C1 solutions

"Temporary Papa"

31 August, 2015 (at *10:17*)

C1: With $\mathbf{V}$ a vectorspace over field $\mathbf{F}$, suppose $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \ldots \subset \mathbf{V}$ are subspaces. Define sets

$$\mathbf{X} := \mathbf{W}_1 \cup \mathbf{W}_2\,; \quad \mathbf{S} := \left\{ \mathbf{u}_1 + 2\mathbf{u}_2 \;\middle|\; \begin{matrix} \mathbf{u}_1 \in \mathbf{W}_1 \text{ and} \\ \mathbf{u}_2 \in \mathbf{W}_2 \end{matrix} \right\};$$

$$\mathbf{Y} := \bigcap_{n=1}^{\infty} \mathbf{W}_n\,.$$

OYOSOP, *prove*, or give an *explicit CEX* (field, VS and vectors/scalars) to: "*Set* $\mathbf{X}$ *is a VS.*" Ditto $\mathbf{S}$ and $\mathbf{Y}$.

1: Assertion. *"The union,* $\mathbf{X}$*, of two subspaces is always subspace."* ♢

This assertion is False. *It is a Canard, a Snare and Delusion, say I, Josephine Student. I shall reveal the fallacy with a glorious CEX!*

CEX. Over field $\mathbf{F} := \mathbb{R}$, consider VS $\mathbf{X} := \mathbb{R}\times\mathbb{R}$ and subspaces $\mathbf{W}_1 := \mathbb{R}\times\{0\}$ and $\mathbf{W}_2 := \{0\}\times\mathbb{R}$. Note that

$$(5,0) \in \mathbf{W}_1 \subset \mathbf{X} \quad \text{and}$$
$$(0,7) \in \mathbf{W}_2 \subset \mathbf{X}\,.$$

Yet the sum $(5,7) \stackrel{\text{note}}{=\!=} (5,0) + (0,7)$ is *not* in $\mathbf{X}$. ♦

Pf that $\mathbf{S}$ *is a VS.* Fix two general vectors $\mathbf{u}_1 + 2\mathbf{u}_2$ and $\mathbf{v}_1 + 2\mathbf{v}_2$ in $\mathbf{S}$, and a scalar α. By associativity and commutativity of vec-addition, and that SVM distributes over vec-addition, their sum equals

$$*: \qquad [\mathbf{u}_1 + \mathbf{v}_1] + 2[\mathbf{u}_2 + \mathbf{v}_2]\,.$$

Since each $\mathbf{W}_j$ is sealed under addition, sum $\mathbf{u}_1 + \mathbf{v}_1$ lies in $\mathbf{W}_1$, and $\mathbf{u}_2 + \mathbf{v}_2$ lies in $\mathbf{W}_2$. Hence $(*)$ lies in $\mathbf{S}$.

Similarly, the product $\alpha[\mathbf{u}_1 + 2\mathbf{u}_2]$ equals

$$**: \qquad [\alpha\mathbf{u}_1] + 2[\alpha\mathbf{u}_2] \quad =: \quad \mathbf{s}\,.$$

Since $\mathbf{W}_j$ is a VS, necessarily $\mathbf{W}_j \ni \alpha\mathbf{u}_j$. So $\mathbf{S} \ni \mathbf{s}$.

Lastly, $\widehat{\mathbf{0}} = \widehat{\mathbf{0}} + [2\cdot\widehat{\mathbf{0}}] \stackrel{\text{note}}{\in} \mathbf{S}$. ♦

Pf that $\mathbf{Y}$ *is a VS.* Fix vectors $\mathbf{u},\mathbf{v} \in \mathbf{Y}$ and scalar α. Sum $\mathbf{u}+\mathbf{v}$ lies in $\mathbf{Y}$ IFF for *each* $n \in \mathbb{Z}_+$, space $\mathbf{W}_n$ owns $\mathbf{u}+\mathbf{v}$. Ditto for the product $\alpha\mathbf{u}$.

So fix an $n \in \mathbb{Z}_+$. Since $\mathbf{W}_n \supset \mathbf{Y}$, our $\mathbf{W}_n$ owns $\mathbf{u}$ and $\mathbf{v}$. Being a VS, $\mathbf{W}_n$ owns $\mathbf{u}+\mathbf{v}$ and $\alpha\mathbf{u}$.

Lastly, each $\mathbf{W}_n$ owns $\widehat{\mathbf{0}}$, so $\mathbf{Y}$ also owns $\widehat{\mathbf{0}}$. ♦

Filename: `Classwork/LinearAlg/LinA2010t/c1-soln.LinA2010t.latex`
As of: *Monday 31Aug2015.* Typeset: *31Aug2015* at *10:17.*