C1: 195 For $m \in [1,\infty)$, let Ω_m denote the equi-angular spiral which crosses the x-axis at $Q_1 := (1,0)$ and, one wrap later, at $Q_m := (m,0)$. (When $m=1$, the "spiral" degenerates into a circle.) Let $\mathbf{P}_m = (\alpha_m, \beta_m)$ be the parm'tion of Ω_m st. $\mathbf{P}_m(0) = Q_1$, $\mathbf{P}_m(2\pi) = Q_m$ and $\mathbf{P}_m(t)$ wraps once whenever t increases by 2π.

So $\alpha_m(t) = m^{t/[2\pi]} \cdot \cos(t)$.

& $\beta_m(t) = \mathrm{e}^{\mathsf{K}t} \cdot \sin(t)$, where $\mathsf{K} := \frac{\log(m)}{2\pi}$.

Viewing $\mathbf{P}_m$ as *position*, we get the *speed fnc*

$$*0: \quad \|\mathbf{P}'_m(t)\| = \mathsf{B}\mathrm{e}^{\mathsf{K}t}, \text{ where } \mathsf{B} := \sqrt{\mathsf{K}^2 + 1^2}\,.$$

a Length $L_m := \operatorname{Len}\binom{Q_1 \smile Q_m}{\text{wrap}} \stackrel{\text{note}}{=} \int_0^{2\pi} \mathsf{B}\mathrm{e}^{\mathsf{K}t}\,\mathrm{d}t$, which equals

$$*1: \quad \begin{aligned} \tfrac{\mathsf{B}}{\mathsf{K}} \cdot [m-1] &= \sqrt{1^2 + \left[\tfrac{2\pi}{\log(m)}\right]^2} \cdot [m-1] \\ &= T_m \cdot [m-1]\,. \quad \text{Alternatively,} \\ \tfrac{\mathsf{B}}{\mathsf{K}} \cdot [m-1] &= \tfrac{\sqrt{\mathsf{K}^2+1^2}}{\mathsf{K}} \cdot \left[\mathrm{e}^{2\pi\mathsf{K}} - 1\right]. \end{aligned}$$

Total-len of Ω_m going in to the origin from Q_1, is

$$*2: \quad T_m = \int_{-\infty}^{0} \mathsf{B}\mathrm{e}^{\mathsf{K}t}\,\mathrm{d}t = \tfrac{\mathsf{B}}{\mathsf{K}} = \sqrt{1^2 + \left[\tfrac{1}{\mathsf{K}}\right]^2}\,.$$

An alternative derivation is

$$T_m \stackrel{\text{note}}{=} L_m \cdot \sum_{j=1}^{\infty} \left[\tfrac{1}{m}\right]^j = \tfrac{\mathsf{B}}{\mathsf{K}} = \sqrt{1^2 + \left[\tfrac{2\pi}{\log(m)}\right]^2}\,.$$

As $m \searrow 1$. Geometry tells us that one wrap of Ω_m tends to the unit-circle. So WExpect $L_m \to 2\pi \cdot 1$. And T_m approaches the length of ∞ly many trips around the circum of a circle; so WExpect $T_m \to \infty$.

Letting $\mathsf{R}_m := \frac{m-1}{\log(m)}$, note $[L_m]^2$ equals

$$[m-1]^2 + [2\pi]^2 \cdot [\mathsf{R}_m]^2\,.$$

By l'Hôpital's theorem

$$*3: \quad \lim_{m\to 1} \mathsf{R}_m = \lim_{m\to 1} \frac{1}{1/m} \stackrel{\text{note}}{=} 1\,.$$

Thus

$$\left[\lim_{m\to 1} L_m\right]^2 = 0^2 + \left[[2\pi]^2 \cdot 1^2\right].$$

Since each L_m is non-negative, $\lim_{m\to 1} L_m = 2\pi$. □

As $m \nearrow \infty$. Imagine m enormous. Then the portion of Ω_m going in from Q_1 looks like the line-segment from $(1,0)$. to $(0,0)$. So WExpect $\lim_{m\nearrow\infty} T_m = 1$, And indeed, $\lim_{m\nearrow\infty}$ RhS(**??**) equals $\sqrt{1^2}$.

With m huge, look at the plane from a distance m. From that distance, curve L_m appears to be a line-segment from the origin to $(m,0)$. So *WExpect L_m to be asymptotic to m.* And from RhS(**??**),

$$\lim_{m\nearrow\infty} \frac{L_m}{m} = \left[\lim_{m\nearrow\infty} T_m\right] \cdot \left[\lim_{m\nearrow\infty} \frac{m-1}{m}\right] = 1 \cdot 1\,,$$

which equals 1, as predicted. □

b Fnc $\mathbf{F}(t) := (x(t), y(t))$ parametrizes the *Friendly* spiral $\mathbf{\Phi}$ by

$$*4: \quad x(t) = \mathrm{e}^t \cdot \cos(t) \quad \text{and} \quad y(t) = \mathrm{e}^t \cdot \sin(t)\,. \quad \text{Thus} \quad \tfrac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t[\cos(t) - \sin(t)]\,.$$

Note $\mathbf{F}$ is $\mathbf{P}_m$ when $\mathsf{K} = 1$. So, courtesy (**??**), the *speed function* for $\mathbf{F}$ is

$$*5: \quad \|\mathbf{F}'(t)\| = \mathsf{B}\mathrm{e}^t, \quad \text{where } \mathsf{B} = \sqrt{2}.$$

For $\tau \in [-\infty, 0)$, let $\mathbf{\Lambda}\langle\tau\rangle$ be the section of $\mathbf{\Phi}$ from $\mathbf{F}(\tau)$ to Q_1. Rotate $\mathbf{\Lambda}\langle\tau\rangle$ about the $y = x+3$ line $\mathbb{L}$, generating a surface-of-revolution whose area, $\mathcal{A}\langle\tau\rangle$, we now compute.

Note that all of $\mathbf{\Lambda}\langle\tau\rangle$ lies south-east of $\mathbb{L}$. And, measuring in the north-west direction, the ***signed-distance*** to $\mathbb{L}$ from a point $P := (a,b)$ in the plane, is

$$\mathcal{D}(P) := [3 + a - b]\big/\sqrt{2}$$

So the area of the surface is

$$\begin{aligned} \mathcal{A}\langle\tau\rangle &= \int_\tau^0 \overbrace{2\pi \cdot \mathcal{D}(\mathbf{F}(t))}^{\text{Circum}} \cdot \overbrace{\|\mathbf{F}'(t)\|\,\mathrm{d}t}^{\mathrm{d}s} \\ &= 2\pi \int_\tau^0 \left[3 + x(t) - y(t)\right] \cdot \mathrm{e}^t\,\mathrm{d}t \\ &= \frac{2\pi}{5}\left[18 - \mathrm{e}^\tau\left[15 + \mathrm{e}^\tau[3\cos(\tau) - \sin(\tau)]\right]\right]. \end{aligned}$$

(Geometrically, $\mathcal{A}\langle\tau\rangle \xrightarrow{\text{must}} 0$ as $\tau \nearrow 0$, and this formula indeed has this property.) Also, $\mathcal{A}\langle-\infty\rangle = \frac{2\pi}{5} \cdot 18 = \frac{36\pi}{5}$.

c Let $\mathbf{\Upsilon}$ be the part of $\mathbf{\Phi}$ from $B := \mathbf{F}(-\pi/2)$ to Q_1. Let R be the region up-from $\mathbf{\Upsilon}$ to the x-axis. Let $\mathbb{S}$ be the SoR obtained by rotating R about the x-axis.. Compute the volume of this SoR.

Using "CrSecArea" to label the area of the cross-section at time t, formulas (**??**) yield

$$*6: \quad \begin{aligned} \operatorname{Vol}(\mathbb{S}) &= \int_{-\pi/2}^{0} \overbrace{\pi \cdot y(t)^2}^{\text{CrSecArea}} \cdot \overbrace{\tfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t}^{\text{Width}} \\ &= \frac{\pi}{15} \cdot \left[1 - 3 \cdot \exp\left(\frac{-3\pi}{2}\right)\right]. \end{aligned}$$

d Showing the interesting steps, compute from $\mathbf{F}()$ the arclength parametrization $\mathbf{A}(s) = (x(s), y(s))$, of the spiral, satisfying that $\mathbf{A}(0) = \mathbf{F}(0)$.

Using speed fnc (**??**), the distance traveled by time τ is

$$*7: \qquad s(\tau) \;=\; \int_0^\tau \mathsf{B}\mathsf{e}^t\,dt \;=\; [\mathsf{e}^\tau - 1]\cdot\mathsf{B}\,.$$

To obtain the *time* at which the we have traveled a given *distance*, we write the inverse-fnc of $s()$ as

$$*8: \qquad \tau(s) \;:=\; \log\Big(1 + \frac{s}{\mathsf{B}}\Big)\,.$$

So the arclen-parametrization is $\mathbf{A}(s) = \mathbf{F}(\tau(s))$. Looking at the horiz. coord,

$$\begin{aligned} x(s) = \alpha(\tau(s)) &= [1 + \tfrac{s}{\mathsf{B}}]\cdot\cos(\log(1 + \tfrac{s}{\mathsf{B}})) \\ &= \Big[1 + \tfrac{s}{\sqrt{2}}\Big]\cdot\cos\Big(\log(1 + \tfrac{s}{\sqrt{2}})\Big). \end{aligned}$$

C2: 95 α The quotient and remainder polynomials,

$$q(x) = 2x^3 + x^2 - \frac{1}{3}$$

$$\& \; r(x) = 5x^2 - x + 2,$$

satisfy $B = [q\cdot C] + r$ and $\mathrm{Deg}(r) < \mathrm{Deg}(C)$, where $B(x) := 6x^6 + 3x^5 - 6x^4 + 2x^3 + 8x^2 + 1$ and $C(x) := 3x^3 - 3x + 3$.

β (Solve prob. #16[P]548, having replaced "70ft" by "80ft". We use $\mathbf{\Gamma} :: \frac{\text{lb}}{\text{ft}^3}$ for the weight-density of water.)

The dam is inclined at angle $\theta := 30°$ from vertical. Let $\widehat{\mathbf{F}}$ denote the total hydrostatic force on the tilted dam. Then

$$\widehat{\mathbf{F}} \;=\; \cos(\theta)\cdot\mathbf{F} \;=\; \frac{\sqrt{3}}{2}\cdot\mathbf{F}\,,$$

where $\mathbf{F}$ is the total force on the dam, *were the dam vertical*. We henceforth compute $\mathbf{F}$.

The dam is an isosceles trapezoid, $\mathbb{T}$, with bottom and top edges parallel; their lengths are $B := 50\text{ft}$ and $B + 2E := 100\text{ft}$, respectively. So $E = 25\text{ft}$.

The slant-height of $\mathbb{T}$ is $S := 80\text{ft}$. Thus the height of $\mathbb{T}$ (distance between parallel edges) is

$$\begin{aligned} H &:= \sqrt{S^2 - E^2}\,. \quad \text{And} \\ A &:= \mathrm{Area}(\mathbb{T}) \\ &= \mathrm{Area}(\text{Rect}) + \mathrm{Area}(\text{Triangles}) = [B\cdot H] + [E\cdot H]\,. \end{aligned}$$

The centroid of the $B\times H$ rectangle is distance $\frac{1}{2}H$ below the top edge. The centroid of the two triangles is distance $\frac{1}{3}H$ below the top edge. So the 1st-moment of $\mathbb{T}$ w.r.t the top edge is

$$\begin{aligned} M &:= \tfrac{1}{2}H\cdot BH + \tfrac{1}{3}H\cdot EH \\ &= [\tfrac{1}{2}B + \tfrac{1}{3}E]\cdot H^2\,. \end{aligned}$$

So the distance of Centroid($\mathbb{T}$) from the top edge is

$$Y \;:=\; \frac{M}{A} \;=\; \frac{\frac{1}{2}B + \frac{1}{3}E}{B + E}\cdot H\,.$$

The *pressure* at depth Y is $\mathbf{\Gamma}Y$. Thus the total force on the (vertical) dam is

$$\mathbf{F} \;=\; \text{Pressure}\cdot\text{Area} \;=\; \mathbf{\Gamma}Y\cdot A \;=\; \mathbf{\Gamma}M\,.$$

Consequently,

$$\begin{aligned} \mathbf{F} &= \mathbf{\Gamma}\cdot[\tfrac{1}{2}B + \tfrac{1}{3}E]\cdot H^2 \\ &= \mathbf{\Gamma}\cdot\Big[\frac{B}{2} + \frac{E}{3}\Big]\cdot[S^2 - E^2] \\ &= \mathbf{\Gamma}\cdot 192500\,\text{ft}^3\,. \qquad \text{Thus,} \end{aligned}$$

$$\begin{aligned} \widehat{\mathbf{F}} &= 96250\cdot\sqrt{3}\cdot[\text{ft}^3\cdot\mathbf{\Gamma}] \\ &\approx 166709.8903\cdot[\text{ft}^3\cdot\mathbf{\Gamma}]\,. \end{aligned}$$

γ For $M \in \mathbb{R}$, let $\overline{y_M}$ denote the y-coord of the centroid of R_M, the region in the lying above

parabola $y = x^2 =: \mathcal{B}(x)$ *and below*
line $y = 1 + Mx =: \mathcal{T}(x)$,

whose 1st quadrant-intersection has x-coordinate

$$‡1: \qquad \mathbf{U} = \mathbf{U}_M \;=\; \tfrac{1}{2}\Big[M + \sqrt{M^2 + 4}\Big]\,.$$

ITOf M and $\mathbf{U}{=}\mathbf{U}_M$, we see that

$$\begin{aligned} ‡2: \qquad \mathrm{Area}(R_M) &= \int_0^{\mathbf{U}} \overbrace{[\mathcal{T}(x) - \mathcal{B}(x)]}^{\text{Height}}\;\overbrace{dx}^{\text{Width}} \\ &= \mathbf{U} + \tfrac{1}{2}M\mathbf{U}^2 - \tfrac{1}{3}\mathbf{U}^3\,. \end{aligned}$$

To compute the *torque* about the $y{=}0$ line, note that the vertical cross-section of R_M at x has centroid at height $\frac{\mathcal{T}(x) + \mathcal{B}(x)}{2}$. So the torque equals

$$\begin{aligned} ‡3: \qquad \mathrm{Tor}(R_M) &= \int_0^{\mathbf{U}} \overbrace{\frac{\mathcal{T}(x) + \mathcal{B}(x)}{2}}^{\text{Lever arm}}\cdot\overbrace{[\mathcal{T}(x) - \mathcal{B}(x)]\,dx}^{\text{Area}} \\ &= \tfrac{1}{2}\mathbf{U} - \tfrac{1}{10}\mathbf{U}^5 + \tfrac{1}{6}M^2\mathbf{U}^3 + \tfrac{1}{2}M\mathbf{U}^2\,. \end{aligned}$$

Consequently, $\overline{y_M} \;=\; \frac{\mathrm{Tor}(R_M)}{\mathrm{Area}(R_M)}$.

Case: $M = 0$ From (**??**), the x-coord of intersection is $\mathbf{U} = \frac{1}{2}\cdot\sqrt{4} = 1$. Formulas (**??**) and (**??**) give us:

$$\begin{aligned} \mathrm{Tor}(R_0) &= \tfrac{1}{2} - \tfrac{1}{10} + 0 + 0 \overset{\text{note}}{=\!=} 2/5\,; \\ \mathrm{Area}(R_0) &= 1 + 0 - \tfrac{1}{3} \overset{\text{note}}{=\!=} 2/3\,. \end{aligned}$$

Thus $\overline{y_0} = \frac{2/5}{2/3} = 3/5$. (The $[M{=}0]$–case can easily be done directly, without computing general (**??**),(**??**) formulas.)

C3: [z] A multivariate polynomial, where each monomial has the same degree, is **homogeneous**.

[e] $\frac{A}{x-4}+\frac{B}{x+3}+\frac{C}{x+1} \stackrel{*}{=} \frac{h(x)}{[x-4][x+3][x+1]}$, where

$$h(x) \quad := \quad 2x^2 \,-\, 7x \,+\, 2\,.$$

To compute C, multiply both sides of $(*)$ by $x+1$, then evaluate at $x=$-1. This gives

$$C \;=\; \frac{h(\text{-}1)}{[\text{-}1-4][\text{-}1+3]} \;=\; \frac{11}{[\text{-}5]\cdot 2} \;=\; \frac{\text{-}11}{10}\,.$$

[f] Having fixed a number $G \geqslant 1$, let

3a:
$$y(t) \quad := \quad \mathsf{e}^{t\cdot G}\sin(t) \quad \text{and}$$
$$x(t) \quad := \quad \mathsf{e}^{t\cdot G}\cos(t)\,.$$

Fnc $\mathbf{P}() := \big(x(), y()\big)$ parametrizes a spiral Ω. The length of one wrap of Ω from $\big(1,0\big)$ to $\big(\mathsf{e}^{2\pi G},0\big)$ is, from (**??**),

$$\text{OneWrapLen} \quad = \quad \frac{\sqrt{G^2+1^2}}{G}\cdot\big[\mathsf{e}^{2\pi G} \,-\, 1\big]\,.$$

Let S_G denote the slope of Ω at $Q := \mathbf{P}(0) \stackrel{\text{note}}{=\!=} \big(1,0\big)$. When G=0 then our "spiral" is a circle, with vertical slope at Q. So $\boxed{\text{WExpect }\big[\lim_{G\to 0} S_G\big] = \infty}$. Conversely, as $G \to$ +∞ the spiral flattens out near Q; so $\boxed{\text{WExpect }\big[\lim_{G\to +\infty} S_G\big] = 0}$.

Having done the preparations, we use (**??**) to compute:

3b:
$$\frac{\mathrm{d}y}{\mathrm{d}x} \;=\; \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \;=\; \frac{G{\cdot}\sin(t) \;+\; \cos(t)}{G{\cdot}\cos(t) \;-\; \sin(t)}\,.$$

Evaluating at t=0 gives $S_G \;=\; \frac{1}{G}$.

[g] For the trapezoid-centroid problem, all the ideas are present in the above solution to the trapezoidal dam problem.

[h] The quotient and remainder polynomials,
$q(x) \;=\; 2x-6$ and $r(x) \;=\; 14x+12$,
satisfy $B \;=\; [q\cdot C]+r$ and $\operatorname{Deg}(r)<\operatorname{Deg}(C)$, where
$B(x) := 2x^3$ and $C(x) := x^2+3x+2$.

[i] That a spiral Ω is ***equi-angular*** precisely means that there is an angle, α, with the following property: *For each point $P \in \Omega$, the ray from the center of the spiral through P makes angle α with the Ω-tangent-line at P.*

[j] Triangle $\mathbf{T}$, with vertices at $\big(0,\pm 3\big)$ and $\big(9,0\big)$, has area

$$\mathcal{A} \;:=\; 3\cdot 9 \;=\; 27 \;=\; 3^3\,.$$

Along a median M of a triangle, the centroid occurs one-third of the way along M from the edge where the median terminates. Hence $\text{Centroid}(\mathbf{T}) = \big(3,0\big)$. The closest point on $\mathbb{L}$, the $[y = 3{+}x]$–line, is $\big(0,3\big)$, at distance $\mathcal{D} := 3\sqrt{2}$. Rotating $\mathbf{T}$ about $\mathbb{L}$ produces a SoR with volume

$$\text{Vol(SoR)} \;\stackrel{\text{Pappus}}{=\!=\!=}\; \text{Circum}\big(\mathcal{D}\text{-circle}\big)\cdot\mathcal{A}$$
$$=\; 2\pi\mathcal{D}\cdot\mathcal{A} \;=\; \pi\cdot 3^4\cdot 2\cdot\sqrt{2}\,.$$

End of Home&Class-C-ANS