Chromatic polynomial of a graph

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
17 November, 2017 (at *12:08*)

Polynomial preliminaries. A poly(nomial) such as $f(x) = 8x^2 - 7x + 2$ can be written as $8\mathbf{e}_2 - 7\mathbf{e}_1 + 2\mathbf{e}_0$, where $\mathbf{e}_j(x) := x^j$. As a linear combination,

$$\dagger: \qquad f = \sum_{j=0}^{2} \alpha_j \mathbf{e}_j\,,$$

where $\vec{\boldsymbol{\alpha}} := (\alpha_0, \alpha_1, \alpha_2) = (2, \text{-}7, 8)$. The product of f with a degree-3 polynomial $g = \sum_{k=0}^{3} \beta_k \mathbf{e}_k$ is

$$f \cdot g = \sum_{n=0}^{5} \gamma_n \mathbf{e}_n\,,$$

where, summing over natnum-pairs (j, k), each

$$1\text{a}: \qquad \gamma_n = \sum_{(j,k):\, j+k=n} [\alpha_j \cdot \beta_k]\,.$$

This $\vec{\boldsymbol{\gamma}} = (\gamma_0, \ldots, \gamma_5)$ is $\vec{\boldsymbol{\alpha}} \circledast \vec{\boldsymbol{\beta}}$, the ***convolution*** of $\vec{\boldsymbol{\alpha}}$ with $\vec{\boldsymbol{\beta}}$. An alternative notation to (†) is

$$\ddagger: \qquad f = \sum_{j=0}^{\infty} \alpha_j \mathbf{e}_j\,,$$

where $\alpha_j = 0$ when $j > 2$. Such an $\vec{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \ldots)$ is an ***eventually-zero*** sequence.

We'll also need the ***falling-factorial*** polynomials,

$$[\![x \downarrow N]\!] := x \cdot [x-1] \cdot [x-2] \cdots [x - [N-1]]$$

(a product of N many terms), where $N \in \mathbb{N}$. How to write a poly such as $F(x) := 2x^3 - x^2 - 3x + 3$ in terms of falling factorials? The coeff of x^3 is 2, so we subtract,

$$F(x) - 2\cdot[\![x \downarrow 3]\!] = 5x^2 - 7x + 3\,.$$

From this we subtract $5\cdot[\![x \downarrow 2]\!]$, producing

$$\text{-}2x + 3\,.$$

From *this* we subtract $\text{-}2[\![x \downarrow 1]\!]$, yielding 3. From 3 we subtract $3\cdot[\![x \downarrow 0]\!]$, ending with zero. Thus

$$F(x) = 3\cdot[\![x \downarrow 0]\!] + \text{-}2\cdot[\![x \downarrow 1]\!] + 5\cdot[\![x \downarrow 2]\!] + 2\cdot[\![x \downarrow 3]\!]\,.$$

Letting $\mathbf{t}_n(x) := [\![x \downarrow n]\!]$, we have equality

$$1\text{b}: \qquad \begin{aligned} F &= 3\mathbf{e}_0 + \text{-}3\mathbf{e}_1 + \text{-}1\mathbf{e}_2 + 2\mathbf{e}_3 \\ &= 3\mathbf{t}_0 + \text{-}2\mathbf{t}_1 + 5\mathbf{t}_2 + 2\mathbf{t}_3\,. \end{aligned}$$

We have done a change-of-basis computation, from basis $(\mathbf{e}_j)_{j=0}^{\infty}$ to the $(\mathbf{t}_n)_{n=0}^{\infty}$ basis. □

Chromatic form. Looking ahead, consider a monic intpoly $\boldsymbol{\mathcal{P}}()$ with $R \in \mathbb{N}$ many [not necessarily distinct] integer-roots $Z_1, Z_2, \ldots, Z_R$. Writing $\boldsymbol{\mathcal{P}}(x)$ in ***chromatic form*** means writing

$$1\text{c}: \qquad \boldsymbol{\mathcal{P}}(x) = \Big[\prod_{j=1}^{R} [x - Z_j]\Big] \cdot f(x)\,,$$

where [either "$\cdot f(x)$" is absent or] f is a monic intpoly with no integral roots. Indeed, courtesy the Gauss Lemma for polynomials, our f has *no rational roots.*

When $\boldsymbol{\mathcal{P}}$ is a chromatic polynomial, its integer-root part $\prod_j [x - Z_j]$ should be written in form

$$*: \qquad x^{e_0} \cdot [x-1]^{e_1} \cdot [x-2]^{e_2} \cdots [x - [K-1]]^{e_{K-1}}\,,$$

often mixed with falling-factorials, e.g

$$**: \qquad [\![x \downarrow 2]\!]^5 \cdot [\![x \downarrow 4]\!] \cdot [x-1]^7\,.$$

See (4d), the Chromatic-polynomial Corollary. □

Graph terminology.

Use K_n for the ***complete graph*** on n vertices, and $K_{j,k}$ for the ***complete (j,k)-bipartite graph***. Use C_n for the ***cyclic graph*** with n vertices and n edges [C_1 has a single self-edge; C_2 has 2 edges between 2 vertices; for $n \geqslant 3$, our C_n is a simple graph]. Use Emp_n for the n-vertex graph with *no* edges; an "*Emp*ty graph".

Coloring. Consider $G = (\mathbb{V}, \mathbb{E})$, a finite graph, [loops, multiple-edges ok] with $N := |\mathbb{V}|$, and $L := |\mathbb{E}|$. Use $\mathbf{c}(G)$ for the # of ***connected components*** of G.

For $k \in \mathbb{N}$, a "***k-coloring*** of G" means to assign a "color", an element of $[1\,..\,k]$, to each vertex, so that no two ***neighbors*** [the end-vertices of an edge] have the same color. A k-coloring is ***full*** if it uses all k colors. The ***chromatic number*** of G, written $\chi(G)$, is the minimum number of colors needed. I.e, it is the unique k so that there *is* a k-coloring of G, and every k-coloring of G is full.

For $k = 0, 1, 2, \ldots$, let $\boldsymbol{\mathcal{P}}_G(k)$ be the *number* of k-colorings of G. Note $\boldsymbol{\mathcal{P}}_{Emp_0}(k) = 1$ since the ***void graph*** [no vertices] has exactly one k-coloring. Chromatic number $\chi(G)$ is the smallest natnum k for which $\boldsymbol{\mathcal{P}}_G(k)$ is positive.[♡1]

[♡1] If G has a loop, then $\boldsymbol{\mathcal{P}}_G() \equiv 0$ and $\chi(G) := \infty$.

If G has multiple edges between vertices $\mathbf{u}, \mathbf{v}$, then replacing them by a single edge will not change the chromatic poly/number. Without announcement we will do this; effectively, we compute chromatic polys only of *simple* graphs.

Deletion-contraction. Consider an N-vertex G and edge $\alpha \in \mathbb{E}$. Let $G \setminus \{\alpha\}$ mean to ***delete*** edge α; no vertices are removed, so $G \setminus \{\alpha\}$ still has N vertices. In contrast, use G/α to mean the graph with $N-1$ vertices, where we have ***contracted*** α, so that its two endpoints[♡2] become a single vertex, and α is gone.

2: Deletion-contraction Thm. *On N-vertex simple graph $G = (\mathbb{V}, \mathbb{E})$, function $\mathcal{P}_G()$ is a degree-N monic polynomial. For each $\alpha \in \mathbb{E}$,*

2a: $$\mathcal{P}_G(x) = \mathcal{P}_{G \setminus \{\alpha\}}(x) - \mathcal{P}_{G/\alpha}(x),$$

as polynomials. Also, $\mathcal{P}_G(x)$ has no constant term, except when G is the void graph. ♢

Pf. The only $N{=}0$ graph is void, and $\mathcal{P}_{Emp_0}(x)$ is constant-1, which is monic and of degree zero. Fixing $N \geqslant 1$, we induct on the number, L, of edges. The $L{=}0$ case is trivial, since $\mathcal{P}_{Emp_N}(x)$ is x^N.

As $G \setminus \{\alpha\}$ has $L-1$ edges, poly $\mathcal{P}_{G \setminus \{\alpha\}}$ is monic of degree-N. And G/α has $N-1$ vertices, so $\mathcal{P}_{G/\alpha}$ is a degree-$[N-1]$ poly. Since their difference is a monic degree-N poly, ISTEstablish the (2a) recurrence. We will show that

2b: $$\mathcal{P}_G(k) + \mathcal{P}_{G/\alpha}(k) = \mathcal{P}_{G \setminus \{\alpha\}}(k),$$

for each posint k. This implies equality as polynomials, since we will have equal outputs, for $N+1$ many values of k.

The endpoints of α, call them $\mathbf{u}$ and $\mathbf{v}$. Consider a coloring of G/α, but split apart the combined vertex back into separate vertices $\mathbf{u}$ and $\mathbf{v}$ [and don't put in edge α]. This is now a coloring of $G \setminus \{\alpha\}$ that gives $\mathbf{u}$ and $\mathbf{v}$ the *same* color. In contrast, each coloring of G gives distinct colors to $\mathbf{u}$ and $\mathbf{v}$; so removing α gives a coloring of $G \setminus \{\alpha\}$ with $\mathbf{u}$ and $\mathbf{v}$ having *different* colors. Hence (2b). ♦

[♡2] If $\mathbf{u}, \mathbf{v}$ have multiple edges, then contracting a $\mathbf{u}$—$\mathbf{v}$ edge creates a loop, hence a graph with $\mathcal{P}() \equiv 0$. This is ok, but inefficient; typically, first collapse each multi-edge to a single edge.

Properties of $\mathcal{P}_G$. Initially, $N \in \mathbb{N}$.

3a: $$\mathcal{P}_{Emp_N}(x) = x^N. \qquad \mathcal{P}_{K_N}(x) = [\![x \downarrow N]\!].$$

Now, $N \geqslant 1$. The N-vertex ***path graph***, P_N, is a special case of a tree. Below, T_N is an arbitrary tree on N vertices. Easily,

3b: $$\mathcal{P}_{P_N}(x) = \mathcal{P}_{T_N}(x) = x \cdot [x-1]^{N-1}.$$

For $N \geqslant 2$,

3c: $$\mathcal{P}_{C_N}(x) = [x-1]^N + [\text{-}1]^N [x-1].$$

In particular,

3c′: $$\mathcal{P}_{C_4}(x) = x \cdot [x-1] \cdot [x^2 - 3x + 3].$$

Proof of (3c). Note C_2 becomes the path P_2, after collapsing the multi-edge, hence has chrom-poly $x \cdot [x-1]$, which is what $(3c)_{N=2}$ equals; the base case.

Applying (2a) to $G := C_{N+1}$ and an edge α, gives $G \setminus \{\alpha\} = P_{N+1}$ and $G/\alpha = C_N$. So $\mathcal{P}_G(x)$ equals

$$\Big[x \cdot [x-1]^N\Big] - \Big[[x-1]^N + [\text{-}1]^N [x-1]\Big].$$

And this reduces to $[x-1]^{N+1} + [\text{-}1]^{N+1}[x-1]$. ♦

From each vertex of C_N, attach an edge to a common vertex, $\mathbf{u}_{N+1}$. This ***wheel graph*** W_{N+1}, has $N+1$ vertices and $2N$ edges. For $N \geqslant 2$, then,

3d: $$\mathcal{P}_{W_{N+1}}(x) = x \cdot \mathcal{P}_{C_N}(x-1) \overset{\text{note}}{=\!=} x\Big[[x-2]^N + [\text{-}1]^N [x-2]\Big].$$

(Fixing a posint x, there are x choices to color vertex $\mathbf{u}_{N+1}$, hence $x-1$ colors available for the embedded C_N.) E.g, (3c′) gives $\mathcal{P}_{C_4}(x-1) = [x-1] \cdot [x-2] \cdot [x^2 - 5x + 7]$. So

3d′: $$\mathcal{P}_{W_5}(x) = [\![x \downarrow 3]\!] \cdot [x^2 - 5x + 7].$$

Exer E1: Suppose $\mathcal{P}_G(x) = x \cdot [x-1]^{N-1}$. Prove that G is a tree.

Defn. An ***alternating polynomial*** $h(x)$ has form

4: $$B_N x^N - B_{N-1} x^{N-1} + B_{N-2} x^{N-2} - B_{N-3} x^{N-3} + \ldots + [\text{-}1]^j B_{N-j} x^{N-j} + \ldots + [\text{-}1]^{N-K} B_K x^K,$$

where $N \geqslant K$ are natnums, and each $B_j > 0$. Call index K "the ***low-degree*** of h", written $\mathrm{LD}(h)$. Here is an easy exercise. *For f and g alternating-polys:*

4a: *Product $f \cdot g$ is alternating, and* $\mathrm{LD}(f \cdot g) = \mathrm{LD}(f) + \mathrm{LD}(g)$.

4b: *If* $\mathrm{Deg}(f) = 1 + \mathrm{Deg}(g)$, *then $f - g$ is alternating, with* $\mathrm{LD}(f - g) = \mathrm{Min}(\mathrm{LD}(f), \mathrm{LD}(g))$. □

4c: Chromatic polynomial Theorem. *For a non-void simple graph $G = (\mathbb{V}, \mathbb{E})$, write its chromatic polynomial $\mathcal{P}_G(x)$ in form (4). Then*

∗: *$\mathcal{P}_G$ is a monic alternating intpoly, with $N = |\mathbb{V}|$, $B_{N-1} = |\mathbb{E}|$ and $K = \mathbf{c}(G)$.* ◊

[This $\mathbf{c}(G)$ is the number of connected-components.]

Proof. [We have "monic" and "$N = |\mathbb{V}|$" from (2).] First suppose G decomposes into (non-void) disjoint subgraphs $H_1 \sqcup H_2$. Let $N_j := |\mathbb{V}_{H_j}|$, $L_j := |\mathbb{E}_{H_j}|$, $K_j := \mathbf{c}(H_j)$ and $f_j := \mathcal{P}_{H_j}$. So f_j has form

$$f_j(x) = x^{N_j} - L_j x^{N_j - 1} + \ldots + C_j x^{K_j}$$

with $C_j \neq 0$. Easily, $\mathcal{P}_G = f_1 \cdot f_2$, hence is alternating, by (4a), with low-degree $K_1 + K_2 \stackrel{\text{note}}{=\!=} \mathbf{c}(G)$. The penultimate coeff of $f_1 \cdot f_2$ is $-[L_1 + L_2]$, which indeed is the number of G-edges. So *WLOG, G is connected.*

When G is connected. [Recall $N \geqslant 1$ since G is non-void.] Pick a G-edge, α, whose removal does not disconnect G; if there is none such, then G is a tree [possibly the edgeless tree], where (∗) evidently holds.

Hence both $G \smallsetminus \{\alpha\}$ and G/α are connected, Thus $f := \mathcal{P}_{G \smallsetminus \{\alpha\}}$ and $g := \mathcal{P}_{G/\alpha}$. each satisfy (∗). So $f - g$ is alternating, by (4b), and $\mathrm{LD}(f - g) = \mathrm{Min}(1, 1) = 1$, which is indeed the number of connected-comps of G.

Counting edges. Let $L := |\mathbb{E}|$. Our $G \smallsetminus \{\alpha\}$ has $L - 1$ edges, thus $f(x) = x^N - [L-1]x^{N-1} + \ldots$. And g is monic, $g(x) = x^{N-1} - \ldots$. The difference thus has form $\mathcal{P}_G(x) = x^N - Lx^{N-1} + \ldots$, as desired. ♦

4d: Chromatic-polynomial Corollary. *Polynomial $\mathcal{P}_G$ has no negative roots. Setting $K := \chi(G)$, we can therefore write $\mathcal{P}_G(x)$ in chromatic form as*

$$x^{e_0} \cdot [x-1]^{e_1} \cdot [x-2]^{e_2} \cdots [x - [K-1]]^{e_{K-1}} \cdot f(x)\,,$$

with each $e_j \in \mathbb{Z}_+$. Moreover, f is [either absent, or] a monic intpoly, with no negative real[♡3]*roots, and no rational roots.* ◊

Proof. An alternating-poly evaluated at a negative real, yields a sum of posreals, hence is positive. Finally, since f is ***primitive*** (the GCD of its coeffs is 1) each rational root must be integral, by the Gauss Lemma for polynomials.

Exer E3.1415: *"The composition of two chromatic-polys is always a chromatic-poly."* Prove, or CEX.

Gluing

The next result uses

5: *. . . two graphs H_j, for $j = 1,2$, with N_j many vertices and L_j many edges. Let $h_j() := \mathcal{P}_{H_j}()$.*

5a: Gluing lemma. *When G is built from non-void simple graphs H_1 and H_2 by. . .*

*0: **disjoint union**, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)$.*

*1: picking a vertex $\mathbf{u}_j$ in H_j and identifying the two vertices, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) \big/ x$. This G has $N_1 + N_2 - 1$ many vertices and $L_1 + L_2$ many edges. Call this G a **point-gluing** of H_1 and H_2.*

2: picking an edge α_j in H_j and identifying the two edges (choose an orientation), then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) \big/ x[x-1]\,.$$

*This G has $N_1 + N_2 - 2$ vertices and $L_1 + L_2 - 1$ edges. Call this G an **edge-gluing** of H_1 with H_2.*

[♡3] However, f *can* have *complex* roots with negative real-part.

3: *picking a vertex $\mathbf{u}_j$ in H_j and putting in a (new) edge between them, then*

$$\mathcal{P}_G(x) \;=\; h_1(x)\cdot h_2(x)\cdot \tfrac{x-1}{x}\,. \qquad \diamondsuit$$

This G has N_1+N_2 many vertices and L_1+L_2+1 many edges. This G is a ***new-edge-attaching*** *of H_1 and H_2.*

Proof. Exercise. ♦

5b: *Gluing on a subgraph.* Graph $M = (\mathbb{V},\mathbb{E})$ is a ***subgraph*** of $H = (\mathbb{V}',\mathbb{E}')$ if there exist *injections* $\Phi{:}\mathbb{V}\hookrightarrow\mathbb{V}'$ and $\Psi{:}\mathbb{E}\hookrightarrow\mathbb{E}'$ so that:

For each $\alpha\in\mathbb{E}$ with endpoints $\mathbf{u},\mathbf{v}\in\mathbb{V}$, necessarily, the endpoints of $\Psi(\alpha)$ are $\Phi(\mathbf{u})$ and $\Phi(\mathbf{v})$.

Henceforth, *for both injections we'll use a common symbol* (usually Φ) and write $\boxed{\Phi{:}M\hookrightarrow H}$.

Consider graphs H_j as in (5), as well as a graph M. Suppose $\Phi_1{:}M\hookrightarrow H_1$ and $\Phi_2{:}M\hookrightarrow H_2$. Define the

gluing of H_1 with H_2, over (Φ_1,Φ_2)

as the graph G which is the "union" of H_1 and H_2, where for each vertex $\mathbf{u}$ and edge α of M:

5c: *Vertex $\Phi_1(\mathbf{u})$ is identified with $\Phi_2(\mathbf{u})$ and edge $\Phi_1(\alpha)$ is identified with $\Phi_2(\alpha)$.*

So G has $N_1+N_2-|\mathrm{Vertices}(M)|$ many vertices, and $L_1+L_2-|\mathrm{Edges}(M)|$ many edges. When we don't need the details of the gluing, we will refer to G as

a gluing of H_1 with H_2, over M.

NOTATION: I use symbol $\sqcup$ for "***disjointunion***", so let's use $H_1\sqcup H_2$ for gluing over the *void graph*. More generally, use

5d: $$H_1\sqcup_M H_2 \quad \textit{or} \quad H_1\sqcup^{\Phi_1}_{\Phi_2} H_2$$

for gluing over M; the latter, if the details are needed.

Say that M is ***gluing-good*** if, for all graphs H_1, H_2 having M as a subgraph, necessarily

5e: $$\mathcal{P}_G(x) \;=\; \mathcal{P}_{H_1}(x)\cdot\mathcal{P}_{H_2}(x)\Big/\mathcal{P}_M(x)\,,$$

whenever G is a gluing of H_1 with H_2 over M. □

Exer E2.1: Find an infinite family of connected graphs which are gluing-good.

Exer E2.2: *"Each connected graph M is gluing-good".* Find a proof, or CEX.

Bipartite graphs. For natural numbers B and G, the complete bipartite graph $K_{\mathrm{B},\mathrm{G}}$ has all edges between $\mathbf{B} := [1\,..\,\mathrm{B}]$ and $\mathbf{G} := [1\,..\,\mathrm{G}]$, the "Boys" and "Girls", and no other edges.

For natnums B and ℓ, the "***Stirling number*** *of the second kind*", $\mathcal{S}(\mathrm{B},\,\ell)$, is the number of partitions of $[1\,..\,\mathrm{B}]$ into ℓ many non-empty atoms. (I.e, $\mathcal{S}(\mathrm{B},\,\ell)$ is the number of equivalence relations on $[1\,..\,\mathrm{B}]$ that have ℓ many [non-void] equivalence classes.)

6: Sara's Lemma. *Let $f_{\mathrm{B},\mathrm{G}}$ abbreviate $\mathcal{P}_{K_{\mathrm{B},\mathrm{G}}}$. Then*

6a: $$f_{\mathrm{B},\mathrm{G}}(x) \;=\; \sum_{\ell=0}^{\mathrm{B}} \mathcal{S}(\mathrm{B},\,\ell)\cdot[\![x\downarrow\ell]\!]\cdot[x-\ell]^{\mathrm{G}}\,. \qquad \diamondsuit$$

Computing. At $\mathrm{B}=0$, the RhS is $1\cdot 1\cdot[x-0]^{\mathrm{G}}$ which is x^{G}, which is correct.

Once $\mathrm{B}\geqslant 1$ we can start the sum at $\ell=1$, since there are no partitions of the empty set into positively many atoms.

$\boxed{\text{CASE: } \mathrm{B}=1.}$ The RhS is $1\cdot[\![x\downarrow 1]\!]\cdot[x-1]^{\mathrm{G}}$, i.e, $x\cdot[x-1]^{\mathrm{G}}$, which is what (3b) says, as $K_{1,\mathrm{G}}$ is a tree.

$\boxed{\text{CASE: } \mathrm{B}=2.}$ At $\mathrm{B}=2$, our RhS(6*a*) is

$$\begin{array}{c} \mathcal{S}(2,\,1)\cdot[\![x\downarrow 1]\!]\cdot[x-1]^{\mathrm{G}} \\ + \\ \mathcal{S}(2,\,2)\cdot[\![x\downarrow 2]\!]\cdot[x-2]^{\mathrm{G}} \end{array} \;=\; \begin{array}{c} 1\cdot x\cdot[x-1]^{\mathrm{G}} \\ + \\ 1\cdot x[x{-}1]\cdot[x-2]^{\mathrm{G}}\,. \end{array}$$

So $f_{2,\mathrm{G}}(x)$ equals $x\cdot\big[[x-1]^{\mathrm{G}}+[x{-}1]\cdot[x-2]^{\mathrm{G}}\big]$.

Once $\mathrm{G}\geqslant 1$, we have that

6b: $$f_{2,\mathrm{G}}(x) \;=\; x\cdot[x{-}1]\cdot\Big[[x-1]^{\mathrm{G}-1}+[x-2]^{\mathrm{G}}\Big]\,.$$

Plugging in G=2 yields RhS($3c'$), which is reassuring seeing as $K_{2,2}$ equals C_4.

$\boxed{\text{CASE: } \mathrm{B}=3.}$ RhS(6*a*) is a sum of three terms:

$$\begin{aligned} \mathcal{S}(3,\,1)\cdot[\![x\downarrow 1]\!]\cdot[x-1]^{\mathrm{G}} &= 1\cdot x[x-1]^{\mathrm{G}}\,; \\ \mathcal{S}(3,\,2)\cdot[\![x\downarrow 2]\!]\cdot[x-2]^{\mathrm{G}} &= 3\cdot x[x-1][x-2]^{\mathrm{G}}\,; \\ \mathcal{S}(3,\,3)\cdot[\![x\downarrow 3]\!]\cdot[x-3]^{\mathrm{G}} &= 1\cdot x[x-1][x-2][x-3]^{\mathrm{G}}. \end{aligned}$$

As soon as $\mathrm{G}\geqslant 1$, ratio $f_{3,\mathrm{G}}(x)\big/x[x-1]$ equals

6c: $$[x-1]^{\mathrm{G}-1}+3[x-2]^{\mathrm{G}}+[x-2][x-3]^{\mathrm{G}}\,.$$

Using colors *orange*&*blue* there are 2 colorings of $K_{3,\mathrm{G}}$ [since G>0], so (6c) at $x{=}2$ *better* equal 1. Does it?

When G $=1$, the $K_{3,1}$ is a 4-vertex tree, so (6c) with G=1 *better darn well* be $[x-1]^2$. Is it?

FWIW, $(6c)_{G=2}$ is $[x^3-5x^2+10x-7]$; irred.

Evaluating, $(6c)_{G=3}$ gives

6d: $$\frac{f_{3,3}(x)}{x\cdot[x-1]} = x^4-8x^3+28x^2-47x+31\,.$$

This last is irreducible over the rationals. □

Graph-Stirling numbers

For a generalization of "bipartite graph", consider two graphs $H_j = (\mathbb{V}_j, \mathbb{E}_j)$ with N_j vertices and L_j edges. The ***full product*** $G := H_1 \odot H_2$ is their disjoint union, augmented by an edge from each H_1-vertex, to each H_2-vertex. Thus

7a: *$H_1 \odot H_2$ has N_1+N_2 many vertices, and $L_1+L_2+[N_1N_2]$ many edges.*

We'll get a formula for its chrom-poly, in terms of the following type of polynomial product.

Given two polynomials f and g, we define their ***falling product*** $h := f \mathbin{\lrcorner} g$, as following.

❶ Write each w.r.t the *falling-factorial* basis, i.e
$$f = \sum_{j=0}^{J}\alpha_j\mathbf{t}_j \quad\text{and}\quad g = \sum_{k=0}^{K}\beta_k\mathbf{t}_k\,,$$
as shown in (1b).

❷ Compute $\vec{\boldsymbol\gamma} := \vec{\boldsymbol\alpha} \circledast \vec{\boldsymbol\beta} = (\gamma_0,\gamma_1,\ldots,\gamma_{J+K})$, the convolution.

❸ Define $h := \sum_{n=0}^{J+K}\gamma_n\mathbf{t}_n$.

Defn. A ***Stirling partition*** of graph $G=(\mathbb{V},\mathbb{E})$, is a partition of $\mathbb{V}$ into non-empty subsets (called the ***atoms*** of the ptn) so that no two adjacent vertices are in the same atom. [I.e, each atom is an "independent set".]

For natnum ℓ, define the ***graph-Stirling number***

7b: $$\boldsymbol{\mathcal{S}}(G,\,\ell)$$

to be the number of ℓ-atom Stirling partitions of $\mathbb{V}$.

7c: Graph-Stirling Thm. *For an N-vertex*[♡4] *graph G,*
$$\boldsymbol{\mathcal{P}}_G(x) = \sum_{\ell=0}^{N}\boldsymbol{\mathcal{S}}(G,\,\ell)\cdot[\![x\downarrow\ell]\!]\,. \qquad\Diamond$$

Proof. Exercise. ♦

7d: Full-product Thm. *Consider a graph $G := H_1 \odot H_2$. Then*
$$\boldsymbol{\mathcal{P}}_G = \boldsymbol{\mathcal{P}}_{H_1} \mathbin{\lrcorner} \boldsymbol{\mathcal{P}}_{H_2}\,. \qquad\Diamond$$

Proof idea. Fix ℓ. The ℓ-atom Stirling partitions of G are in 1-to-1 correspondence with: *Pick natnums with $j_1+j_2=\ell$, then take a j_1-atom Stirling ptn of H_1, and a j_2-atom Stirling ptn of H_2.* ♦

7e: *Example.* Let H be $P_2 \sqcup P_1$. So $\boldsymbol{\mathcal{S}}(H,\,1)=0$, $\boldsymbol{\mathcal{S}}(H,\,2)=2$ and $\boldsymbol{\mathcal{S}}(H,\,3)=1$. Our (7c) asserts that $\boldsymbol{\mathcal{P}}_H = \sum_{j=0}^{3}\alpha_j\mathbf{t}_j$, where $\vec{\boldsymbol\alpha} = (0,0,2,1)$. I.e,
$$\begin{aligned}\boldsymbol{\mathcal{P}}_H(x) &= 2\cdot[\![x\downarrow 2]\!] + [\![x\downarrow 3]\!]\\ &= [\![x\downarrow 2]\!]\,[2+[x-2]] \stackrel{\text{note}}{=\!=} x^2\cdot[x-1]\,.\end{aligned}$$

This agrees with (3b) and with (5a.0).

Let's compute $G := H \odot K$, where $K := K_1$. Our G is a K_3 with a new edge *attached* to a vertex. So (5a.3) says $\boldsymbol{\mathcal{P}}_G(x) = [\![x\downarrow 3]\!]\cdot[x-1]$. What does (7d) say?

Our K has chr-poly $x = [\![x\downarrow 1]\!]$. We set $\vec{\boldsymbol\beta} := (0,1)$, then compute convolution $\vec{\boldsymbol\gamma} := \vec{\boldsymbol\alpha}\circledast\vec{\boldsymbol\beta} = (0,0,0,2,1)$. Thm (7d) asserts that
$$\begin{aligned}\boldsymbol{\mathcal{P}}_G(x) &= 2\cdot[\![x\downarrow 3]\!] + [\![x\downarrow 4]\!]\\ &= [\![x\downarrow 3]\!]\,[2+[x-3]] \stackrel{\text{note}}{=\!=} [\![x\downarrow 3]\!]\,[x-1]\,.\end{aligned}$$

This agrees with our derivation via (5a.3). □

7f: *Bipartite ex.* Graph $H := \mathit{Emp}_3$ has $\boldsymbol{\mathcal{S}}(H,\,1)=1$, $\boldsymbol{\mathcal{S}}(H,\,2)=3$, $\boldsymbol{\mathcal{S}}(H,\,3)=1$, and so $\vec{\boldsymbol\alpha} = (0,1,3,1)$. Thus
$$\begin{aligned}\boldsymbol{\mathcal{P}}_H(x) &= [\![x\downarrow 1]\!] + 3[\![x\downarrow 2]\!] + [\![x\downarrow 3]\!]\\ &= x\cdot\Big[1+[x-1]\,[3+[x-2]]\Big]\\ &= x\cdot[1+[x^2-1]] = x^3\,.\end{aligned}$$

[♡4] We can start the sum at $\ell=1$ *except* when G is the void graph. After all, when N is positive then $\boldsymbol{\mathcal{S}}(G,\,0)$ is zero.

This indeed agrees with (3a).

Since $K_{3,3}$ is the full-product of H with H, we compute $\vec{\gamma} := \vec{\alpha} \circledast \vec{\alpha} = \big(0,0,1,6,11,6,1\big)$. Polynomial

$$[\![x\downarrow 2]\!] + 6[\![x\downarrow 3]\!] + 11[\![x\downarrow 4]\!] + 6[\![x\downarrow 5]\!] + [\![x\downarrow 6]\!]$$

simplifies (thanks, *Maple*) to

$$x\cdot[x-1]\cdot[x^4 - 8x^3 + 28x^2 - 47x + 31]\,.$$

Happily, this agrees with (6d). □

Spanning subgraphs

Fix a $G = (\mathbb{V},\mathbb{E})$ with N vertices and L edges. Each subset $S \subset \mathbb{E}$ can be interpreted as $(\mathbb{V}, S)$, a spanning subgraph of G. [Thus G has 2^L many spanning subgraphs.] Let $\mathbf{c}(S)$ denote the number of connected components of S.

8: CPSS Thm (Chromatic-Poly Spanning Subgraph). *The chromatic polynomial of G satisfies*

$$*:\qquad \mathcal{P}_G(x) \;=\; \sum_{S:\ S\subset\mathbb{E}} (-1)^{|S|}\cdot x^{\mathbf{c}(S)}\,. \qquad \lozenge$$

Exer E4: Prove this. Think *Inclusion-Exclusion.* [*Hint:* See the pamphlet on our TEACHING PAGE.]

Comparison. We can paraphrase Theorems (8) and (7c) as saying: *Spanning subgraphs* express $\mathcal{P}_G$ w.r.t the standard basis $(\mathbf{e}_j)_{j=0}^{\infty}$ [see (1b)], whereas *Graph-Stirling numbers* write $\mathcal{P}_G$ w.r.t the falling-factorial. basis $(\mathbf{t}_n)_{n=0}^{\infty}$. □

8a: *Ex.* As K_3 has 3 edges, it has $2^3 = 8$ spanning subgraphs. The *no-edge* graph has 3 CCs [Connected Components], hence contributes a $[-1]^0\cdot x^3 = x^3$ term.

The three *one-edge* spanning subgraphs each have 2 CCs; these contribute $3\cdot[-1]^1\cdot x^2 = -3x^2$.

The three *two-edge* spanning subgraphs, each have 1 CC, contributing $3\cdot[-1]^2\cdot x^1 = 3x$. Finally, the unique *three-edge* spanning subgraph has 1 CC, hence contributing a $[-1]^3\cdot x^1 = -x$ term. Adding,

$$\begin{aligned}\mathcal{P}_{K_3}(x) &= x^3 - 3x^2 + 3x - x \\ &= x^3 - 3x^2 + 2x \overset{\text{note}}{=\!=} [\![x\downarrow 3]\!]\,.\end{aligned} \qquad □$$

Applications/Extensions

Here is one:

9a: *Orientations.* An ***orientation*** of $G = (\mathbb{V},\mathbb{E})$ is putting a direction on each edge, creating a digraph; so G has $2^{|\mathbb{E}|}$ many orientations. An orientation of G is ***acyclic*** if it has no directed-cycles. Use $\mathcal{A}(G)$ for the number of acyclic orientations of G.

An orientation of K_N is called an N***-tournament***; there are $2^{\binom{N}{2}}$ of them. E.g, K_4 has $2^6 = 64$ orientations. Exer E5: Prove that $\mathcal{A}(K_N)$ equals $[N!]$. □

9b: Acyclic-count Thm (Richard Stanley). *On N-vertex G,*

$$\mathcal{A}(G) \;=\; [-1]^N\cdot\mathcal{P}_G(-1)\,. \qquad \lozenge$$

Sketch. The idea is to establish an analog of (2a),

$$\textbf{9c:}\qquad \mathcal{A}(G) \;=\; \mathcal{A}(G\smallsetminus\{\alpha\}) + \mathcal{A}(G/\alpha)\,,$$

for each edge α of G, as follows. Let $M := G\smallsetminus\{\alpha\}$.

- Each acyclic-orient of M extends to *at least one* acyclic-orient of G.
- The number of acyclic-orients of M which give rise to *two* acyclic-orients of G, is $\mathcal{A}(G/\alpha)$. ♦

10: *Generalizing full-product.* Let $\mathcal{G}$ be the set of all graphs. Consider a graph $S = (\mathbb{V},\mathbb{E})$ and a fnc $\mathcal{F}{:}\,\mathbb{V}\to\mathcal{G}$; we'll often write $\mathcal{F}(\mathbf{u})$ as $\mathcal{F}_{\mathbf{u}}$. Use $N_{\mathbf{u}}|L_{\mathbf{u}}$ for the number of vertices|edges of $\mathcal{F}_{\mathbf{u}}$.

The "***full product*** of $\mathcal{F}$ over S"

$$G := \bigodot_S(\mathcal{F})$$

is S, but where each S-vertex $\mathbf{u}$ has been replaced by a copy of graph $\mathcal{F}_{\mathbf{u}}$. Moreover, for each S-edge $\mathbf{v}$—$\mathbf{w}$:

> *Each $\mathcal{F}_{\mathbf{v}}$-vertex $\mathbf{v}'$ and each $\mathcal{F}_{\mathbf{w}}$-vertex $\mathbf{w}'$, are the endpoints of a G has an edge.*

Thus $\bigodot_S(\mathcal{F})$ has $\sum_{\mathbf{u}\in\mathbb{V}} N_{\mathbf{u}}$ many vertices. It has

$$\textbf{10a:}\qquad \Big[\sum_{\mathbf{u}\in\mathbb{V}} L_{\mathbf{u}}\Big] + \sum_{\mathbf{v},\mathbf{w}}[N_{\mathbf{v}}\cdot N_{\mathbf{w}}]$$

where the second sum is over all S-edges $\mathbf{v}$—$\mathbf{w}$. In the case that S is a single edge, P_2, we recover the full-product as defined above (7a).

Full-product from (7a) is associative, so expressions such as $H_1 \odot H_2 \odot \ldots \odot H_5$ make sense. If we define $\mathcal{F}(j) \coloneqq H_j$, then

10b: $$H_1 \odot H_2 \odot \ldots \odot H_5 \quad = \quad \bigodot_{K_5}(\mathcal{F}),$$

regarding $[1 .. 5]$ as the vertex-set of K_5. $\square$

Exer: Is there a formula for the chromatic number/polynomial of $\bigodot_S(\mathcal{F})$, in terms of corresponding information about S and function $\mathcal{F}$? What about special cases, e.g $S = P_3$? Or $S = C_{\text{Even}}$?

Filename: `Problems/GraphTheory/chromatic-num.latex`
As of: *Wednesday 12Jan2005*. Typeset: *17Nov2017* at *12:08*.