrow 3][x - 1].
\]

This agrees with our derivation via \((5a.3)\). \(\square\)

\textbf{7f: Bipartite ex.} Graph \( H := \text{Emp}_3 \) has \( \mathcal{S}(H, 1) = 1 \), \( \mathcal{S}(H, 2) = 3 \), \( \mathcal{S}(H, 3) = 1 \), and so \( \vec{\alpha} = (0, 1, 3, 1) \).

Thus

\[
\mathcal{P}_H(x) = x \cdot [1 + [x - 1][3 + [x - 2]]] = x \cdot [1 + [x^2 - 1]] = x^3.
\]

\(^{74}\)We can start the sum at \( \ell=1 \) except when \( G \) is the void graph. After all, when \( N \) is positive then \( \mathcal{S}(G, 0) \) is zero.
This indeed agrees with (3a).

Since $K_{3,3}$ is the full-product of $H$ with $H$, we compute $\gamma := \vec{\alpha} \oplus \vec{\alpha} = (0, 0, 1, 6, 11, 6, 1)$. Polynomial

$$[x \downarrow 2] + 6[x \downarrow 3] + 11[x \downarrow 4] + 6[x \downarrow 5] + [x \downarrow 6]$$

simplifies (thanks, Maple) to

$$x \cdot [x-1] \cdot [x^4 - 8x^3 + 28x^2 - 47x + 31].$$

Happily, this agrees with (6d). □

### Spanning subgraphs

Fix a $G = (V, E)$ with $N$ vertices and $L$ edges. Each subset $S \subset E$ can be interpreted as $(V, S)$, a spanning subgraph of $G$. [Thus $G$ has $2^L$ many spanning subgraphs.]

Let $c(S)$ denote the number of connected components of $S$.

#### 8: CPSS Thm (Chromatic-Poly Spanning Subgraph).

The chromatic polynomial of $G$ satisfies

$$p_G(x) = \sum_{S : S \subset \mathcal{P}} (-1)^{|S|} \cdot x^{c(S)}.$$  

Exer E4: Prove this. Think Inclusion-Exclusion. [Hint: See the pamphlet on our Teaching Page.]

**Comparison.** We can paraphrase Theorems (8) and (7c) as saying: Spanning subgraphs express $p_G$ w.r.t the standard basis $(e_j)_{j=0}^{n}$ [see (1b)], whereas Graph-Stirling numbers write $p_G$ w.r.t the falling-factorial basis $(t_n)_{n=0}^{\infty}$.

#### 9a: Orientations.

An orientation of $G = (V, E)$ is putting a direction on each edge, creating a digraph; so $G$ has $2^{|E|}$ many orientations. An orientation of $G$ is acyclic if it has no directed-cycles. Use $A(G)$ for the number of acyclic orientations of $G$.

An orientation of $K_N$ is called an $N$-tournament; there are $2^N$ of them. E.g., $K_4$ has $2^6 = 64$ orientations. Exer E5: Prove that $A(K_N)$ equals $|N!|$. □

#### 9b: Acyclic-count Thm (Richard Stanley).

On $N$-vertex $G$,

$$A(G) = [-1]^N \cdot P_G(-1).$$  

**Sketch.** The idea is to establish an analog of (2a),

$$A(G) = A(G \setminus \{\alpha\}) + A(G / \alpha),$$

for each edge $\alpha$ of $G$, as follows. Let $M := G \setminus \{\alpha\}$.

- Each acyclic-orient of $M$ extends to at least one acyclic-orient of $G$.
- The number of acyclic-orient of $M$ which give rise to two acyclic-orientations of $G$, is $A(G / \alpha)$.

#### 10: Generalizing full-product.

Let $G$ be the set of all graphs. Consider a graph $S = (V, E)$ and a fnc $F : V \rightarrow G$; we’ll often write $F(u)$ as $F_u$. Use $N_u | L_u$ for the number of vertices/edges of $F_u$.

The “full product of $F$ over $S$”

$$G := \bigodot_S(F)$$

is $S$, but where each $S$-vertex $u$ has been replaced by a copy of graph $F_u$. Moreover, for each $S$-edge $v \rightarrow w$:

Each $F_v$-vertex $v'$ and each $F_w$-vertex $w'$, are the endpoints of a $G$-edge.

Thus $\bigodot_S(F)$ has $\sum_{u \in V} N_u$ many vertices. It has

$$\bigoplus_{u \in V} L_u + \sum_{v \neq w} [N_v \cdot N_w]$$

many edges, where the second sum is over all $S$-edges $v \rightarrow w$. In the case that $S$ is a single edge, $p_2$, we recover the full-product as defined above (7a).
Full-product from (7a) is associative, so expressions such as $H_1 \odot H_2 \odot \ldots \odot H_5$ make sense. If we define $\mathcal{F}(j) := H_j$, then

10b: $H_1 \odot H_2 \odot \ldots \odot H_5 = \odot_{K_5}(\mathcal{F})$,

regarding $[1..5]$ as the vertex-set of $K_5$. $\square$

Exer: Is there a formula for the chromatic number/polynomial of $\odot_S(\mathcal{F})$, in terms of corresponding information about $S$ and function $\mathcal{F}$? What about special cases, e.g. $S = P_3$? Or $S = C_{\text{Even}}$?